Topological Characterization of Classical Waves: The Topological Origin of Magnetostatic Surface Spin Waves

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We propose a topological characterization of Hamiltonians describing classical waves. Applying it to the magnetostatic surface spin waves that are important in spintronics applications, we settle the speculation over their topological origin. For a class of classical systems that includes spin waves driven by dipole-dipole interactions, we show that the topology is characterized by vortex lines in the Brillouin zone in such a way that the symplectic structure of Hamiltonian mechanics plays an essential role. We define winding numbers around these vortex lines and identify them to be the bulk topological invariants for a class of semimetals. Exploiting the bulk-edge correspondence appropriately reformulated for these classical waves, we predict that surface modes appear but not in a gap of the bulk frequency spectrum. This feature, consistent with the magnetostatic surface spin waves, indicates a broader realm of topological phases of matter beyond spectrally gapped ones.

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The principle of bulk-edge correspondence is a cornerstone in the field of topological phases of matter [1]: at the boundary of a system whose bulk frequency spectrum is topologically nontrivial, there should appear localized edge modes with eigenfrequencies in a gap of the bulk spectrum. This principle underlies the unconventional stability of chiral edge states in quantum Hall insulators [2] and Dirac surface states of topological insulators [3], and has more recently led to predictions of edge modes in various classical systems [4–6]. The bulk system topology is usually characterized by a topological invariant defined for Hamiltonians describing spatially unbounded systems with specified symmetry operations. It dictates the existence and number of topologically protected edge modes. The corresponding hallmark of these edge states is their robustness against symmetry-preserving perturbations.

The insensitiveness of edge states to material parameters strikes a chord in the field of magnetism. Since their discovery in 1960 [7], ferromagnetic spin waves known as "magnetostatic surface waves" (MSSWs) have been a subject of various experimental and theoretical studies. These edge modes owe their intrinsic chiral structure to dipole-dipole interactions. MSSWs propagate perpendicular to the ordered magnetization regardless of the sample geometry, be it a slab [8] or a sphere [9]. They are known to be anomalously robust against back scatterings [10,11], hinting toward a topological origin. The chirality and robustness render them interesting for many fundamental studies, e.g., for nonreciprocal transport of spin [12] and heat [13]. Today, in the context of magnon spintronics [14], MSSWs are almost exclusively used in studies of spin-wave transport in microstructures since they offer the largest decay length of all available modes and are easily excited by the commonly used inductive microwave antennas. It is therefore of fundamental interest whether MSSWs are indeed topologically protected or not.

In this Letter, we show that the bulk Hamiltonian of spin waves in the presence of dipole-dipole interactions is characterized by a topological invariant. A pair of *vortex lines* in the Brillouin zone acts as extended Dirac monopoles, which cannot be removed by small continuous changes in system parameters. We demonstrate that these topological vortex lines lead to MSSWs via the notion of class CI semimetals, where CI denotes the symmetry class formally defined by the presence of two symmetry operators Γ and $\mathcal{T}$ [15]. Even though they are conventionally called chiral and even time-reversal symmetry, respectively, these mathematical operations are realized for MSSWs as the symplectic structure [16] and the reality condition that are both inherent to classical mechanics. We first show that in a quantum mechanical context, class CI semimetals have edge states which appear in a band gap. The dipolar Hamiltonian has a topologically nontrivial class CI semimetal structure. Because it describes classical waves, however, the topological edge states have instead eigenfrequencies above the bulk spectrum, in agreement with MSSWs. Motivated by this example, we establish a new type of bulk-edge correspondence for a general class of classical mechanical systems [Fig. 1(a)].

It is instructive to first visualize the setup [Fig. 1(b)]. The three-dimensional (3D) Brillouin zone in class CI can be sliced up into 1D subsystems [green straight lines in Fig. 1(b)], which generically possess only Γ symmetry and thus belong to class AIII. As in the Su-Schrieffer-Heeger model [17], the bulk topological invariant of class AIII in 1D is the integer winding number over the 1D Brillouin zone. Its nonzero value guarantees topologically protected dangling edge modes [18,19] even in the presence of disorder [20,21]. For dipolar spin waves, each subsystem gives winding number ± 1 which remains constant as the slice is varied, unless a vortex line is crossed, forcing a discontinuous jump by 2. This topological structure is analogous to Weyl semimetals, where the slice-wise 2D Chern number stays constant away from band-crossings (Weyl points) in the 3D Brillouin zone, but changes discontinuously when a Weyl point is traversed [22]. While Weyl semimetals are characterized by the Dirac monopole charges of the Weyl points (along with the Dirac strings connecting them [23,24]), the dipolar spin wave Hamiltonian features vortex lines of 1D "extended monopoles" in 3D, i.e., topological defects of codimension two.

We elaborate on this structure by elementary winding number analysis augmented with $\mathcal{T}$ symmetry, following ideas in Refs. [23,25]. We assume that the system is periodic on a 3D lattice $\mathbb{Z}^3$ and denote the Brillouin zone by $\mathbb{T}^3$. By definition of a class CI Hamiltonian H [15], given are a unitary Γ and an antiunitary $\mathcal{T}$ such that $\{H,\Gamma\} = [H,\mathcal{T}] = \{\Gamma,\mathcal{T}\} = 0$, $\Gamma^2 = \mathcal{T}^2 = 1$ where $[\cdot,\cdot]$ ($\{\cdot,\cdot\}$) denotes (anti-)commutator. In the Brillouin zone, Γ symmetry means

$$H_{\boldsymbol{k}} = \begin{pmatrix} 0 & U_{\boldsymbol{k}} \\ U_{\boldsymbol{k}}^{\dagger} & 0 \end{pmatrix}, \qquad \boldsymbol{k} = (k_x, k_y, k_z) \in \mathbb{T}^3, \quad (1)$$

on a basis in which $\Gamma = \mathbb{1} \otimes \sigma_3$ ($\sigma_{1,2,3}$ denote Pauli matrices), while time-reversal symmetry $\mathcal{T}$ relates the Hamiltonian at $\boldsymbol{k}$ and $-\boldsymbol{k}$ by $U_{-\boldsymbol{k}} = U_{\boldsymbol{k}}^{t}$. Suppose $H_{\boldsymbol{k}}$ is *gapped*; i.e., its eigenvalues are all nonzero, on $\mathbb{T}^3 \backslash L$ where $L = \{\boldsymbol{k} \in \mathbb{T}^3 | k_x = 0, \pi, k_y = 0, \pi\}$ is a set of four vortex lines parallel to k_z. More general line defects are obtained by either deforming or splitting the four straight lines passing through the time-reversal invariant momenta (TRIMs) on $k_z = 0$ plane [26]. Here, we focus on the straight line configuration realized by MSSWs for readability. The gap condition means $\det U_{\boldsymbol{k}} \neq 0$ on $\mathbb{T}^3 \backslash L$. Let us first examine the slice $\mathbb{T}^2 = \{\boldsymbol{k} \in \mathbb{T}^3 | k_z = 0\}$, which the vortex lines intersect at its four TRIMs. Take a small but otherwise arbitrary loop ℓ_a encircling only the ath TRIM [labeling in Fig. 2(a)], oriented counterclockwise. Define its winding number by

$$w_a = \frac{1}{2\pi i} \oint_{\ell_a} d\{\ln(\det U_{\boldsymbol{k}})\}, \qquad a = 0,1,2,3. \quad (2)$$

The winding number is an integer *topological invariant*, insensitive to perturbations of U (thus of H that respects Γ and the gap condition), and deformations of ℓ_a (avoiding the vortices). As graphically proven in Fig. 2(a), there is a "charge cancellation" consistency condition $\sum_{a=0}^{3} w_a = 0$

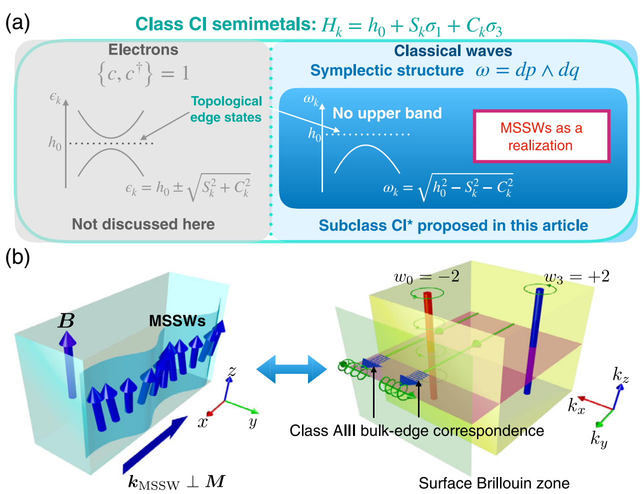

FIG. 1. (a) The class of systems presented in the Letter. The symmetry class is CI with Γ and $\mathcal{T}$, but we further focus on a subclass, denoted CI*, in which ω plays the role of Γ. (b) The real space setup (left) and the corresponding Fourier space (right) structure for MSSWs. The green straight lines along k_y axis belong to class AIII with well-defined winding numbers.

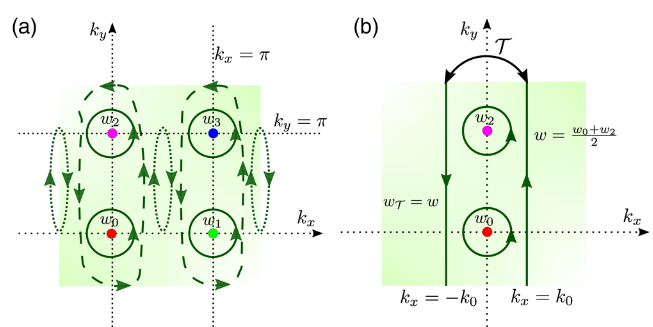

FIG. 2. (a) Graphical proof of the charge cancellation. Because of 2π periodicity in k_x, k_y, the solid, dashed, and dotted loops are continuously deformable to each other without crossing TRIMs. The dotted loops are contractible to two points, hence trivial with zero winding. (b) Determination of the winding of large loops by a deformation of small loops. Because of $\mathcal{T}$ symmetry, large loops $k_x = k_0$ and $k_x = -k_0$ have the same windings $w = w_{\mathcal{T}}$.

because the sum may be evaluated in a second way which is manifestly trivial.

By $\mathcal{T}$ symmetry, the w_a are enough to determine the winding numbers along "large" loops of $\mathbb{T}^2$ [say, at constant k_x or k_y, Fig. 2(b)]. First, any small loop ℓ_a can be deformed into a symmetric one which is mapped onto itself under $\mathcal{T}: \boldsymbol{k} \to -\boldsymbol{k}$. In Eq. (2), the integrand for one half of ℓ_a is repeated on the other half, so that the total line integral should be an *even* integer. Similarly, a large winding around k_y at a fixed $k_x = k_0 \neq 0, \pi$ must equal that at $k_x = -k_0$ evaluated along the opposite orientation, and they are constrained by their sum equaling that of the enclosed small windings. As for the 2D slices with $k_z \neq 0, \pi$ which do not respect $\mathcal{T}$ individually, continuity along k_z forces on them the same topological structure as the $k_z = 0$ slice [Fig. 1(b)]. To summarize, Hamiltonians in class CI with 1D line defects L are topologically characterized by three independent small even windings $(w_0, w_1, w_2) \in (2\mathbb{Z})^3$. If $w_a \neq 0$, there is a corresponding vortex line of topologically protected gapless points or singularities of $H_{\boldsymbol{k}}$.

To obtain the CI semimetal bulk-edge correspondence, consider for some fixed k_x, $k_z \neq 0, \pi$, the two class AIII 1D subsystems $\mathbb{T}_{(k_x,k_z)}$ and $\mathbb{T}_{(-k_x,-k_z)}$ along the y direction. Their (large) winding numbers are equal and opposite by $\mathcal{T}$ symmetry, and if nonzero, the 1D bulk-edge correspondence of class AIII ensures that when a surface is cut along x-z plane, there appear surface-localized eigenstates of H with zero eigenvalue. A similar argument holds with x replaced by y. If at least one w_a is nonzero, then some $\mathbb{T}_{k_x,k_z}$ or $\mathbb{T}_{k_y,k_z}$ has nonzero winding number, implying the existence of edge eigenstates.

The application of the CI semimetal setup presented above requires a Hamiltonian *operator* acting on a complex Hilbert space. To introduce such a structure for classical mechanical systems on a lattice $\mathbb{Z}^3$, a metric plays a crucial role; below we explain why [26]. In classical mechanics [16], one starts from a real symplectic vector space V whose coordinates are canonical variables $v = (\{p_{\boldsymbol{n}}\}, \{q_{\boldsymbol{n}}\})^t \in V$, $\boldsymbol{n} \in \mathbb{Z}^3$. The symplectic two-form $\omega = \sum_{\boldsymbol{n}} dp_{\boldsymbol{n}} \wedge dq_{\boldsymbol{n}}$ can be regarded as a linear map identifying V with the dual space V^*. In linearized problems, the dynamics is determined by a positive definite quadratic energy function $E(v)$, i.e., another linear map $V \to V^*$. Hamilton's equations of motion read $dv/dt = I \circ E(v)$, where $I = \omega^{-1}: V^* \to V$ is the Poisson bracket and $\circ$ denotes composition of maps. Note that E is not an operator (a map $V \to V$). One way of promoting the energy $E: V \to V^*$ to an operator is to assume that a preferred metric $g: V \to V^*$ is given on V and define $H = g^{-1} \circ E: V \to V$. Indeed, H defined in this way is what one calls Hamiltonian in problems where V comes with a natural Euclidean metric. Now the equations of motion may be rewritten as $dv/dt = JHv$ where $J = I \circ g: V \to V$ satisfies $J^t = -J$ (transpose with respect to g). By rescaling $g' = g \circ \sqrt{-(I \circ g)^{-2}}$, we can further arrange for $J^2 = -1$.

For a given classical system with a (rescaled) metric as above, i.e., maps $\omega, E, g: V \to V^*$, we shall say $H = g^{-1} \circ E$ belongs to *class CI** if there exists a positive $h_0 \in \mathbb{R}^+$ such that $\{H - h_0, J\} = 0$ with $J = \omega^{-1} \circ g$. To recognize the connection to the definition of class CI, we complexify V to $V^{\mathbb{C}} = V \oplus iV$ and extend g, H, J complex linearly to $V^{\mathbb{C}}$. This step is usually implicit when one carries out Fourier transforms. Here, H always has the even "time-reversal" symmetry $\mathcal{T}$ of complex conjugation, reflecting the reality of the original problem. One can also introduce a chiral symmetry $\Gamma = iJ$, which is unitary and satisfies $\{H - h_0, \Gamma\} = \{\mathcal{T}, \Gamma\} = 0$, $\Gamma^2 = 1$. Therefore, a classical H in class CI* has the complexified $H - h_0$ in class CI. If $H - h_0$ is semimetallic with vortex lines L, the winding numbers (w_0, w_1, w_2) topologically characterize H. On a basis where $\Gamma = \mathbb{1} \otimes \sigma_2$, the classical Hamiltonian takes its canonical form

$$H = h_0 + S \otimes \sigma_1 + C \otimes \sigma_3, \tag{3}$$

with some *real* operators S, C. A basis transform by $Q = \{1 + i(\sigma_1 + \sigma_2 + \sigma_3)\}/2$ brings Γ into $Q^\dagger \Gamma Q = \mathbb{1} \otimes \sigma_3$ and $H - h_0$ into the off-diagonal form as in Eq. (1) with the Fourier transform of $U = C - iS$ providing the winding numbers, Eq. (2). If some $w_a \neq 0$, the CI semimetal bulk-edge correspondence predicts edge states in the gap of H at h_0, i.e., $Hv_{n_0} = h_0 v_{n_0}$.

We now reveal that the edge states v_{n_0} appear above the physical bulk frequency spectrum. Although the eigenvalues of H do not equal physical eigenfrequencies in general, there is a one-to-one correspondence between them within class CI*. Suppose $0 \neq v_{n+} \in V$ is an eigenvector of H with eigenvalue $h_0 + \epsilon_n > 0$. The class CI* condition $\{H - h_0, J\} = 0$ implies that $v_{n-} \equiv Jv_{n+} \in V$ satisfies $Hv_{n-} = (h_0 - \epsilon_n) v_{n-}$. Whether $\epsilon_n = 0$ or not, $v_{n-} \not\propto v_{n+}$ because the eigenvalues of J are $\pm i$ while $v_{n\pm}$ are both real. Hence, all eigenvectors of H come in pairs $v_{n\pm}$ mutually related by J with respective eigenvalues $h_0 \pm \epsilon_n$. One can choose the label n such that $\epsilon_n \geq 0$. Because $v_{n\pm}$ form a complete set of basis vectors, the general solution of Hamilton's equations $dv/dt = JHv$ is given by $v = \sum_{n,\pm} c_{n\pm}(t) v_{n\pm}$ with the time-dependent coefficients satisfying

$$\frac{d}{dt}\begin{pmatrix} c_{n+} \\ c_{n-} \end{pmatrix} = \begin{pmatrix} 0 & -h_0 + \epsilon_n \\ h_0 + \epsilon_n & 0 \end{pmatrix}\begin{pmatrix} c_{n+} \\ c_{n-} \end{pmatrix}. \tag{4}$$

This yields

$$c_{n\pm} = A_n (h_0 \pm \epsilon_n)^{-1/2} \cos(\omega_n t + \alpha_n \mp \pi/4), \tag{5}$$

where A_n, α_n are constants and $\omega_n = \sqrt{h_0^2 - \epsilon_n^2}$ is the physical eigenfrequency. This clearly shows that the edge states with $\epsilon_{n_0} = 0$ have the physical frequency $\omega_{n_0} = h_0$ higher than those of the bulk modes with $\epsilon_n \neq 0$.

While our topological characterization of class CI Hamiltonians is interpreted in the classical mechanical framework, previous studies of topological spin waves [5,27,28] focused on eigenvalues of iJH in the Bogoliubov-de Gennes formalism. To the best of our knowledge, their approach seems to always predict gapless edge modes, and consequently Ref. [5] missed the topological nature of MSSWs.

To summarize, classical problems with a metric have a natural candidate for chiral symmetry in $\Gamma = iJ = i\omega^{-1} \circ g$. If H up to a constant shift anticommutes with Γ, the (real) eigenvectors of H do coincide with the physical eigenstates, while its eigenvalues $h_0 \pm \epsilon_n$ correspond to the physical eigenfrequencies $\omega_n = \sqrt{h_0^2 - \epsilon_n^2}$. If there is a "gapless" edge "state" of H ($\epsilon_{n_0} = 0$) protected by a CI semimetal structure, there exists an edge-localized physical eigenstate whose frequency ($\omega_{n_0} = h_0$) appears above the bulk frequency spectrum.

The general framework presented above requires only the specified symmetry conditions. We now demonstrate that all those assumptions are almost faithfully respected by dipolar spin waves traveling perpendicular to the magnetization [8]. Consider a simple cubic lattice of classical spins interacting only with an external magnetic field $B > 0$ along z direction and between each other via dipole-dipole interactions. The ground state satisfies $\boldsymbol{s_n} = (0, 0, 1)$, $\boldsymbol{n} \in \mathbb{Z}^3$ where $\boldsymbol{s_n}$ is the normalized spin vector at site $\boldsymbol{n}$. The energy function of spin waves in terms of the linearized spin components $\boldsymbol{s}_n \approx (s^x_{\boldsymbol{n}}, s^y_{\boldsymbol{n}}, 1 - \{(s^x_{\boldsymbol{n}})^2 + (s^y_{\boldsymbol{n}})^2\}/2)$ yields [29]

$$E = B' \sum_{\boldsymbol{n}} s^\alpha_{\boldsymbol{n}} s^\alpha_{\boldsymbol{n}} - G \sum_{\boldsymbol{n} \neq \boldsymbol{n}'} \frac{\partial^2}{\partial n^\alpha \partial n'^\beta} \left(\frac{1}{|\boldsymbol{n} - \boldsymbol{n}'|} \right) s^\alpha_{\boldsymbol{n}} s^\beta_{\boldsymbol{n}'}, \quad (6)$$

where sums over $\alpha, \beta = x, y$ are implicit, $B' = B + 4\pi G/3$ [30], and the constants B and G are appropriately normalized. $s^x_{\boldsymbol{n}}$ and $s^y_{\boldsymbol{n}}$ are identified to be $p_{\boldsymbol{n}}, q_{\boldsymbol{n}}$ respectively, with the area two form of the sphere (phase space of s_n) acting as the symplectic two-form ω [32]. The system comes with the Euclidean metric $g = \delta_{\alpha\beta}$ of the spin configuration space, with which the Hamiltonian H is identical to E as a matrix. Applying spatial Fourier transform, H decomposes over the Brillouin zone as two-by-two matrices $H_{\boldsymbol{k}} = (B + D_{\boldsymbol{k}})\mathbb{1} + S_{\boldsymbol{k}}\sigma_1 + C_{\boldsymbol{k}}\sigma_3$ ($\mathbb{1}$ is the unit matrix) each acting on $(p_{\boldsymbol{k}}, q_{\boldsymbol{k}})$. For $\boldsymbol{k} \approx 0$, i.e., in the long-range limit, the coefficient functions are approximated by

$$D_{\boldsymbol{k}} = 2\pi G \frac{k_x^2 + k_y^2}{|\boldsymbol{k}|^2}, \qquad S_{\boldsymbol{k}} = 2\pi G \frac{2k_x k_y}{|\boldsymbol{k}|^2},$$
$$C_{\boldsymbol{k}} = 2\pi G \frac{k_x^2 - k_y^2}{|\boldsymbol{k}|^2}. \quad (7)$$

$H_{\boldsymbol{k}}$ is already in the class CI* canonical form Eq. (3) with $h_0 = B + D_{\boldsymbol{k}}$ and has complex conjugation as a $\mathcal{T}$ symmetry. σ_2 is identified with a chiral symmetry, which is exact when $D_{\boldsymbol{k}}$ is constant. To compute the winding number, note $U_{\boldsymbol{k}} = C_{\boldsymbol{k}} - iS_{\boldsymbol{k}}$ as stated below Eq. (3). Substituting it into Eq. (2) yields $w_0 = -2$ around the origin (and k_z axis), proving that the dipolar Hamiltonian is topologically nontrivial. Although expressions for $D_{\boldsymbol{k}}$, $S_{\boldsymbol{k}}$, $C_{\boldsymbol{k}}$ away from the origin are not available in a closed analytical form, they can be numerically evaluated by Ewald's method [33] as plotted in Fig. 3. One confirms $U_{\boldsymbol{k}} \neq 0$ along $\boldsymbol{k} = (0, \pi, k_z), (\pi, 0, k_z)$ and $U_{\boldsymbol{k}} = 0$ (i.e., a vortex line is located) along $\boldsymbol{k} = (\pi, \pi, k_z)$ [26]. Thus, the topology of the dipolar spin wave Hamiltonian is characterized by $(w_0, w_1, w_2) = (-2, 0, 0)$. All the 1D slices for fixed k_x, $k_z \neq 0, \pi$ have winding numbers ± 1. Note that the slices $\pm(k_x, k_z)$ are paired by the reality condition ("$\mathcal{T}$ symmetry") and represent the same physical degrees of freedom. Therefore, when a surface is cut along x-z plane, one surface mode for each k_x, k_z is expected.

Strictly speaking, the bulk-edge correspondence is valid only if $D_{\boldsymbol{k}}$ is constant. It is satisfied on the $k_z = 0$ slice in the long-range limit as $D_{\boldsymbol{k}} \to 2\pi G$ and the eigenfrequency of the corresponding edge mode should be $\omega = B + 2\pi G$, which is precisely the frequency of MSSWs for $k_z = 0$. Although $D_{\boldsymbol{k}}$ deviates from $2\pi G$ for $k_{x,y}$ of order unity, the numerical calculation shows the $\boldsymbol{k}$ dependence is weak so that the chiral symmetry is approximately satisfied for $k_z = 0$ (Fig. 3). In contrast, on planes with constant $k_z \neq 0$, $D_{\boldsymbol{k}}$ varies as much as $S_{\boldsymbol{k}}$ or $C_{\boldsymbol{k}}$ does and chiral symmetry is violated. This can explain the lack of robustness of obliquely traveling MSSWs. Physically, we expect the chiral symmetry breaking term $D_{\boldsymbol{k}}$ to shift the frequency of the edge modes relative to that of bulk modes, eventually causing them to merge with the bulk band and disappear. To our knowledge, the fate of the class AIII bulk-edge correspondence when strict chiral symmetry is broken while the bulk winding is still well defined is an open mathematical problem.

Finally, we discuss the chiral, unidirectional propagation of MSSWs. When a surface is cut in the y direction as in Fig. 1(b), edge states appear on the surface Brillouin zone except for the projections of the bulk vortex lines $k_x = 0, \pi$. Thus, the edge states always have nonzero components of k_x and one can define their chirality with respect to the

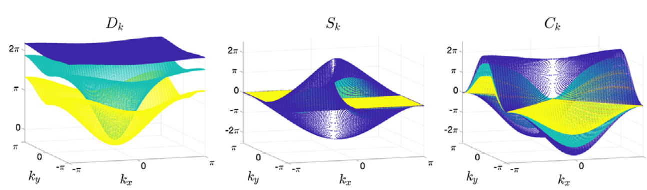

FIG. 3. Numerically evaluated Fourier transform of the Hamiltonian (6) for constant k_z slices (we set $G = 1$): $D_{\boldsymbol{k}}$ (left), $S_{\boldsymbol{k}}$ (center), and $C_{\boldsymbol{k}}$ (right) with $k_z = 0$ (blue), 0.2π (green), and 0.5π (yellow).

x direction. The reality condition $\mathcal{T}$ means the pair of edge states at $\pm(k_x, k_z)$ are physically identical so that the sign of k_x itself cannot decide the direction of propagation. This however also implies there is one propagating mode for the pair of states, which is thus necessarily chiral [i.e., it can propagate in only one of $\pm(k_x, k_z)$ directions]. The "chiral symmetry" Γ is indeed correlated with the direction of propagation in the following way. Recall that class AIII edge states are eigenstates of Γ with their eigenvalues $s = \pm 1$ for windings ± 1 [18,19,21]. Because of the $\mathcal{T}$ symmetry, edge states with $s = \pm 1$ are paired up and form a single physical eigenstate. An explicit computation [26] shows that $s = +1$ for $k_x \gtrless 0$ gives edge modes traveling in the positive and negative x directions, respectively.

In conclusion, we have established the notion of class CI semimetals characterized by even windings around vortex defect lines and explained how they arise in certain classical mechanical systems. We constructed a chiral symmetry operator from the symplectic two form and a metric. We showed that the corresponding chiral symmetric classical systems can support topologically protected edge modes with their eigenfrequencies appearing above the bulk spectrum. The framework is applicable to MSSWs for $k_z = 0$ and reproduces all of their characteristic features.

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