From the Weyl Anomaly to Entropy of Two-Dimensional Boundaries and Defects

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We study whether the relations between the Weyl anomaly, entanglement entropy (EE), and thermal entropy of a two-dimensional (2D) conformal field theory (CFT) extend to 2D boundaries of 3D CFTs, or 2D defects of $D \geq 3$ CFTs. The Weyl anomaly of a 2D boundary or defect defines two or three central charges, respectively. One of these, b, obeys a c theorem, as in 2D CFT. For a 2D defect, we show that another, d_2, interpreted as the defect's "conformal dimension," must be non-negative if the averaged null energy condition holds in the presence of the defect. We show that the EE of a sphere centered on a planar defect has a logarithmic contribution from the defect fixed by b and d_2. Using this and known holographic results, we compute b and d_2 for 1/2-Bogomol'nyi-Prasad-Sommerfield surface operators in the maximally supersymmetric (SUSY) 4D and 6D CFTs. The results are consistent with b's c theorem. Via free field and holographic examples we show that no universal "Cardy formula" relates the central charges to thermal entropy.

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Introduction.—CFTs play a central role in many branches of physics. In condensed matter physics, they describe critical points. In string theory, the worldsheet theory is a CFT. In quantum field theory, CFTs are fixed points of renormalization group (RG) flows.

CFTs in two-dimensional Minkowski space, i.e., 2D CFTs, enjoy Virasoro symmetry with central charge c. Unitarity plus ground state normalizability implies $c \geq 0$. For an RG flow from ultraviolet (UV) to infrared (IR) CFTs with central charges $c_{\rm UV}$ and $c_{\rm IR}$, respectively, unitarity, locality, and Poincaré symmetry imply the c theorem: $c_{\rm UV} \geq c_{\rm IR}$ [1]. These properties suggest c measures the effective number of massless degrees of freedom (DOF), which is expected to be non-negative and to decrease along RG flows.

Virasoro symmetry also implies that c determines at least four other following quantities that can count DOF: (i) c fixes the normalization of the two-point function of the stress-energy tensor, $T^{\mu\nu}$. (ii) On a background with a nontrivial spacetime metric $g_{\mu\nu}$ with Ricci scalar $\mathcal{R}$, quantum effects can produce the Weyl anomaly, $T^\mu_\mu = -(c/24\pi)\mathcal{R}$ [2–5]. (iii) The EE of a spatial interval of length ℓ, which measures the strength of vacuum correlations, is $S_{\rm EE} = (c/3)\ln(\ell/\varepsilon) + \mathcal{O}(\varepsilon^0)$ [6,7], with UV cutoff $0 < \varepsilon \ll 1$. (iv)At nonzero temperature T the CFT's entropy density, s, which measures the number of thermodynamic microstates, is $s = (\pi/6)cT$ [8,9].

While CFTs have an infinite correlation length, no real system is infinite: boundary conditions (BCs) will always be important. Moreover, no real system is perfect: defects such as impurities, domain walls between differently ordered phases, and so on will always be important. Constructing and classifying CFTs with conformally invariant boundaries (BCFTs) or defects (DCFTs) is thus crucial for describing an enormous number of systems.

In this Letter, we study a 2D boundary of a 3D CFT or 2D conformal defect in a $D \geq 3$ CFT. We assume the boundary or defect is flat, i.e., a static straight line. Such a system does not have Virasoro symmetry in general: the 2D contribution to $T^{\mu\nu}$ is not conserved because energy and momentum can flow between the boundary or defect and the bulk CFT, and these systems have a finite number of symmetry generators following from the CFT's $SO(D,2)$ conformal symmetry being broken to $SO(2,2) \times SO(D-2)$, where $SO(2,2)$ are conformal transformations leaving the static line invariant and $SO(D-2)$ are rotations about the static line [10].

We find that the logarithmic term in $S_{\rm EE}$ of a spherical region centered on a defect is fixed by T^μ_μ, while in general no simple relation exists between T^μ_μ and s. The boundary or defect contribution to T^μ_μ includes multiple central charges [11,12]. Assuming the averaged null energy condition (ANEC) holds in the presence of the defect, we conjecture a positivity bound on one of the defect central charges. In Ref. [13], we also find new central charges allowed in 4D if parity is broken. We use the method of Refs. [21,22] to show that the logarithmic term in general depends on *two* defect central charges—confirming and extending a key result of Ref. [23]. Using this and known holographic results, we compute these central charges for certain 1/2-Bogomol'nyi-Prasad-Sommerfield (BPS) surface operators in the maximally SUSY 4D and 6D CFTs. Finally, for the free massless scalar and fermion 3D BCFTs and for 2D defects holographically dual to probe branes, we calculate $s \propto T$ at the boundary or defect, with no universal relation between the proportionality coefficient and central charges.

Conventions.—We start with a local, unitary, Lorentzian CFT on a $D \geq 3$ spacetime $\mathcal{M}$ with coordinates x^μ ($\mu = 0, 1, \ldots, D-1$) and metric $g_{\mu\nu}$, which we will call the "bulk" CFT. We introduce a codimension $D-2$ defect along a static 2D submanifold Σ with coordinates y^a ($a = 0, 1$). We parametrize $\Sigma \hookrightarrow \mathcal{M}$ by embedding functions $X^\mu(y)$ such that Σ's induced metric is $\gamma_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$. We denote $\mathcal{M}$'s covariant derivative as ∇_μ and Σ's induced covariant derivative as $\hat{\nabla}_a$, which acts on a mixed tensor $\mathcal{T}^\mu_a$ as $\hat{\nabla}_a \mathcal{T}^\mu_b = \partial_a \mathcal{T}^\mu_b + \Gamma^\mu_{\nu a} \mathcal{T}^\nu_b - \hat{\Gamma}^c_{ab} \mathcal{T}^\mu_c$. The second fundamental form is then $\Pi^\mu_{ab} = \hat{\nabla}_a \partial_b X^\mu$, with traceless part $\mathring{\Pi}^\mu_{ab} \equiv \Pi^\mu_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}\Pi^\mu_{cd}$.

Physically, the defect can arise from 2D DOF coupled to the bulk CFT and/or BCs imposed on bulk CFT fields [24]. In 3D the defect is a domain wall between two CFTs, and if one of these is the "trivial" CFT, then the defect is a boundary.

The Weyl anomaly.—We denote the DCFT partition function as Z and the generating functional of connected correlators $W \equiv -i \ln Z$, which are both functionals of $g_{\mu\nu}$ and X^μ. Varying W, we define the stress-energy tensor, $T_{\mu\nu}$, and displacement operator, $\mathcal{D}_\mu$

$$\delta W = \frac{1}{2}\int d^D x \sqrt{-g}\,\delta g_{\mu\nu} \langle T^{\mu\nu} \rangle + \int d^2 y \sqrt{-\gamma}\,\delta X^\mu \langle \mathcal{D}_\mu \rangle,$$

where $g \equiv \det g_{\mu\nu}$ and $\gamma \equiv \det \gamma_{ab}$. Invariance of W under reparametrizations of y^a implies $\mathcal{D}_\mu$'s components along Σ vanish [29]. Invariance of W under reparametrizations of x^μ implies $\nabla_\nu \langle T^{\nu\mu} \rangle = -\delta^{D-2} \langle \mathcal{D}^\mu \rangle$, with δ^{D-2} a delta function that restricts to Σ [29]. Physically, $\langle T^{\mu\nu} \rangle$ is not conserved at Σ because the defect and bulk can exchange transverse energy momentum.

Our DCFTs are invariant under infinitesimal Weyl transformations, $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta_\omega X^\mu = 0$, up to the Weyl anomaly [2–5]: $\delta_\omega W = \int d^D x \sqrt{-g}\,\omega \langle T^\mu_\mu \rangle$, where $\langle T^\mu_\mu \rangle$ is built from external sources, such as $g_{\mu\nu}$. We consider contributions to $\langle T^\mu_\mu \rangle$ from $g_{\mu\nu}$ and $\partial_a X^\mu$ only. Determining $\langle T^\mu_\mu \rangle$'s most general form requires solving the Wess-Zumino (WZ) consistency condition, $[\delta_{\omega_1}, \delta_{\omega_2}]W = 0$, and then fixing any local counterterms that contribute to $\langle T^\mu_\mu \rangle$. In our DCFTs, $\langle T^\mu_\mu \rangle = \langle T^\mu_\mu \rangle_{\text{bulk}} + \delta^{D-2} \langle T^\mu_\mu \rangle_{\text{def}}$, where $\langle T^\mu_\mu \rangle_{\text{bulk}}$ and $\langle T^\mu_\mu \rangle_{\text{def}}$ are bulk CFT and defect Weyl anomalies, respectively, and we fixed local counterterms to cancel terms with normal derivatives of δ^{D-2}. For $\langle T^\mu_\mu \rangle_{\text{bulk}}$ we will only need to know that $\langle T^\mu_\mu \rangle_{\text{bulk}} = 0$ in odd D, but can be nonzero in even D, which defines the bulk central charge(s). For a 2D defect in a $D \geq 3$ DCFT [11,12,30]

$$\langle T^\mu_\mu \rangle_{\text{def}} = -\frac{1}{24\pi}\left(b\mathcal{R}_\Sigma + d_1 \mathring{\Pi}^\mu_{ab} \mathring{\Pi}^{ab}_\mu - d_2 W^{ab}_{ab}\right), \quad (1)$$

where $\mathcal{R}_\Sigma$ is Σ's intrinsic scalar curvature, W_{abcd} is the pullback of the bulk Weyl tensor to Σ, and b, d_1, and d_2 are defect central charges. When $D = 3$, $W_{abcd} = 0$ identically, so d_2 exists only for $D \geq 4$.

Bounds on central charges.—As mentioned above, in a 2D CFT c determines various observables and with reasonable assumptions, such as unitarity, obeys $c \geq 0$ and the c theorem. By comparison, less is known about b, d_1, and d_2. Under Weyl transformations $\sqrt{-\gamma}\mathcal{R}_\Sigma$ changes by a total derivative, while both $\sqrt{-\gamma}\mathring{\Pi}^\mu_{ab}\mathring{\Pi}^{ab}_\mu$ and $\sqrt{-\gamma}W^{ab}_{ab}$ are invariant [4]. As a result, in the Euclidean DCFT on $\mathbb{S}^D$ of radius r with bulk partition function Z_{CFT} and with defect along a maximal $\mathbb{S}^2$, $Z/Z_{\text{CFT}} \propto (r\Lambda)^{b/3}$ [29]. For a local, unitary defect RG flow, b obeys a c theorem, suggesting b counts defect DOF [29]. WZ consistency forces b to be independent of any marginal couplings. The normalization of $\mathcal{D}^\mu$'s two-point function is fixed by d_1, such that unitarity implies $d_1 \geq 0$ [31,32].

Table I shows b, d_1, and d_2 in the 3D BCFTs of a free, massless real scalar or Dirac fermion [29,33,34], and in CFTs holographically dual to Einstein gravity in $(D+1)$-dimensional anti–de Sitter space, AdS_{D+1}, with metric G_{MN} ($M, N = 0, 1, \ldots, D$) and defect dual to a probe brane along AdS_3 whose action $S_{\text{probe}} = -T_{\text{br}} \int d^3\xi \sqrt{-\det(P[G_{MN}])}$ with tension T_{br} and brane coordinates ξ [30]. In all of these theories, the central charges are ≥ 0 with the exception of the scalar with Dirichlet BC, which has $b < 0$. This example proves that unitarity does not require $b \geq 0$.

For a unitary 3D BCFT with unique stress-energy tensor at the boundary [35], Ref. [32] conjectured

$$b = \frac{2\pi^2}{3}\epsilon(1) - \frac{2}{3}d_1, \quad (2)$$

TABLE I. Central charges b, d_1, and d_2 of Eq. (1) for 3D BCFTs of free, massless real scalars with Dirichlet or Robin BC, or Dirac fermion with the unique conformal "mixed" BC [29,33,34], and for a 2D defect dual to a probe brane of tension T_{br} along AdS_3 inside AdS_{D+1} of radius L [30].

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Robin	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A
Probe brane	N/A	$6\pi L^3 T_{\text{br}}$	$6\pi L^3 T_{\text{br}}$	$6\pi L^3 T_{\text{br}}$

where $\epsilon(v)$ is a contribution to $T^{\mu\nu}$'s two-point function from exchange of spin-2 boundary operators, with $v \in [0,1]$ the BCFT conformal cross ratio, with boundary at $v = 1$ [31,36,37]. Unitarity implies $\epsilon(v) \geq 0$ [31]. However, if the BCFT has any spin-2 boundary operators of dimension $\Delta \in [2,3)$, then $\epsilon(v)$ diverges as $(1-v)^{\Delta-3}$ when $v \to 1$. In that case, unitarity does not constrain the sign of $\epsilon(1)$, the $(1-v)^0$ term in $\epsilon(v)$'s expansion about $v = 1$, and so b has no definite lower bound. On the other hand, in the absence of such operators $\epsilon(v)$ is regular as $v \to 1$, unitarity implies $\epsilon(1) \geq 0$, and hence $b \geq -\frac{2}{3}d_1$. All the examples in Table. I obey this bound, and the free scalar with Dirichlet BC saturates it.

We can prove a new bound assuming the ANEC holds in the presence of the defect. The ANEC states that for any null direction u, $\int_{-\infty}^{\infty} du \langle T_{uu}\rangle \geq 0$. Proofs of the ANEC for CFTs appear in Refs. [38,39]. Though these proofs have not yet been extended to BCFTs or DCFTs, they rely mainly on unitarity and causality, which in a BCFT or DCFT should suffice to guarantee that a lightlike observer's total energy is ≥ 0.

While $SO(D,2)$ symmetry forces $\langle T_{\mu\nu}\rangle = 0$ for the undeformed CFT, the $SO(2,2)\times SO(D-2)$ symmetry preserved by Σ allows $\langle T_{\mu\nu}\rangle \neq 0$. When $D = 4$, Refs. [25,27] showed that by writing the most general form of $\langle T_{\mu\nu}\rangle$ such that it is well defined as a distribution and comparing its variation under constant Weyl transformations to the variation of $\langle T^{\mu}_{\mu}\rangle_{\rm def}$ with respect to $g_{\mu\nu}$, $\langle T_{\mu\nu}\rangle$ completely determined by d_2. Generalizing to arbitrary $D \geq 4$ is straightforward: with coordinates x^i transverse to Σ ($i = 2,3,\ldots,D-1$), and for a point a distance $|x^i| > 0$ from Σ

$$\langle T^{ab}\rangle = -\frac{h_D}{2\pi}\frac{\eta^{ab}}{|x^i|^D}, \qquad \langle T^{ai}\rangle = 0,$$
$$\langle T^{ij}\rangle = \frac{h_D}{2\pi(D-3)}\frac{3\delta^{ij}|x^k|^2 - D x^i x^j}{|x^l|^{D+2}},$$
$$h_D \equiv \frac{1}{3\mathrm{vol}(\mathbb{S}^{D-3})}\frac{D-3}{D-1}d_2, \tag{3}$$

where h_D is the defect's "conformal dimension" (see, e.g., Ref. [40]). Using $SO(2,2)\times SO(D-2)$ transformations, any null geodesic a distance R from Σ can be mapped to

$$t = Ru, \quad x^1 = Ru\cos\psi, \quad x^2 = Ru\sin\psi, \quad x^3 = R, \tag{4}$$

and $x^{i>3} = 0$, where ψ is the angle between Σ and the null geodesic. Plugging Eqs. (3) and (4) into the ANEC gives

$$\int_{-\infty}^{\infty} du\langle T_{uu}\rangle = \frac{1}{6\sqrt{\pi}R^D}\frac{|\sin\psi|}{\mathrm{vol}(\mathbb{S}^{D-3})}\frac{\Gamma(\frac{D-1}{2})}{\Gamma(\frac{D}{2})}d_2 \geq 0, \tag{5}$$

which immediately implies $d_2 \geq 0$.

EE and central charges.—Consider a CFT in D-dimensional Minkowski space with a flat 2D defect. We will compute the EE of a sphere of radius ℓ centered on the defect, using the method of Refs. [21,22].

We parametrize the Minkowski metric as

$$\eta = -dt^2 + (dx^1)^2 + (d|x^i|)^2 + |x^i|^2 ds^2_{\mathbb{S}^{D-3}}, \tag{6}$$

with the defect along t and x^1 and located at $|x^i| = 0$. Defining $r^2 = (x^1)^2 + |x^i|^2$, the sphere's causal development is given by $r \pm t \leq \ell$. The change of coordinates

$$t = \frac{\ell\cos\theta\sinh(\frac{\tau}{\ell})}{1+\cos\theta\cosh(\frac{\tau}{\ell})}, \qquad r = \frac{\ell\sin\theta}{1+\cos\theta\cosh(\frac{\tau}{\ell})},$$
$$x^1 = r\cos\phi, \qquad |x^i| = r\sin\phi. \tag{7}$$

maps the sphere's causal development to the static patch of D-dimensional de Sitter space, dS_D, with metric

$$\Omega^2\eta = -\ell^{-2}\cos^2\theta d\tau^2 + d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi ds^2_{\mathbb{S}^{D-3}}),$$
$$\Omega = 1 + \cos\theta\cosh(\tau/\ell), \tag{8}$$

with $\tau \in (-\infty,\infty)$, $\theta \in [0,\pi/2]$, and $\phi \in [0,\pi]$. The defect is then along τ and θ and is located at $\phi = 0,\pi$, i.e., along a maximal dS_2.

The reduced density matrix of the sphere's causal development maps to $e^{-\beta H_\tau}$ modulo normalization, with $\beta = 2\pi\ell$ and H_τ the generator of τ translations. As a result, $S_{\rm EE}$ maps to thermal entropy in dS_D at inverse temperature $\beta = 2\pi\ell$, $S_{\rm EE} = \beta E - F$, with E the Killing energy corresponding to H_τ and $F = -\ln\mathrm{tr}(e^{-\beta H_\tau})$ the dimensionless free energy. We define the defect's contribution as $S^{\rm def}_{\rm EE} \equiv S_{\rm EE} - S^{\rm CFT}_{\rm EE}$, with $S^{\rm CFT}_{\rm EE}$ the EE of a sphere of radius ℓ in the bulk CFT, with the same UV cutoff, and similarly for $E^{\rm def}$ and $F^{\rm def}$.

Analytically continuing to Euclidean time, $\tau = -i\tau_E$ with $\tau_E \sim \tau_E + \beta$, Eq. (8) becomes the metric of $\mathbb{S}^D$, with the defect wrapping a maximal $\mathbb{S}^2$. As a result, $F = -\ln Z$, with Z the DCFT's Euclidean partition function on $\mathbb{S}^D$. Using $Z/Z_{\rm CFT} \propto (r\Lambda)^{b/3}$ with $\Lambda = 1/\varepsilon$, we find

$$F^{\rm def} = -\ln(Z/Z_{\rm CFT}) = -\frac{b}{3}\ln\left(\frac{\ell}{\varepsilon}\right) + \mathcal{O}(\varepsilon^0). \tag{9}$$

The defect's contribution to the Killing energy is

$$E^{\rm def} = \int dS_\mu K_\nu \langle T^{\mu\nu}\rangle_{\rm def}, \tag{10}$$

with $dS_\mu = \ell^{-1}\cos\theta(\sin\theta)^{D-2}(\sin\phi)^{D-3}d\theta d\phi ds_{\mathbb{S}_{D-3}}\delta^\tau_\mu$ the volume element on a constant time slice, and $K^\mu\partial_\mu = \partial_\tau$ the time translation Killing vector. We thus need $\langle T^\tau{}_\tau\rangle_{\rm def}$, which we obtain by Weyl transformation from the Minkowski-space $\langle T^{\mu\nu}\rangle$ in Eq. (3), with the result

$$\langle T^{\tau}_{\tau}\rangle_{\rm def} = -\frac{(\sin\theta\sin\phi)^{-D}}{6\pi {\rm vol}(\mathbb{S}^{D-3})}\frac{D-3}{D-1}d_2 - \frac{b}{24\pi}\delta^{D-2}, \quad (11)$$

where the first term comes from Weyl-rescaling $T^{\mu\nu}$ and the second term comes from $T^{\mu\nu}$'s anomalous Weyl transformation law. The integral of the second term is finite, but the integral of the first diverges at $\theta = 0$ and the defect $\phi = 0, \pi$. Using a regulator ε/ℓ, we thus find

$$\beta E^{\rm def} = -\frac{1}{3}\frac{D-3}{D-1}d_2 \ln\left(\frac{\ell}{\varepsilon}\right) + \mathcal{O}(\varepsilon^0). \quad (12)$$

Plugging Eqs. (9) and (12) into $S_{\rm EE}$ then gives [41]

$$S^{\rm def}_{\rm EE} = \frac{1}{3}\left(b - \frac{D-3}{D-1}d_2\right)\ln\left(\frac{\ell}{\varepsilon}\right) + \mathcal{O}(\varepsilon^0). \quad (13)$$

We have also derived Eq. (13) via the replica method, generalizing the result of Ref. [34] for 3D BCFTs to $D \geq 3$ DCFTs [44].

Further checking Eq. (13), we consider the holographic DCFT given by a probe brane along an AdS_3 submanifold inside AdS_{D+1} of radius L, with action $S_{\rm probe}$ above. In that case, Ref. [29] found

$$S^{\rm def}_{\rm EE} = \frac{4\pi T_{\rm br}L^3}{D-1}\ln\left(\frac{\ell}{\varepsilon}\right) + \mathcal{O}(\varepsilon^0), \quad (14)$$

which agrees with Eq. (13), using b and d_2 from Table I.

Equation (13) agrees with a key result of Ref. [23] that the coefficient of $\ln(\ell/\varepsilon)$ in $S^{\rm def}_{\rm EE}$ includes a contribution from $\langle T_{\mu\nu}\rangle$, which we have shown is proportional d_2.

For defects of various dimensions, Refs. [23,45,46] found that the universal part of $S^{\rm def}_{\rm EE}$ need not obey a defect c theorem. For a 2D defect in a $D \geq 4$ CFT, Eq. (13) shows although b obeys a defect c theorem, the combination of b and d_2 in Eq. (13) need not and does not [46].

Holographic examples.—First, we consider 4D $\mathcal{N} = 4$ SUSY $U(N)$ YM theory at large N and large 't Hooft coupling λ, dual to 10D type IIB supergravity (SUGRA) on $\mathrm{AdS}_5 \times \mathbb{S}^5$ [47]. SUGRA solutions describing the most general 1/2-BPS 2D surface operators appear in Ref. [48]. Generically such a surface operator breaks $U(N) \to \prod_{k=1}^{n} U(N_k)$ with $\sum_{k=1}^{n} N_k = N$ and produces a nonzero expectation value for one adjoint complex scalar field, Φ, which decomposes into the block diagonal form

$$\langle\Phi\rangle = \frac{e^{-i\phi}}{\sqrt{2}|x^i|}{\rm diag}(z_1\mathbb{1}_{N_1}, z_2\mathbb{1}_2, \ldots, z_n\mathbb{1}_{N_n}), \quad (15)$$

with $\mathbb{S}^1$ angular coordinate ϕ around the defect, $z_k \in \mathbb{C}$ dimensionless parameters, and $\mathbb{1}_{N_k}$ the $N_k \times N_k$ identity matrix [48–51]. For such a defect Ref. [50] holographically computed $\langle T^{\mu\nu}\rangle$ for $\mathcal{M} = \mathrm{AdS}_3 \times \mathbb{S}^1$, though the result is scheme dependent. We fix the scheme by conformally mapping $\mathrm{AdS}_3 \times \mathbb{S}^1$ to Minkowski space and demanding that without a defect $\langle T^{\mu\nu}\rangle = 0$. Reference [52] holographically computed $S^{\rm def}_{\rm EE}$ for a sphere centered on the defect. Using these results, Eqs. (3) and (13) give

$$b = 3\left(N^2 - \sum_{k=1}^{n} N_k^2\right),$$
$$d_2 = 3\left(N^2 - \sum_{k=1}^{n} N_k^2\right) + \frac{24\pi^2 N}{\lambda}\sum_{k=1}^{n} N_k|z_k|^2. \quad (16)$$

Both of these are manifestly positive, and b is independent of the marginal parameters λ and z_k.

As discussed in Ref. [50] the one-loop $\langle T^{\mu\nu}\rangle$ on $\mathrm{AdS}_3 \times \mathbb{S}^1$ in the presence of the surface operator matches the term $\propto N/\lambda$ in Eq. (16). Given that the other terms in Eq. (16) are independent of the marginal parameters at large λ, and that $T^{\mu\nu}$ on $\mathrm{AdS}_3 \times \mathbb{S}^1$ is scheme dependent, b and d_2 may in fact be one-loop exact.

Second, we study 1/2-BPS Wilson surface defects in the 6D A_{M-1} $\mathcal{N} = (2,0)$ SUSY CFT specified by a representation, $\mathfrak{R}$, of A_{M-1} with highest weight w and a 2D surface [53]. When $M \gg 1$ the theory is holographically dual to 11D SUGRA on $\mathrm{AdS}_7 \times \mathbb{S}^4$ [47], and Wilson surfaces are dual to M2 branes, or M5 branes with M2-brane flux, reaching the AdS_7 boundary at Σ [54–58].

Using the holographic results for $\langle T^{\mu\nu}\rangle$ and $S^{\rm def}_{\rm EE}$ in the presence of a flat Wilson surface [59,60], Eqs. (3) and (13) give

$$b = 24(w,\varrho) + 3(w,w), \qquad d_2 = 24(w,\varrho) + 6(w,w), \quad (17)$$

where ϱ is the gauge algebra's Weyl vector, and $(\cdot,\cdot)$ is the scalar product on the weight space. Both b and d_2 are ≥ 0 for all $\mathfrak{R}$ and are invariant under the action of the Weyl group. In the defect, RG flows triggered by the expectation value of a marginal Wilson surface operator studied holographically in Refs. [46,61,62], each of b and d_2 is larger in the UV than in the IR, consistent with b's c theorem.

Thermal entropy.—For a 2D CFT on $\mathbb{S}^1$ of radius r, in the thermodynamic limit $rT \to \infty$, c determines the thermal entropy: $S = (\pi/6)cT(2\pi r)$ [63,64]. Do b, d_1, and d_2 similarly determine a 2D boundary or defect's contribution to S?

Consider the 3D BCFTs of free, massless real scalar or Dirac fermion on a hemisphere of radius r. In the Supplemental Material [13], we calculate the boundary contribution to thermal entropy, S_∂. When $rT \to \infty$, we find

$$S^{R/D}_\partial = \pm\frac{\pi}{12}T(2\pi r), \qquad S^{\rm f}_\partial = 0 \quad (18)$$

where the superscripts denote the Robin scalar, Dirichlet scalar, and Dirac fermion, respectively. Table I shows the

Dirac fermion has $d_1 \neq 0$, so $S_\partial^{\mathrm{f}} = 0$ proves that S_∂ cannot have a term $\propto d_1$ with universal nonzero coefficient. Instead, Table I and Eq. (18) suggest $S_\partial \stackrel{?}{=} (4\pi/3) bT(2\pi r)$, which, if true, looks like eight times a Cardy entropy.

However, consider the holographic DCFT given by a probe brane along an asymptotically AdS_3 submanifold inside an AdS_{D+1}-Schwarzschild black hole of radius L and temperature T, with action S_{probe} above. In the Supplemental Material [13], we compute this defect's contribution to S

$$S_{\mathrm{def}} = \frac{16\pi^2}{D^2} L^3 T_{\mathrm{br}} T(2\pi r), \tag{19}$$

which via Table I we can write as $S_{\mathrm{def}} \stackrel{?}{=} (1/D^2)(8\pi/3) bT(2\pi r)$, but without further input this choice is arbitrary as $b = d_1 = d_2$ [30]. We can compare to a DCFT given by gluing two free-field 3D BCFTs along their boundaries, with no boundary interactions, whose S_{def} is simply a sum of the S_∂ in Eq. (18). When $D = 3$, no such sum can produce the $1/D^2 = 1/9$ factor in the holographic S_{def}. This proves that if $S_{\mathrm{def}} \propto bT(2\pi r)$, then the coefficient cannot be universal.

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