Universal Uhrig Dynamical Decoupling for Bosonic Systems

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We construct efficient deterministic dynamical decoupling schemes protecting continuous-variable degrees of freedom. Our schemes target decoherence induced by quadratic system-bath interactions with analytic time dependence. We show how to suppress such interactions to Nth order using only N pulses. Furthermore, we show how to homogenize a 2^m-mode bosonic system using only $(N+1)^{2m+1}$ pulses, yielding—up to the Nth order—an effective evolution described by noninteracting harmonic oscillators with identical frequencies. The decoupled and homogenized system provides natural decoherence-free subspaces for encoding quantum information. Our schemes only require pulses which are tensor products of single-mode passive Gaussian unitaries and SWAP gates between pairs of modes.

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Decoherence due to unwanted system-environment interactions is a major obstacle on the road towards robust quantum information processing. Although quantum error correction and fault-tolerance provide general mechanisms to combat such sources of error, they are highly demanding in terms of resources. In near-term quantum devices, simpler strategies targeting a reduction of effective error rates at the physical level are more realistic. Dynamical decoupling (DD) is one of the success stories in this direction: originally developed in the context of NMR [1–3], it has been demonstrated in a wide range of systems [4–10]. In DD, unitary control pulses are instantaneously applied to the system at specific times. The goal is to average out the effect of the system-environment interaction, irrespective of its specific form.

A DD scheme is described by control pulses $\{U_j\}_{j=1}^{L}$ applied to the system S at times $\{t_j\}_{j=1}^{L} \subset [0,T]$ resulting in the evolution

$$U^{\mathsf{res}}(T) = U(T,t_L)\prod_{j=1}^{L}(U_j \otimes I_E)U(t_j,t_{j-1}). \quad (1)$$

Here $U(t+\Delta t, t)$ describes the uncontrolled time evolution generated by the decoherence Hamiltonian from time t to $t+\Delta t$. The pulse sequence achieves Nth order decoupling if there is an environment unitary U_E such that $\|U^{\mathsf{res}}(T) - I_S \otimes U_E\| = O(T^{N+1})$. The (nested) Uhrig dynamical decoupling (NUDD) scheme [11–13] is currently state of the art and has been experimentally demonstrated [14]: it is remarkably efficient, requiring only $(N+1)^{2M}$ Pauli pulses to suppress generic interactions between M qubits and their environment to order N. This scaling is significantly more efficient than what could be achieved by concatenating [15] first order schemes [16] where pulses from a unitary 1-design are applied at equidistant times.

DD for bosonic systems.—One may seek to construct similar protocols for bosonic systems. A natural class of system-environment interactions are Hamiltonians

$$H^{\mathsf{or}}(t) = \frac{1}{2}\sum_{j,k=1}^{2n} A_{j,k}(t)R_jR_k, \quad (2)$$

which are quadratic in the mode operators $R = (Q_1^S,\ldots,Q_{n_S}^S,P_1^S,\ldots,P_{n_S}^S,Q_1^E,\ldots,Q_{n_E}^E,P_1^E,\ldots,P_{n_E}^E)$ of system and environment; here $A(t) = A(t)^T \in \mathbb{R}^{2n\times 2n}$ is symmetric and $n = n_S + n_E$ is the total number of modes. These Hamiltonians generate one-parameter groups of Gaussian unitaries $U(t)$. Motivated by earlier work on decoherence suppression in a specific system-environment model [25], Arenz, Burgarth, and Hillier [26] have pioneered the systematic study of dynamical decoupling for infinite-dimensional systems: they showed that, even in this restricted context, decoupling cannot be achieved in the same strong sense as for qudit systems. While the application of unitary pulses at specific times can approximately decouple system and environment such that $U^{\mathsf{res}}(T) \approx U_S^{\mathsf{res}}(T) \otimes U_E^{\mathsf{res}}(T)$, no pulse sequence can render the system's evolution $U_S^{\mathsf{res}}(T)$ trivial for an arbitrary initial Hamiltonian [Eq. (2)]. One may, however, find sequences which simplify the system's evolution over time T to be of the form $U_S^{\mathsf{res}}(T) \approx e^{i\omega T H_0}$ where $H_0 = \frac{1}{2}\sum_{j=1}^{n_S}(Q_j^2 + P_j^2)$, a process referred to as *homogenization*. In other words, after applying a homogenization sequence, the effective decoupled and homogenized evolution

$$U^{\mathrm{res}}(T) \approx e^{i\omega T H_0} \otimes U_E^{\mathrm{res}}(T) \tag{3}$$

is simply that of noninteracting identical oscillators rotating with the same frequency. This evolution [Eq. (3)] is still highly beneficial for fault-tolerant quantum information processing as the eigenspaces of the number operator H_0 become decoherence free. By combining such schemes with very simple error-correcting codes spanned by tensor products of number states with fixed total number, such as those constructed in Ref. [27], reduced logical error rates can be achieved.

Here we construct new deterministic schemes that achieve the decoupling and homogenization of quadratic system-bath interactions to the Nth order using only a polynomial number (in N) of pulses. We note that such interactions are ubiquitous in physics, accurately describing present-day quantum optical setups [28]. Furthermore, they provide approximate descriptions of opto- and nanomechanical oscillators [29,30], atomic ensembles [31], and ion traps [32] in suitable parameter regimes. In particular, quadratic Hamiltonians generate typically considered and often dominant sources of noise such as single photon loss induced by resonator energy damping, or more generally thermal Gaussian noise. It is conceivable that the constructed schemes provide benefits for systems with additional nonlinear interactions (e.g., of Kerr type), but this is beyond the scope of this Letter. We note that our results extend to the case where the Hamiltonian [Eq. (2)] includes addition terms that are linear terms in the mode operators (cf. Ref. [17]).

Our analysis proceeds in the language of the symplectic group $\mathsf{Sp}(2n) = \{S \in \mathbb{R}^{2n\times 2n} | S J_n S^T = J_n\}$ and its Lie algebra $\mathfrak{sp}(2n) = \{X \in \mathbb{R}^{2n\times 2n} | X J_n + X^T J_n = 0\}$, using the fact that every Gaussian unitary generated by a Hamiltonian [Eq. (2)] is in one-to-one correspondence with a symplectic matrix $S^{\mathrm{or}}(t) \in \mathsf{Sp}(2n)$ generated by $X^{\mathrm{or}}(t) = A(t) J_n \in \mathfrak{sp}(2n)$. Here $J_n = J_{n_S} \oplus J_{n_E}$ is the $2n \times 2n$ matrix defining the symplectic form.

Instead of Eq. (1) we analyze the associated symplectic evolution

$$S^{\mathrm{res}}(T) = S^{\mathrm{or}}(T, t_L) \prod_{j=1}^{L} (S_j \oplus I_{2n_E}) S^{\mathrm{or}}(t_j, t_{j-1}), \tag{4}$$

where $S^{\mathrm{or}}(t_j, t_{j-1})$ is generated by $X^{\mathrm{or}}(t)$ from time t_{j-1} to t_j and the pulses $S_j \in \mathsf{Sp}(2n_S)$ are associated with Gaussian unitaries.

A bosonic decoupling scheme.—We propose the following pulse sequence: the passive Gaussian unitary U_S defined by its action

$$U_S Q_i U_S^* = -Q_i, \quad U_S P_i U_S^* = -P_i, \quad \text{for } i = 1, \ldots, n_S \tag{5}$$

on system mode operators is applied at times

$$t_j^{\mathrm{UDD}} = T\Delta_j, \Delta_j = \sin^2 \frac{j\pi}{2(N+1)} \tag{6}$$

for $j = 1, \ldots, N$. We note that U_S is a tensor product of single-mode phase flips.

Theorem 1: (Bosonic decoupling sequence) For any analytic generator $X^{\mathrm{or}}: [0, T] \to \mathfrak{sp}[2(n_S + n_E)]$, there are $S_S \in \mathbb{R}^{2n_S \times 2n_S}$ and $S_E \in \mathbb{R}^{2n_E \times 2n_E}$ such that the resulting evolution [Eq. (4)] after applying N pulses satisfies

$$\|S^{\mathrm{res}}(T) - S_S \oplus S_E\| = O(T^{N+1}). \tag{7}$$

Because of property [Eq. (7)], we call the pulse sequence an Nth order decoupling scheme. Note that in Ref. [26] a single application of the unitary U_S was shown to achieve first order decoupling. For higher orders, remarkably, the number of required applications is independent of the number of system and environment modes. The times [Eq. (6)] are those associated with UDD [33] for a single qubit.

In Theorem 1 (and throughout the main body of this Letter), we use the Frobenius norm $\|A\| = \sqrt{\mathrm{tr}(A^T A)}$. We state our bounds without detailed estimates on the constants in $O(T^{N+1})$. For concrete estimates on the required DD control rate $1/T$, a more refined analysis is necessary. As an example, we provide a rudimentary bound in the Supplemental Material [17] that involves the energy scales J_z and J_0 set by the system-bath interaction and the bath Hamiltonian, respectively: it takes the form $O\{[1/(N+1)!][(J_z + J_0)T]^{N+1}\}$. This mirrors some of the analysis conducted for single qubit UDD in Ref. [34], but we note that the reference also provides more detailed estimates.

Bosonic decoupling with arbitrary pulse times.—The original derivation of UDD [33,35] discusses the effect of π pulses (i.e., Pauli-σ_y) applied at *a priori* arbitrary times $t_1, \ldots, t_L$ to a system qubit coupled to a bosonic bath. The author focuses on a particular figure of merit defined as the overlap of the time-evolved qubit state with the original state. He finds that this "signal" is the inverse exponential of a parameter

$$\chi(T) = \int_0^\infty \frac{S_\beta(\omega)}{\omega^2} |y_L(\omega T)|^2 d\omega, \tag{8}$$

which depends on the noise spectrum $S_\beta(\omega)$ of the system-bath coupling, as well as the pulse times $t_1, \ldots, t_L$ via $y_L(z) = 1 - e^{iz} + 2\sum_{m=1}^{L} (-1)^m e^{izt_m/T}$. Equation (8) is then used to find optimal pulse times by minimizing the quantity $\chi(T)$. Furthermore, Eq. (8) permits us to compare the efficiency of different pulse sequences in a variety of regimes. In particular, for hard high-frequency cutoffs in $S_\beta(\omega)$, UDD pulse times are optimal, whereas for soft high-frequency cutoffs, the optimal sequences resemble periodic DD [36].

We argue in the Supplemental Material [17] that the Eq. (8) also completely characterizes bosonic decoupling for a single mode coupled to a bath of oscillators at inverse temperature β: assuming that the initial state is a product state (with the thermal state of the environment), and the pulse unitary [Eq. (5)] is applied at times $t_1, \ldots, t_L$, we find that the system's resulting evolution is described by a Gaussian quantum channel whose nonunitary component is fully specified by the quantity [Eq. (8)]. This provides a complementary justification for the pulse sequence considered in Theorem 1. Also, all statements about the optimality of pulse sequences and the temperature dependence of the decoupling efficiency translate immediately from the spin-boson setting to the one considered here.

Bosonic homogenization schemes.—We propose a pulse sequence that homogenizes an already decoupled evolution. Assume that the system has $n_s = 2^m$ modes labeled by bitstrings $\nu = (\nu_1, \ldots, \nu_m) \in \{0,1\}^m$. Let us introduce the passive Gaussian unitaries U_{y_0}, U_{x_i}, and U_{z_i} for $i = 1, \ldots, m$ that make up the control pulses. They are defined by their action on mode operators, i.e., by

$$\begin{aligned} U_{y_0} Q_\nu U_{y_0}^* &= P_\nu, \qquad U_{y_0} P_\nu U_{y_0}^* = -Q_\nu, \\ U_{x_i} Q_{(\nu_1,\ldots,\nu_m)} U_{x_i}^* &= Q_{(\nu_1,\ldots,\nu_{i-1},1-\nu_i,\nu_{i+1},\ldots,\nu_m)}, \\ U_{x_i} P_{(\nu_1,\ldots,\nu_m)} U_{x_i}^* &= P_{(\nu_1,\ldots,\nu_{i-1},1-\nu_i,\nu_{i+1},\ldots,\nu_m)}, \\ U_{z_i} Q_{(\nu_1,\ldots,\nu_m)} U_{z_i}^* &= (-1)^{\nu_i} Q_{(\nu_1,\ldots,\nu_m)}, \\ U_{z_i} P_{(\nu_1,\ldots,\nu_m)} U_{z_i}^* &= (-1)^{\nu_i} P_{(\nu_1,\ldots,\nu_m)}, \end{aligned} \tag{9}$$

for all $\nu = (\nu_1, \ldots, \nu_m) \in \{0,1\}^m$. We set $U_{y_i} = U_{x_i} U_{z_i}$. The unitary U_{y_0} acts as the same single-mode passive Gaussian unitary on all modes, U_{z_i} as single-mode phase flips on half of the modes, and U_{x_i} (respectively U_{y_i}) as two-mode SWAP (respectively, beam splitter) gates between pairs of modes. Depending on the experimental setup, the difficulty of realizing two-mode SWAP gates may differ significantly from that associated with single-mode passive gates. Unlike in the case of multiqubit DD schemes (which only require single-qubit Pauli gates), this fact needs to be taken into account when analyzing, e.g., the effect of finite pulse widths. For example, a SWAP gate in quantum optics may be realized simply by crossing fiberoptic wires, whereas a realization in circuit electrodynamics requires considerably more effort, see e.g., Ref. [37]. We emphasize, however, that unlike more recent bosonic fault-tolerance proposals [38], our schemes do not require any nonlinear gates, which are typically considered more challenging.

Using the unitaries [Eq. (9)], we show how to construct a multimode homogenization scheme from a multiqubit DD scheme: assume that an $(m+1)$-qubit DD scheme with qubits labeled from 0 to m uses pulses which are products of single-qubit Pauli matrices $(\sigma_w)_k$ where $w \in \{x, y, z\}$ and $k = 0, \ldots, m$. Then we construct the bosonic pulses by replacing Pauli factors (retaining their order in the product) according to the substitution rules

$$\begin{aligned} (\sigma_x)_0 &\mapsto U_{y_0}, \qquad (\sigma_x)_i \mapsto U_{x_i}, \\ (\sigma_y)_0 &\mapsto U_{y_0}, \qquad (\sigma_y)_i \mapsto U_{y_i}, \\ (\sigma_z)_0 &\mapsto I, \qquad (\sigma_z)_i \mapsto U_{z_i}, \end{aligned} \tag{10}$$

where $i = 1, \ldots, m$ and I means that no pulse is applied. Our homogenization scheme is obtained by applying the substitution rule [Eq. (10)] to the NUDD scheme [11–13] for $m+1$ qubits.

Theorem 2: (Bosonic homogenization sequence) The described pulse sequence consists of $(N+1)^{2m+1}$ passive Gaussian pulses. For any analytic generator $X^{\mathrm{or}}: [0,T] \to \mathfrak{sp}[2(2^m + n_E)]$ of the form $X_S^{\mathrm{or}}(t) \oplus X_E^{\mathrm{or}}(t)$, there are $\omega \in \mathbb{R}$ and $S_E \in \mathsf{Sp}(2n_E)$ such that the resulting evolution [Eq. (4)] satisfies

$$\|S^{\mathrm{res}}(T) - e^{\omega T J_{2^m}} \oplus S_E\| = O(T^{N+1}).$$

Theorem 2 assumes that the system and environment are already decoupled, i.e., that $H^{\mathrm{or}}(t) = H_S(t) \otimes I_E + I_S \otimes H_E(t)$; it guarantees a homogenized evolution since $J_{2^m} \in \mathfrak{sp}(2 \times 2^m)$ is the symplectic matrix associated with H_0. Correspondingly, we call the pulse sequence constructed here a bosonic homogenization sequence of order N. Combining decoupling and homogenization schemes (by concatenation [39]) leads to an effective evolution of the form of Eq. (3). In the remainder of this Letter, we sketch the proofs of Theorems 1 and 2.

Bosonic decoupling using Uhrig times.—To prove Theorem 1, we use the direct-sum structure of the matrix $X^{\mathrm{or}}(t) \in \mathfrak{sp}[2(n_S + n_E)]$, that is

$$X^{\mathrm{or}}(t) = \begin{pmatrix} X_{SS}(t) & X_{SE}(t) \\ X_{ES}(t) & X_{EE}(t) \end{pmatrix}.$$

Here $X_{BC}(t)$ are analytic functions of real $2n_B \times 2n_C$ matrices for $B, C \in \{S, E\}$ by assumption; $X_{SE}(t)$ and $X_{ES}(t)$ are responsible for system-environment interactions. We define the piecewise constant function $\sigma: [0,1] \to \{-1, 1\}$ that satisfies $\sigma(0) = 1$ and switches its sign at each $\{t_j/T\}_{j=1}^N$. For our analysis, we change into the toggling frame [40] with evolution $S^{\mathrm{tf}}(T)$ generated by

$$X^{\mathrm{tf}}(t) = \begin{pmatrix} X_{SS}(t) & \sigma(t/T) X_{SE}(t) \\ \sigma(t/T) X_{ES}(t) & X_{EE}(t) \end{pmatrix}.$$

Direct computation of the Dyson series of $S^{\mathrm{tf}}(T)$ shows that a sufficient condition for Nth order decoupling is the following [17].

The function $\sigma(t)$ is (or equivalently the times t_j are) a solution to the integral equations

$$\mathcal{F}^{r_1,\ldots,r_s}_{\gamma_1,\ldots,\gamma_s}(\sigma) = 0 \quad \text{if} \begin{cases} s + \sum_{k=1}^s r_k \leq N \text{ and} \\ \bigoplus_{k=1}^s \gamma_k = 1 \end{cases} \tag{11}$$

for all $s \in \mathbb{N}$, $r_1, \ldots, r_s \in \mathbb{N}_0$ and $\gamma_1, \ldots, \gamma_s \in \mathbb{Z}_2$, where $\mathcal{F}^{r_1,\ldots,r_s}_{\gamma_1,\ldots,\gamma_s}(\sigma) = \int_0^1 d\tau_s \ldots \int_0^{\tau_2} d\tau_1 \prod_{k=1}^s \sigma(\tau_k)^{\gamma_k} \tau_k^{r_k}$ and where $\oplus$ denotes addition modulo 2.

The same integral Eq. (11) appears in the analysis of the UDD scheme for a single qubit [13,41]; in particular, it is known that the times [Eq. (6)] are a solution. Thus we obtain a bosonic decoupling scheme (for an arbitrary number of modes) by using the times of the single-qubit UDD sequence.

Multiple qubits and multiple bosonic modes.—The proof of Theorem 2 relies on a connection between multiqubit systems and bosonic systems: we identify elements of $\mathsf{Sp}(2 \times 2^m)$ and a basis of its Lie algebra $\mathfrak{sp}(2 \times 2^m)$ which satisfy commutation relations analogous to those obeyed by the Pauli matrices. We associate mode operators with basis vectors of $\mathbb{R}^{2\times 2^m} \cong \mathbb{R}^2 \otimes (\mathbb{R}^2)^{\otimes m}$ by

$$Q_{(\nu_1,\ldots,\nu_m)} \leftrightarrow |q\rangle \otimes |e_{\nu_1}\rangle \otimes \cdots \otimes |e_{\nu_m}\rangle$$
$$P_{(\nu_1,\ldots,\nu_m)} \leftrightarrow |p\rangle \otimes |e_{\nu_1}\rangle \otimes \cdots \otimes |e_{\nu_m}\rangle.$$

Here we use an orthonormal basis $|q\rangle, |p\rangle$ of $\mathbb{R}^2$ for the first factor (which we will later identify with "qubit 0"), as well as an orthonormal basis $|e_0\rangle, |e_1\rangle$ for each of the remaining m factors (which will be identified with "qubits 1 to m"). On $\mathbb{R}^2$, let us define the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

that we also write as $S_{(0,0)}$, $S_{(1,0)}$, $S_{(1,1)}$, $S_{(0,1)}$, respectively. For $\alpha = (a_0, a_1, \ldots, a_m) \in (\mathbb{Z}_2^2)^{m+1}$ the matrix $S_\alpha = S_{a_0} \otimes S_{a_1} \otimes \cdots \otimes S_{a_m}$ on $\mathbb{R}^2 \otimes (\mathbb{R}^2)^{\otimes m}$ satisfies: (i) there is a subset $\Gamma \subset (\mathbb{Z}_2^2)^{m+1}$ such that $\{S_\alpha\}_{\alpha\in\Gamma}$ is a basis of the Lie algebra $\mathfrak{sp}(2 \times 2^m)$. And (ii) let $\tilde{\Gamma}$ be the set of $\alpha = (a_0, a_1, \ldots, a_m) \in (\mathbb{Z}_2^2)^{m+1}$ such that $a_0 \in \{(0,0), (1,1)\}$. Then S_β is orthogonal symplectic for every $\beta \in \tilde{\Gamma}$. We prove these properties in the Supplemental Material [17]. Using the commutation relations between x, y, z it is straightforward to verify that

$$S_\beta^{-1} S_\alpha S_\beta = (-1)^{\langle\alpha,\beta\rangle} S_\alpha \quad \text{for all } \alpha \in \Gamma, \quad \beta \in \tilde{\Gamma}, \tag{12}$$

where $\langle\alpha,\beta\rangle = \sum_{k=0}^m a_k^T J_1 b_k$ is the symplectic inner product on $(\mathbb{Z}_2^2)^{m+1}$. The relations [Eq. (12)] are analogous to the commutation relations $\sigma_\beta^{-1}\sigma_\alpha\sigma_\beta = (-1)^{\langle\alpha,\beta\rangle}\sigma_\alpha$ of Pauli operators for $m+1$ qubits [42].

Bosonic homogenization from qubit DD.—The close resemblance of the commutation relations [Eq. (12)] with those of Pauli matrices is key to our construction of homogenization schemes. We remark that for qubit DD schemes with Pauli pulses, it is precisely the phases $(-1)^{\langle\alpha,\beta\rangle}$ that lead to a cancellation of unwanted terms in the effective evolution. However, in contrast to the qubit setting, the available pulses in the bosonic setting are restricted to S_β where $S_{b_0} \in \{I, y\}$. This motivates the substitution rules [Eq. (10)], where on qubits $1, \ldots, m$ we replace σ_x, σ_y, σ_z by x, y, z, while on "qubit 0" we only allow I and y.

In the following, we analyze the effect of the resulting pulse sequence on a decoupled evolution. By assumption, the generator of the uncontrolled evolution satisfies $X^{\mathrm{or}}(t) = X_S^{\mathrm{or}}(t) \oplus X_E^{\mathrm{or}}(t) \in \mathfrak{sp}[2(2^m + n_E)]$ and

$$X_S^{\mathrm{or}}(t) = \sum_{\alpha\in\Gamma} B_\alpha(t) S_\alpha, \quad \text{where } B_\alpha(t) = \sum_{r=0}^\infty b_{\alpha,r} t^r$$

for $b_{\alpha,r} \in \mathbb{R}$ and where we use the basis of $\mathfrak{sp}(2 \times 2^m)$ from (i). Since it is decoupled, restricting to the system only (omitting the index S) is sufficient. Consider a general pulse sequence defined by the times $\{t_j\}_{j=1}^L$ and a function $\beta: \{1, \ldots, L\} \to \tilde{\Gamma}$ specifying which pulse $S_{\beta(j)}$ is applied at time t_j.

It is convenient to change into the toggling frame. By exploiting the symplectic group and algebra parametrizations from (ii) and (i) and using the relations Eq. (12), we find that the toggling frame evolution is generated by

$$X^{\mathrm{tf}}(t) = \sum_{\alpha\in\Gamma} F_\alpha(t/T) B_\alpha(t) S_\alpha, \tag{13}$$

where we have defined the functions

$$F_\alpha(t/T) = (-1)^{\sum_{j:t_j\le t}\langle\alpha,\beta(j)\rangle} \quad \text{for } t \in [0, T]. \tag{14}$$

With the generator's form [Eq. (13)], the toggling frame evolution $S^{\mathrm{tf}}(T) = \mathcal{T}\exp\left[\int_0^T X^{\mathrm{tf}}(t)dt\right]$ can be expanded in a Dyson series as

$$S^{\mathrm{tf}}(T) = \sum_{s=0}^\infty \sum_{\vec{\alpha}\in\Gamma^s} \sum_{\vec{r}=0}^\infty \prod_{k=1}^s S_{\alpha_k} b_{\alpha_k,r_k} \mathcal{F}_{\vec{\alpha}}^{\vec{r}} T^{s+\sum_{k=1}^s r_k}, \tag{15}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_s)$, $\vec{r} = (r_1, \ldots, r_s)$ and where

$$\mathcal{F}_{\vec{\alpha}}^{\vec{r}}(\{F_\alpha\}) = \int_0^1 d\tau_s \ldots \int_0^{\tau_2} d\tau_1 \prod_{k=1}^s F_{\alpha_k}(\tau_k)\tau_k^{r_k}.$$

Hence we can directly read off the Nth order term from Eq. (15) and obtain the following Nth order homogenization condition: for $s \in \mathbb{N}$, $r_1, \ldots, r_s \in \mathbb{N}_0$, and $\alpha_1, \ldots, \alpha_s \in \Gamma$, we have

$$\mathcal{F}_{\vec{\alpha}}^{\vec{r}}(\{F_\alpha\}) = 0 \quad \text{if} \begin{cases} s + \sum_{j=1}^s r_j \le N \text{ and} \\ \prod_{k=1}^s S_{\alpha_k} \notin \{\pm I_S, \pm J_{2^m}\}. \end{cases} \tag{16}$$

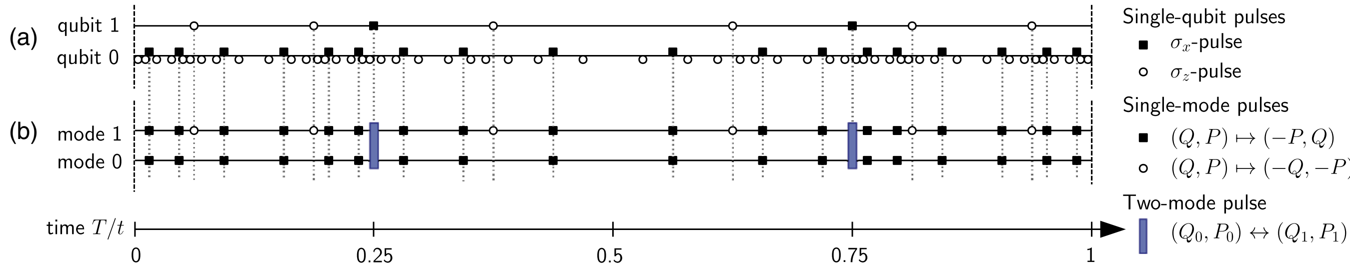

FIG. 1. Bosonic homogenization scheme (b) of suppression order $N = 2$ for two modes and the corresponding order-2 NUDD scheme (a). The evolution describing decoherence (horizontal straight lines) is interleaved with instantaneous control pulses.

The last step follows since a symplectic evolution of the form $c_1 I_S + c_2 J_{2^m}$ for $c_1, c_2 \in \mathbb{R}$ can be written as $e^{\omega T J_{2^m}}$ [17].

A similar analysis applies to $(m+1)$-qubit DD schemes with Pauli pulses. Consider a pulse sequence on $m+1$ qubits defined by $L \in \mathbb{N}$ pulses $U_j = \sigma_{\beta(j)}$ for $\beta: \{1, \ldots, L\} \to (\mathbb{Z}_2^2)^{m+1}$ that are applied to the system at times t_j. Then it achieves Nth order DD [13] if the functions $F_\alpha^{\mathrm{qubit}}$ defined by Eq. (14) for $\alpha \in (\mathbb{Z}_2^2)^{m+1}$ satisfy

$$\mathcal{F}_{\vec{\alpha}}^{\vec{r}}(\{F_\alpha^{\mathrm{qubit}}\}) = 0 \quad \text{if} \begin{cases} s + \sum_{j=1}^{s} r_j \leq N \text{ and} \\ \bigoplus_{k=1}^{s} \alpha_k \neq (0, \ldots, 0) \end{cases} \tag{17}$$

for all $s \in \mathbb{N}$, $r_1, \ldots, r_s \in \mathbb{N}_0$, and $\alpha_1, \ldots, \alpha_s \in (\mathbb{Z}_2^2)^{m+1}$.

The associated bosonic scheme obtained using the substitution rule [Eq. (10)] then has toggling frame generator specified by functions F_α^{bos} defined as

$$F_\alpha^{\mathrm{bos}}(t/T) = (-1)^{\sum_{j:t_j \leq t} \langle \alpha, \beta'(j) \rangle} \quad \text{for } t \in [0, T]$$

for all $\alpha \in \Gamma$. Here $\beta'(j) \in (\mathbb{Z}_2^2)^{m+1}$ differs from $\beta(j) \in (\mathbb{Z}_2^2)^{m+1}$ only in the first entry (associated with qubit 0), where (1,0) [respectively (0,1)] is replaced by (1,1) [respectively (0,0)] as prescribed by Eq. (10). Because the symplectic form is $J_{2^m} = -S_{(1,1,0,\ldots,0)}$, it is straightforward to verify that the property [Eq. (17)] of the functions $F_\alpha^{\mathrm{qubit}}$ implies the desired property [Eq. (16)] for the functions F_α^{bos} (cf. Ref. [17]). In other words, the decoupling property in the qubit setting translates to homogenization of bosonic modes.

Having established a general connection between universal $(m+1)$-qubit DD schemes and bosonic homogenization of 2^m modes, Theorem 2 follows immediately by applying this to the NUDD sequence which achieves Nth order decoupling of $(m+1)$ qubits with $(N+1)^{2(m+1)}$ Pauli pulses [17]. An example is shown in Fig. 1.

Conclusions.—Our Letter introduces novel, highly efficient dynamical decoupling schemes for bosonic systems. Instead of applying finite-dimensional (qubit) decoupling procedures to distinguished subspaces, our schemes are of a genuinely continuous-variable nature. This leads to remarkably simple schemes involving only passive Gaussian unitaries. On a conceptual level, our work establishes a tight connection between qubit- and continuous-variable schemes. In particular, it implies e.g., that considerations related to pulse imperfections such as finite widths (see, e.g., Refs. [41,43]) translate immediately to our bosonic schemes. More generally, this analogy may be used to lift other qubit protocols to the bosonic context. On a practical level, we believe that our protocols could become a powerful tool for continuous-variable quantum information processing as they pose minimal experimental requirements.

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[1] U. Haeberlen and J. S. Waugh, Phys. Rev. **175**, 453 (1968).
[2] J. S. Waugh, L. M. Huber, and U. Haeberlen, Phys. Rev. Lett. **20**, 180 (1968).
[3] E. L. Hahn, Phys. Rev. **80**, 580 (1950).
[4] E. R. Jenista, A. M. Stokes, R. T. Branca, and W. S. Warren, J. Chem. Phys. **131**, 204510 (2009).
[5] G. A. Álvarez, A. Ajoy, X. Peng, and D. Suter, Phys. Rev. A **82**, 042306 (2010).
[6] G. de Lange, Z. H. Wang, D. Ristè, V. V. Dobrovitski, and R. Hanson, Science **330**, 60 (2010).
[7] J. Du, X. Rong, N. Zhao, Y. Wang, J. Yang, and R. B. Liu, Nature (London) **461**, 1265 (2009).
[8] Z.-H. Wang, W. Zhang, A. M. Tyryshkin, S. A. Lyon, J. W. Ager, E. E. Haller, and V. V. Dobrovitski, Phys. Rev. B **85**, 085206 (2012).
[9] C. Barthel, J. Medford, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Phys. Rev. Lett. **105**, 266808 (2010).
[10] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, Nature (London) **458**, 996 (2009).

[11] M. Mukhtar, W. T. Soh, T. B. Saw, and J. Gong, Phys. Rev. A **82**, 052338 (2010).
[12] Z.-Y. Wang and R.-B. Liu, Phys. Rev. A **83**, 022306 (2011).
[13] L. Jiang and A. Imambekov, Phys. Rev. A **84**, 060302(R) (2011).
[14] H. Singh, Arvind, and K. Dorai, Phys. Rev. A **95**, 052337 (2017).
[15] K. Khodjasteh and D. A. Lidar, Phys. Rev. Lett. **95**, 180501 (2005).
[16] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).
[17] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.010501 for a detailed proof of this statement and a revision of multiqubit DD, which includes Refs. [18–24].
[18] W.-J. Kuo and D. A. Lidar, Phys. Rev. A **84**, 042329 (2011).
[19] J. R. West, B. H. Fong, and D. A. Lidar, Phys. Rev. Lett. **104**, 130501 (2010).
[20] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
[21] K. Shiokawa and D. A. Lidar, Phys. Rev. A **69**, 030302(R) (2004).
[22] L. F. Santos and L. Viola, Phys. Rev. A **72**, 062303 (2005).
[23] H. Y. Carr and E. M. Purcell, Phys. Rev. **94**, 630 (1954).
[24] S. Meiboom and D. Gill, Rev. Sci. Instrum. **29**, 688 (1958).
[25] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999).
[26] C. Arenz, D. Burgarth, and R. Hillier, J. Phys. A **50**, 135303 (2017).
[27] Y. Ouyang, arXiv:1809.09801.
[28] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics* Vol. 56 (Springer Science & Business Media, New York, 2004).
[29] S. Pirandola, S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. A **68**, 062317 (2003).
[30] M. Woolley, G. Milburn, and C. M. Caves, New J. Phys. **10**, 125018 (2008).
[31] C. A. Muschik, H. Krauter, K. Hammerer, and E. S. Polzik, Quantum Inf. Process. **10**, 839 (2011).
[32] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).
[33] G. S. Uhrig, Phys. Rev. Lett. **98**, 100504 (2007).
[34] G. S. Uhrig and D. A. Lidar, Phys. Rev. A **82**, 012301 (2010).
[35] G. S. Uhrig, New J. Phys. **10**, 083024 (2008).
[36] S. Pasini and G. S. Uhrig, Phys. Rev. A **81**, 012309 (2010).
[37] L. Chirolli, G. Burkard, S. Kumar, and D. P. DiVincenzo, Phys. Rev. Lett. **104**, 230502 (2010).
[38] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).
[39] We remark that Theorem 2 extends to (not necessarily symplectic) generators $X^{\mathrm{or}}(t)\colon[0,T]\to\mathbb{R}^{2n\times 2n}$. Therefore, it can be applied in conjunction with Theorem 1.
[40] The toggling frame evolution is defined as $S^{\mathrm{tf}}(t)=S^{\mathrm{ctr}}(t)^{-1}S^{\mathrm{or}}(t)$ with generator $X^{\mathrm{tf}}(t)=S^{\mathrm{ctr}}(t)^{-1}X^{\mathrm{or}}(t)S^{\mathrm{ctr}}(t)\in\mathfrak{sp}(2n)$. Here the control evolution $S^{\mathrm{ctr}}(t)$ of the pulse sequence is the product of all pulses applied up to time t.
[41] W. Yang and R.-B. Liu, Phys. Rev. Lett. **101**, 180403 (2008).
[42] For $\alpha=(a_0,a_1,\ldots,a_m)\in(\mathbb{Z}_2^2)^{m+1}$, the multiqubit Pauli operators $\sigma_\alpha=\sigma_{a_0}\otimes\sigma_{a_1}\otimes\cdots\otimes\sigma_{a_m}$ are defined as tensor products of the Pauli matrices $\sigma_{(0,0)}=I$, $\sigma_{(1,0)}=\sigma_x$, $\sigma_{(1,1)}=\sigma_y$, $\sigma_{(0,1)}=\sigma_z$.
[43] G. S. Uhrig and S. Pasini, New J. Phys. **12**, 045001 (2010).