Determinant Formula for the Octagon Form Factor in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

Ivan Kostov,[1] Valentina B. Petkova,[2] and Didina Serban[1]
[1]*Institut de Physique Théorique, DRF-INP 3681, C.E.A.-Saclay, F-91191 Gif-sur-Yvette, France*
[2]*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria*

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We present a closed expression for the octagon form factor which appears as a building block in a class of four-point correlation functions in $\mathcal{N}=4$ supersymmetric Yang-Mills theory considered recently by Coronado. The octagon form factor is expressed, to all loop orders, as the determinant of a semi-infinite matrix. We find that perturbatively at weak coupling the entries of this matrix are linear combinations of ladder functions with simple rational coefficients.

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Introduction.—The discovery of integrability in the planar $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory [1] initiated a "world sheet" approach powered by the analytic tools developed for two-dimensional solvable models. In this approach a single-trace operator is described as a state of a two-dimensional field theory compactified on a circle. By gauge-string duality, this is also a closed string in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background.

The full spectrum of such operators has been obtained for any value of the gauge coupling applying the integrability techniques related to the thermodynamic Bethe ansatz [2–4]. The computation of the three-point functions needed a new theoretical input. It came with the "hexagon proposal" of Basso, Komatsu, and Vieira [5]. The authors of Ref. [5] proposed to split the world sheet of a three-point function into two hexagonal patches, each containing a curvature defect. The observables associated with the two hexagons are special form factors which can be computed using the symmetries of the theory. The prescription using a "hexagonalization" of the world sheet was then extended to the case of the four-point functions [6–8] and to nonplanar corrections [9,10]. The hexagons are glued back by inserting complete sets of virtual states in the intermediate channels.

The contribution of virtual particles in the spectrum of "heavy" operators (i.e., with large dimensions) is suppressed in the weak coupling limit. This is also the case for the three-point functions of such operators. In the strong coupling limit the virtual particles cannot be neglected anymore, and in the cases amenable to analytical treatment their contribution is expressed in terms of Fredholm determinants [11].

In the computation of the four-point functions of heavy operators by hexagonalization, the virtual particles are not suppressed at weak coupling anymore [6] and the evaluation of their contribution represents a challenge. Recently, Coronado obtained some remarkable results for the four-point functions of heavy half-BPS operators with particular polarizations of the R charges [12,13]. In that configuration, the four-point function factorizes into the sum of products of the so called octagon form factors, or *octagons*. An octagon is obtained by gluing together two hexagons by inserting a complete set of virtual particles. The Boltzmann weights of the virtual particles depend on the coordinates and the R-charge polarizations of the two hexagons, as well as on the length ℓ of the "bridge" composed of tree-level propagators (the vertical lines in Fig. 1).

The octagon was expressed in Ref. [12] as an infinite series of nonsingular contour integrals which can be evaluated by residues. It is claimed that full perturbative expansion of the octagon can be recast as a multilinear

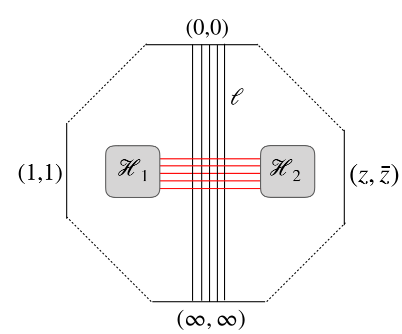

FIG. 1. A sketch of the octagon $\mathbb{O}_\ell$. The red lines symbolize the mirror particles propagating between the two hexagons, each one characterized by a rapidity u and a bound state number a. The two hexagons are separated by a bridge of length ℓ.

combination of conveniently normalized ladder integrals $f_1, f_2, \ldots$ [14], see Eq. (26) for their definition,

$$\mathbb{O}_\ell = 1 + \sum_{n=1}^{\infty} \mathcal{X}_n \sum_{J=n(n+\ell)}^{\infty} g^{2J} \sum_{j_1+\cdots j_n=J} c_{j_1,\cdots,j_n} f_{j_1} \cdots f_{j_n}, \quad (1)$$

where the dependence on the polarizations is carried by the factors

$$\mathcal{X}_n = \frac{1}{2}[(\mathcal{X}^+)^n + (\mathcal{X}^-)^n] \quad (2)$$

and the coefficients $c_{j_1 \ldots j_n}$ are rational numbers to be determined. The conjectured form of the perturbative octagon, Eq. (1), is close in spirit to the result of Basso and Dixon [15] obtained for the fishnet limit of the $\mathcal{N} = 4$ SYM [16]. (The integrability of the fishnet Feynman graphs was first established by A. Zamolodchikov [17].) The analytic expression obtained in Ref. [15] for the fishnet has the form of a single determinant of ladders, while the octagon can be expanded, as shown in Ref. [13], in the minors of the semi-infinite matrix

$$\mathbf{f}_{\infty\times\infty} = [f_{i+j+1}]_{i,j\geq\ell}. \quad (3)$$

In this Letter we report a formal solution for the octagon at finite 't Hooft coupling g in the form of the Pfaffian of a semi-infinite matrix, or equivalently as a determinant of the same matrix. We confirm the Ansatz Eq. (1) to high loop orders observing that the resulting determinant is equivalent perturbatively to the determinant of a simpler matrix whose elements are expressible in terms of the ladder functions alone. This leads to an analytic expression for the coefficients in Eq. (1). Moreover, the exact determinant representation for finite g opens the possibility of analytically accessing the four-point function beyond the perturbative expansion.

We report here the results and the general logic of the derivation, leaving the proofs to an extended paper. We start in the second section with the representation of the contributions of the virtual particles as Fredholm Pfaffians, outlined in [18]. Then in the third section we perform the sum over bound states and give the result of the octagon as a single Fredholm Pfaffian, or the square root of a Fredholm determinant. In the fourth section we use a basis of Bessel functions to transform the Fredholm determinant into the determinant of a semi-infinite matrix. In the fifth section we derive the weak coupling expansion and show that it can be organized as a sum of minors of a semi-infinite matrix of ladders.

The octagon form factor.—In this section we recall the series expansion of the octagon as a sum over virtual particles, which will be our starting point. The virtual particles and their bound states propagate in the mirror dynamics and their energy and momentum are written, with the help of the shift operator $\mathbb{D} = e^{i\partial_u/2}$, as

$$\tilde{p}_a(u) = \frac{1}{2} g(\mathbb{D}^a + \mathbb{D}^{-a})(x - 1/x),$$
$$\tilde{E}_a(u) = (\mathbb{D}^a + \mathbb{D}^{-a}) \log x, \quad a = 1, 2, \ldots, \quad (4)$$

where $x = x(u)$ is given by the Zhukovsky map,

$$u/g = x + 1/x. \quad (5)$$

Our goal is to evaluate the octagon with four physical and four mirror edges with no excitations at the physical edges, as shown schematically in Fig. 1. The octagon is obtained by gluing the hexagons $\mathcal{H}_1$ and $\mathcal{H}_2$ along the common edge $(0,0) - (\infty,\infty)$ by inserting a complete set of virtual states ψ with energies E_ψ. Symbolically

$$\mathbb{O}_\ell = \sum_\psi \langle \mathcal{H}_2|\psi\rangle \tilde{\mu}_\psi e^{-E_\psi \ell} \langle \psi|\mathcal{H}_2\rangle, \quad (6)$$

where $\tilde{\mu}_\psi$ is a measure which will be detailed below. A state ψ may contain any number of fundamental particles and their bound states transforming in the antisymmetric representations of $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$. An n-particle virtual state ψ is completely characterized by the rapidities and the bound state numbers $(u_j,\ a_j)$ of the individual particles $(j = 1, \ldots, n)$.

The four-point function depends on the cross ratios in the Minkowski and in the flavor spaces, parametrized in Ref. [12] by $z, \bar{z}, \alpha, \bar{\alpha}$. Sometimes it will be more convenient to use instead the phases $\xi, \phi, \varphi, \theta$ defined as

$$z = e^{-\xi+i\phi}, \qquad \bar{z} = e^{-\xi-i\phi},$$
$$\alpha = e^{\varphi-\xi+i\theta}, \qquad \bar{\alpha} = e^{\varphi-\xi-i\theta}. \quad (7)$$

Applying the hexagonalization prescription, one obtains the series expansion for the octagon [12]

$$\mathbb{O}_\ell = \frac{1}{2}\sum_{\pm}\sum_{n=0}^{\infty}\sum_{a_1,\ldots,a_n\geq 1} \frac{(\mathcal{X}^\pm/\sqrt{z\bar{z}})^n}{n!} \int \prod_{j=1}^{n} \frac{du_j}{2\pi}$$
$$\times \frac{\sin(a_j\phi)}{\sin\phi} \boldsymbol{\mu}_{a_j}(u_j) \prod_{j<k} \tilde{H}_{a_j,a_k}(u_j, u_k). \quad (8)$$

The different factors in the integrand are defined as follows. The symmetric bilocal factor $\tilde{H}_{ab}(u, v)$ is the product of

$$K(u, v) = \frac{x(u) - x(v)}{x(u)x(v) - 1}. \quad (9)$$

with the four possible shifts of the arguments u and v in $\pm ia/2$ and $\pm ib/2$, respectively, which we write symbolically as

$$\tilde{H}_{ab}(u, v) = K(u, v)^{(\mathbb{D}_u^a + \mathbb{D}_u^{-a})(\mathbb{D}_v^b + \mathbb{D}_v^{-b})}. \quad (10)$$

The local integration measure is

$$\boldsymbol{\mu}_a(u) = \frac{1}{ig} e^{-\tilde{E}_a(u)\ell} e^{2i\xi\tilde{p}_a(u)} \times \frac{1}{(x - x^{-1})^{\mathbb{D}^a + \mathbb{D}^{-a}}} \frac{x(u+i\frac{a}{2}) - x(u-i\frac{a}{2})}{x(u+i\frac{a}{2})x(u-i\frac{a}{2}) - 1}. \quad (11)$$

Finally the dependence on the polarizations is contained in the factors

$$\mathcal{X}^{\pm} = 2[\cos\phi - \cosh(\varphi \mp i\theta)]\sqrt{z\bar{z}}. \quad (12)$$

The octagon as a Fredholm Pfaffian.—The expansion for the octagon, Eq. (8), resembles the grand partition function of a Coulomb gas of dipoles. As it was first pointed out in Ref. [18], the product of the bilocal weights in the n-particle sector can be written as a Pfaffian of a $2n \times 2n$ matrix, and the whole expansion as a sum of two Fredholm Pfaffians [19],

$$\begin{aligned}\mathbb{O}_\ell &= \frac{1}{2}\sum_{\pm}\sum_{n=0}^{\infty}\frac{(\mathcal{X}^{\pm})^n}{n!}\sum_{a_1,\ldots,a_n\geq 1}\int\prod_{j=1}^{n} d\mu(u_j, a_j) \\ &\quad \times \mathrm{pf}[K(u_j + i\epsilon_j a_j/2, u_k + i\epsilon_k a_k/2)]_{\substack{\epsilon_j,\epsilon_k=\pm 1\\ j,k=1,\ldots,n}} \\ &= \frac{1}{2}\sum_{\pm}\mathrm{Pf}[\boldsymbol{J} + \mathcal{X}^{\pm}\boldsymbol{K}]. \end{aligned} \quad (13)$$

In the last line $\boldsymbol{J} = [J^{\epsilon\delta}]_{\epsilon,\delta=\pm}$ is a 2×2 antisymmetric matrix with nonzero elements $J^{+-} = -J^{-+} = 1$, and $\boldsymbol{K}$ is a 2×2 antisymmetric matrix kernel $\boldsymbol{K} = [K^{\epsilon\delta}]$. The kernel elements $K^{\epsilon\delta}$ act in $\mathbb{R}\times\mathbb{Z}_+$ so that the integral in the rapidity $u \in \mathbb{R}$ is accompanied by a sum over the bound state label $a \in \mathbb{Z}_+$. With the help of the shift operator $\mathbb{D}$ the kernel $\boldsymbol{K}$ can be written compactly as

$$\boldsymbol{K}(u, a; v, b) = [K(u, v)^{\mathbb{D}_u^{\epsilon a}\mathbb{D}_v^{\delta b}}]_{\epsilon,\delta=\pm}. \quad (14)$$

The last factor in the integration measure [Eq. (11)] is absorbed into the Pfaffian and the rest gives the integration measure in Eq. (13), which we write in the form

$$d\mu(u, a) = \frac{1}{ig\sqrt{z\bar{z}}}\frac{du}{2\pi}\frac{\sin(a\phi)}{\sin\phi}\Omega_\ell(u)^{\mathbb{D}^a+\mathbb{D}^{-a}}, \quad (15)$$

with

$$\Omega_\ell(u) \equiv \frac{e^{ig\xi[x(u)-1/x(u)]}}{x(u) - 1/x(u)} x(u)^{-\ell}. \quad (16)$$

To compute the Fredholm Pfaffian we first express it as a square root of a Fredholm determinant,

$$\mathrm{Pf}[\boldsymbol{J} + \mathcal{X}^{\pm}\boldsymbol{K}] = \sqrt{\mathrm{Det}[\boldsymbol{I} - \mathcal{X}^{\pm}\boldsymbol{J}\boldsymbol{K}]} = e^{\mathcal{S}^{\pm}}, \quad (17)$$

where $\boldsymbol{I}$ is the 2×2 identity matrix. Then we expand the exponent $\mathcal{S}^{\pm}$ as an infinite series of cyclic integrals or sums

$$\mathcal{S}^{\pm} = -\sum_{n=1}^{\infty}\frac{(\mathcal{X}^{\pm})^n}{2n}\mathcal{I}_n, \quad (18)$$

where the nth integral or sum reads (with $u_{n+1} \equiv u_1$, etc.)

$$\begin{aligned}\mathcal{I}_n &= \frac{1}{(ig\sqrt{z\bar{z}})^n}\sum_{a_1,\ldots,a_n\geq 1}\sum_{\epsilon_1,\ldots,\epsilon_n=\pm}\prod_{j=1}^{n}\int_{-\infty}^{\infty}\prod_{j=1}^{n}\frac{du_j}{2\pi i} \\ &\quad \times \prod_{j=1}^{n}\frac{\sin(a_j\phi)}{\sin\phi}\Omega_\ell\left(u_j - i\epsilon_j\frac{a_j}{2}\right)\Omega_\ell\left(u_j + i\epsilon_j\frac{a_j}{2}\right) \\ &\quad \times \epsilon_j K\left(u_j - \epsilon_j\frac{a_j}{2}, u_{j+1} + i\epsilon_{j+1}\frac{a_{j+1}}{2}\right). \end{aligned} \quad (19)$$

The sum over the bound state labels can be taken into account by the difference operator $[\cos\phi - \cos\partial_u]^{-1}$ applied to the simpler kernel Eq. (9). We will give the details of the computation in a forthcoming work [20].

Discrete basis.—To render the Pfaffian representation useful we have to find a way to also perform the multiple integrations over the rapidities in the expansion of $\mathcal{S}^{\pm}$. The formula we obtain is an infinite-dimensional version of the Pfaffian integration theorem [21,22]. We expand the scalar kernel Eq. (9) with $|x| > 1$ and $|y| > 1$ as a double series

$$K(u, v) = \frac{x - y}{xy - 1} = \sum_{m,n=0}^{\infty} x^{-n}\mathrm{C}_{nm}y^{-m}, \quad (20)$$

where

$$\mathrm{C}_{nm} = \delta_{n+1,m} - \delta_{n,m+1}, \quad m, n \geq 0. \quad (21)$$

Substituting Eq. (20) in the n-fold cyclic integral, we achieve that the latter decouples into a sum of products of simple integrals

$$\begin{aligned}K_{mn} &= \sum_{\epsilon=\pm}\frac{1}{2ig\sqrt{z\bar{z}}}\int\frac{du}{2\pi}\Omega_{\ell+n}(u - \epsilon i\epsilon) \\ &\quad \times \frac{\epsilon}{\cos\partial_u - \cos\phi}\Omega_{\ell+m}(u + \epsilon i\epsilon) \end{aligned} \quad (22)$$

with $m, n \geq 0$. The arguments on the rhs are displaced from the real axis by small amounts $\pm i\epsilon$ with $\epsilon > 0$ to avoid the Zhukovsky cut. Introducing the semi-infinite matrices $\mathbf{C}$ and $\mathbf{K}$ with matrix elements given respectively by Eqs. (21) and (22), the exponents $\mathcal{S}^{\pm}$ take the form

$$\mathcal{S}^{\pm} = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(\mathcal{X}^{\pm})^n}{n}\mathrm{tr}(\mathbf{C}\mathbf{K})^n. \quad (23)$$

The matrix elements of $\mathbf{K}$ can be evaluated by passing to Fourier space, after which the integral in u can be taken and results in a product of two Bessel functions. The remaining integral in the Fourier variable t is

$$K_{mn} = \frac{g}{2i\sqrt{z\bar{z}}}\int_{|\xi|}^{\infty} dt \frac{\left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{m-n} - \left(i\sqrt{\frac{t+\xi}{t-\xi}}\right)^{n-m}}{\cos\phi - \cosh t} \times J_{m+\ell}\left(2g\sqrt{t^2-\xi^2}\right) J_{n+\ell}\left(2g\sqrt{t^2-\xi^2}\right). \quad (24)$$

The transformation to a discrete basis allowed us to write the Fredholm Pfaffian Eq. (13) as a square root of the determinant of a semi-infinite matrix

$$\mathbb{O}_\ell(g,z,\bar{z},\alpha,\bar{\alpha}) = \frac{1}{2}\sum_{\pm}\sqrt{\det[1-\mathcal{X}^{\pm}\mathbf{C}\mathbf{K}]}. \quad (25)$$

Equations (24) and (25) give a formal solution for the octagon for any value of the gauge coupling g.

Determinant formula for the perturbative octagon.—In the rest of this Letter we will focus on the weak coupling expansion of the octagon. We will demonstrate the efficiency of our Eq. (23) or Eq. (25), each of which can be used to reproduce and extend to virtually any loop order the results of Refs. [12,13].

The perturbative expansion of the matrix elements of $\mathbf{K}$, Eq. (24), can be expressed in terms of the (conveniently normalized) ladder Feynman integrals [14]. For the $1\times k$ ladder integrals we will use the notations and the normalization of Ref. [13]

$$f_k = \sum_{j=k}^{2k}\frac{(k-1)!j!}{(j-k)!(2k-j)!} \times |2\xi|^{2k-j}\frac{\mathrm{Li}_j(z)-\mathrm{Li}_j(\bar{z})}{z-\bar{z}}, \quad (26)$$

with $2\xi = -\log z\bar{z}$ defined in Eq. (7). (Up to a factor $v=(1-z)(1-\bar{z})$: $f_n^{\text{Coronado}} = -v f_n^{\text{here}}$.) More precisely, we found that K_{ij} as functions of g, z, $\bar{z}$ are spanned by $\{f_m\xi^n g^{2m+n}\}_{m\geq\ell+1,n\geq0}$.

Substituting this expansion in moments [Eq. (23)] one can easily reconstruct, with the help of *Mathematica*, the perturbative series for the octagon. Remarkably, all positive powers of ξ cancel and the result comes out in the form Eq. (1). This is for now an empirical observation which awaits its analytic proof. It means that the superfluous ξ dependence can be eliminated by a unitary transformation. We can thus simplify drastically the computation by replacing the matrix $\mathbf{K}$ in the traces [Eq. (23)] with the matrix $\mathbf{K}^\circ$ obtained by truncating the expansion of $\mathbf{K}$ to the subset $\{f_m g^{2m}\}_{m\geq\ell+1}$.

The matrix elements of $\mathbf{K}^\circ$ whose indices have the same parity vanish. This property, satisfied also by the constant matrix $\mathbf{C}$, implies that $\det[1-\mathcal{X}^{\pm}\mathbf{C}\mathbf{K}^\circ]$ is equal to the square of another determinant, $\det[1+\mathcal{X}^{\pm}\boldsymbol{\mathcal{R}}]$, with the matrix elements of $\boldsymbol{\mathcal{R}}$ given by

$$\mathcal{R}_{jk} = -K^\circ_{2j+1,2k} + K^\circ_{2j-1,2k}. \quad (27)$$

Now the square root in Eq. (25) gets resolved and the determinant representation of the octagon simplifies to

$$\mathbb{O}_\ell = \frac{1}{2}\sum_{\pm}\det[1+\mathcal{X}^{\pm}\boldsymbol{\mathcal{R}}]. \quad (28)$$

The perturbative series for $\boldsymbol{\mathcal{R}}$ is

$$\mathcal{R}_{ij} = \sum_{p=\max(i+j+\ell,1+j+\ell)}^{\infty}(-1)^{p-\ell}(2p-1)! \times \frac{2p(2i+\ell)-(p-j)(p+j+\ell)\delta_{i,0}}{\prod_{\varepsilon=\pm}[p+\varepsilon(i-j)]![p+\varepsilon(i+j+\ell)]!}f_p g^{2p}. \quad (29)$$

Equations (28) and (29) give the all-loop perturbative solution for the octagon. For actual computations it is convenient to truncate the semi-infinite matrix $\boldsymbol{\mathcal{R}}$ to an $N\times N$ matrix

$$\boldsymbol{\mathcal{R}}_{N\times N} = [\mathcal{R}_{ij}]_{0\leq i,j\leq N-1} \quad (30)$$

and use the approximation formula

$$\mathbb{O}_{\ell=0} = \frac{1}{2}\sum_{\pm}\det(1+\mathcal{X}^{\pm}\boldsymbol{\mathcal{R}})_{N\times N} + o(g^{2N(2N+\ell)}). \quad (31)$$

For example, with $N=3$ the determinant in Eq. (31) gives the result of Coronado for $\ell=0$ up to g^{12} terms:

$$\begin{aligned}\mathbb{O}_{\ell=0} &= 1+\mathcal{X}_1\left(f_1g^2 - f_2g^4 + \frac{1}{2}f_3g^6 - \frac{5}{36}f_4g^8 + \frac{7f_5g^{10}}{288} - \dots\right) \\ &+ \mathcal{X}_2\left(\frac{f_1f_3-f_2^2}{12}g^8 - \frac{f_1f_4-f_2f_3}{24}g^{10} + \dots\right) \\ &+ \mathcal{X}_3\left(\frac{(f_1f_5f_3 - f_3^3 + 2f_2f_4f_3 - f_1f_4^2 - f_2^2f_5)}{34560}g^{18} + \dots\right) \\ &+ \dots\end{aligned}$$

In Ref. [13], the octagon was expanded in a basis of minors of the matrix Eq. (3). In particular, the lowest loop order n-particle contribution is proportional to the determinant of the matrix Eq. (3) restricted to the first n rows and columns,

$$\mathbb{O}_\ell = \sum_{n=0}^{\infty} \mathcal{X}_n g^{2n(n+\ell)} (I_{n+\ell,n} + o(g^2)),$$

$$I_{n+\ell,n}(z,\bar{z}) = \frac{\det([f_{i+j+\ell+1}]_{i,j=0,\dots,n-1})}{\prod_{i=0}^{n-1}(2i+\ell)!(2i+\ell+1)!}. \qquad (32)$$

The lowest term $I_{n+\ell,n}$ is exactly the expression obtained by Basso and Dixon [15] for the Feynman integral for an $(\ell+n)\times n$ fishnet. One can recognize this pattern in Fig. 1 where n virtual particles cross ℓ physical particles. Possibly an interpretation of the higher loop terms in terms of planar Feynman graphs also exists.

This expansion in the minors of the matrix of ladders $\mathbf{f}_{\infty\times\infty}$ is compatible with our determinant representation Eq. (31), which can be written as a sum over all minors of the matrix $\boldsymbol{\mathcal{R}}$, Eq. (29),

$$\mathbb{O}_\ell = \sum_{n=0}^{\infty} \mathcal{X}_n \sum_{\substack{0\le i_1<\dots<i_n \\ 0\le j_1<\dots<j_n}} \det([\mathcal{R}_{i_\alpha j_\beta}]_{\alpha,\beta=1,\dots,n}). \qquad (33)$$

Since the matrix elements of $\boldsymbol{\mathcal{R}}$ behave as $\mathcal{R}_{ij} \sim g^{2(i+j+\ell)}+$ higher powers of g, the lowest loop order contribution is given by the term $\sim g^{2n(n+\ell)}$ of the $n\times n$ minor with $i_\alpha = \alpha+\ell-1$, $j_\beta = \beta+\ell-1$, which is exactly $I_{n+\ell,n}$.

It is straightforward to extract from Eq. (33) the analytic formula for the coefficients in the expansion in the Steinmann basis of minors, but this would go beyond the scope of this short note.

The representations Eq. (23) or Eq. (25) could give for the first time analytic access to the correlation functions at finite g. Remarkably, all the dependence on the gauge coupling is contained in a single integral Eq. (24). We would like to address the subtle problem of the computation of the octagon at finite g in a future work.

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