Edge Wave and Boundary Layer of Vortex Matter

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We show that vortex matter, that is, a dense assembly of vortices in an incompressible two-dimensional flow, such as a fast rotating superfluid or turbulent flows with signlike eddies, exhibits (i) a boundary layer of vorticity (vorticity layer) and (ii) a nonlinear wave localized within the vorticity layer, the edge wave. Both are solely an effect of the topological nature of vortices. Both are lost if vortex matter is approximated as a continuous vorticity patch. The edge wave is governed by the integrable Benjamin-Davis-Ono equation, exhibiting solitons with a quantized total vorticity. Quantized solitons reveal the topological nature of the vortices through their dynamics. The edge wave and the vorticity layer are due to the odd viscosity of vortex matter. We also identify the dynamics with the action of the Virasoro-Bott group of diffeomorphisms of the circle, where odd viscosity parametrizes the central extension. Our edge wave is a hydrodynamic analog of the edge states of the fractional quantum Hall effect.

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Introduction.—In this Letter, we focus on an exemplary problem of hydrodynamics: a blob of vorticity consisting of a dense assembly of signlike point vortices in a 2D incompressible inviscid fluid, vortex matter; see Fig. 1. The question we ask is whether and how a "quantization," or discreteness, of vortices affects large scale flows.

The standard examples of vortex matter are quantum fluids: rotating superfluid helium [1] and Bose-Einstein condensate (see, e.g., [2]), where the circulation of vortices are quantized. In classical fluids, vortex matter arises in the inverse cascade of a turbulent flow when small eddies congregate into large signlike Onsager's vortex clusters [3,4]. There are numerous examples in atmospheric, oceanic, and aeronautic physics, as well as astrophysics (tornadoes, hurricanes, pulsars). Vortex matter is also a subject of what is called quantum turbulence [5]. A closely related topic is active rotor media, where fluid particles possess rotational degrees of freedom (see, e.g., [6]). An understanding of the motion of vortex matter is also important for the "vortex method" in computational hydrodynamics, which approximates vorticity by discrete vortices [7,8]. The fundamental importance of vortex matter emerged in the theory of the fractional quantum Hall effect (FQHE) [9]. There, electrons effectively bound to localized magnetic fluxes move like vortices in an incompressible fluid.

In the listed cases, vortex matter is a liquid. However, it is a special class of liquid whose constituents possess a topological characterization, the circulation. Interactions between topological textures is nonlocal and have a geometric nature. This makes their flows different.

The naive approximation where vortex matter is treated as a uniform vorticity is a classical flow known as a vorticity patch, or a Rankine vortex. It is a domain of a uniform vorticity, surrounded by an irrotational flow. We will show that this approximation fails: the topological characterization of microscale fluid constituents has nowhere to hide. It affects large scale flows.

The dynamics of a vortex patch is the motion of its boundary. It does not possess linear traveling waves. This changes if the vorticity patch is an aggregation of small vortices, a sort of discretized Rankine vortex. We will show that the indestructible discreteness of vortices yields a linear edge mode propagating within a new kind of a boundary layer, the vorticity layer. The vorticity layer and the edge mode would have been lost had vortex matter been treated as a continuous vorticity. The vorticity layer acts towards the stabilization of the violent filaments in the unstable dynamics of the Rankine vortex (see, e.g., [8,10]). We find the dispersion of the edge mode to be nonanalytic

$$E_k = \bar{U}k + 2\eta k|k|, \qquad \eta = \Gamma/8\pi. \tag{1}$$

The wave travels against the overall rotation of the layer with velocity $\bar{U} = \Gamma/\sqrt{16\pi l}$, where l is the mean

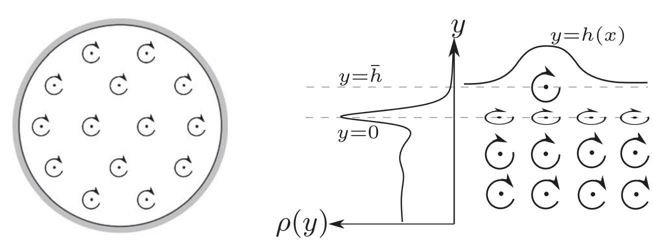

FIG. 1. (Left) The circular blob with a boundary layer. (Right) The blob $y < 0$ is squeezed relative to the continuous vorticity patch $y < \bar{h}$; a vortex trapped in the boundary layer $B: 0 < y \leq h$ illustrates charge one soliton. (Side) Illustration of the overshoot in the vortex density.

intervortex distance and $-\Gamma$ is the circulation of each vortex. The coefficient η has the dimension of viscosity. We identify it with the odd viscosity of [11,12].

Our main result is the dynamics of the edge mode beyond the linear approximation. We show that the edge wave is governed by the celebrated Benjamin-Davis-Ono (BDO) equation (often abbreviated as Benjamin-Ono) [13]. It is an integrable equation and it exhibits solitons. A remarkable property of BDO is that solitons possess a quantized vorticity. This property reveals the topological character of vortices. Our edge mode is a classical prototype of the FQHE edge state [14].

We emphasize two other results: (i) we identify the boundary conditions at the vorticity layer, and (ii) we show that the edge dynamics is the action of the Virasoro algebra, whose central extension is the odd viscosity.

Kirchhoff equations.—We recall the notion of the Rankine vortex (vortex patch), the naive coarse-grained approximation of vortex matter. It is a domain D of uniform vorticity, surrounded by an irrotational flow. The motion of the boundary of the Rankine vortex, also called the vorticity jump, governs by the kinematic boundary condition (KBC). It states that the velocity of the interface (the front) is the velocity of the fluid parcel on the front. We denote the vorticity of a clockwise vortex (anticyclonic) patch by $-2\Omega < 0$ and use the frame rotating anticlockwise with the frequency Ω. Then, if $z(t)$ is the complex coordinate of the front and $u = u_x - iu_y$ is the complex velocity of the flow, the KBC reads

$$\dot{\bar{z}} = u_z|_{z(t)}, \qquad u_z = -i\Omega\bar{z} + i\frac{\Omega}{\pi}\int_{D(t)}\frac{dV'}{z - z'}. \quad (2)$$

This equation, called contour dynamics (or CDE), has been extensively studied (see, e.g., [10]).

We will formulate the problem of the discretized Rankine vortex with the help of the Kirchhoff equations (see, e.g., [15]). Recall that velocity of the form

$$u_z(z,t) = -i\Omega\bar{z} + \frac{i}{2\pi}\sum_{i=1}^{N}\frac{\Gamma_i}{z - z_i(t)} \quad (3)$$

is a solution of the Euler equation

$$\dot{\boldsymbol{u}} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\boldsymbol{\nabla}p - 2\Omega\times\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u} = 0 \quad (4)$$

if and only if the trajectories of the vortices $z_i(t)$ obey the Kirchhoff equations

$$\dot{\bar{z}}_i \equiv v_i = -i\Omega\bar{z}_i + \frac{i}{2\pi}\sum_{i\neq j}^{N}\frac{\Gamma_j}{z_i(t) - z_j(t)}. \quad (5)$$

Both equations are written in the rotating frame. If the circulations of all vortices are set equal (they do not change due to the Kelvin theorem),

$$-\Gamma_i = -\Gamma < 0, \quad (6)$$

which is the case we consider, the Kirchhoff equations are the discrete version of the CDE (2). The question we ask is whether the CDE captures the hydrodynamics of vortex matter. We show that it does not. Even at vanishing spacing, the motion of a patch of tightly packed vortices does not match its naive continuous version.

The approximation fails at the boundary. The bulk vortices can indeed be approximated by a uniform density $\rho_\infty = 2\Omega/\Gamma$ [16]. However, we will see that forces acting within vortex matter squeeze the blob, pushing the boundary inward. Squeezed vortices build a singular boundary layer with a sharp peak of vorticity, the overshoot, schematically shown in Fig. 1. Unlike the known boundary layers, such as the Stokes and Ekman layers, our layer occurs in inviscid fluids. The edge waves that we study are the motion of the overshoot.

To isolate the effect, we focus on the near-circular patch, whose continuous version is a stationary circular Rankine vortex. It shows no dynamics whatsoever. In this limit, the vortex density $\rho(\boldsymbol{r}) = \sum_{i\leq N}\delta(\boldsymbol{r} - \boldsymbol{r}_i)$, is approximated by a step function $\rho_0(\boldsymbol{r}) = \rho_\infty\Theta(R_0 - r)$ supported by a disk of the area $\pi R_0^2 = N\rho_\infty^{-1}$.

Near-circular, near-stationary flows are selected by zero angular impulse [15]. In units of fluid density,

$$L \equiv \sum_i \boldsymbol{r}_i \times v_i = 0 \quad (7)$$

(v_i is the velocity of a vortex). The true vortex matter possesses a scale, the intervortex spacing $l = \rho_\infty^{-1/2}$.

Vorticity layer (overshoot).—The vortex density significantly departs from the step function $\rho_0(\boldsymbol{r})$ within a few spacing units from the boundary of the equivalent Rankine vortex, forming a boundary singularity, the overshoot $\check{\rho} = \rho - \rho_0$ (Fig. 1). Numerical data of the stationary overshoot are available from the studies of the electronic density in FQHE (see, e.g., [17]), a closely related problem. The data show that the overshoot is an asymmetric peak centered inward, with decaying oscillations into the bulk. We outline the properties of the overshoot first, before presenting supportive arguments. Some of them were obtained in [19,20], and some are new.

We will see that forces of vortex matter squeeze the blob, pushing the vorticity jump inward $R_0 \to R = R_0 - \bar{h}$ by the distance $\bar{h} = l/\sqrt{8\pi}$. This is a new effect; previously, the value of the squeezing $\bar{h}$ was correctly estimated via a numerical fit by [18]. We comment that, in a related problem of a superfluid fluid confined in a rotating container, the impact of discreteness of vortices on the size of the blob has been known since the 1960s: vortices repel from the walls of the container [1,21,22].

Inside the squeezed blob $r < R$ we may treat vorticity as uniform $\rho = \rho_\infty$. Hence, the circular vorticity jump $r = R$

does not evolve. We adopt a weakly nonlinear, long-wave approximation, where we neglect the curvature of the blob and treat the blob as the lower half-plane $y < 0$ with local coordinates $y = r - R, dx = -Rd\theta$.

The gap that emerges between the vorticity jump and the irrotational flow is the vorticity layer. It is a strip with a fixed bottom and a varying top $y = h(x)$, the front of the ambient flow

$$B = \{y : 0 < y \leq h(x)\}. \tag{8}$$

The front oscillates about the boundary of the equivalent Rankine vortex $y = \bar{h} \equiv (1/2\pi) \oint h d\theta$. We assume that near-stationary flows are in the state of local equilibrium: the mean radial line density is uniform along the boundary $(\delta/\delta h) \int \rho(x, y) dy = \rho_\infty$.

The first two moments of the overshoot $\check{\rho}(\boldsymbol{r}) = \rho(\boldsymbol{r}) - \rho_0(\boldsymbol{r})$, the line density, and the dipole moment (about the vorticity jump) determine the dynamics

$$n(x) = \int \check{\rho}(x, y) dy, \qquad d(x) = -\int y \check{\rho}(x, y) dy. \tag{9}$$

The line density n is the density of vortices trapped in B: $n(x) = \int_B \rho(x, y) dy$. The local equilibrium yields

$$n(x) = \rho_\infty [\bar{h} - h(x)]. \tag{10}$$

This is our edge mode.

We find the exact mean values of the dipole moment of the dipole moment $\bar{d} = (1/2\pi) \oint d d\theta$ and the squeezing

$$\bar{d} = 1/8\pi, \qquad \bar{h}^2 = l^2 \bar{d}. \tag{11}$$

We see that $\bar{d}$ does not vanish at $l \to 0$. This suggests that the density possesses a singular double layer at the vorticity jump [19,20]. Summing up, we represent the overshoot as a simple and the double singular layers located at the actual vorticity jump $y = 0$

$$\check{\rho}(x, y) = -n(x)\delta(y) + d(x)\delta'(y) + \cdots. \tag{12}$$

The simple layer means a jump of the azimuthal velocity

$$U(x) \equiv (1/2) \underset{y=0}{\mathrm{disc}}[u_x] = \Omega h(x). \tag{13}$$

The layer speeds up by the mean velocity $\bar{U} = \Omega \bar{h}$ with respect to the squeezed blob.

We will also see that the moments are not independent. They move subject to the relation

$$d - h d_x^H = (h/l)^2, \tag{14}$$

where $f^H(x) = \pi^{-1} \int (f(x') - f(x))/(x' - x) dx'$ is the Hilbert transform, and $d_x = \partial_x d$.

Now we support these claims.

The angular impulse.—The value of the mean dipole moment (11) promptly follows from the identity for the angular impulse (7) $L = \Omega \sum r_i^2 - (\Gamma/4\pi) N(N-1)$. The identity follows from the Kirchhoff equations (5) (with all Γ_i equal) after a multiplication by z_i and summation over all vortices. The last term in L is the number of vortex pairs. The term $(\Gamma/4\pi)N^2 = \Omega \int \rho_0 r^2 dV$ is the angular momentum of the continuous Rankine vortex. The subleading term $(\Gamma/4\pi)N$ counts the excluded terms $i = j$ in the sum (5), reflecting the discreteness of vortex matter. With the help of (9) we obtain

$$L = \Omega \int r^2 \check{\rho} dV + (\Gamma/4\pi) N = N\Gamma(-2\bar{d} + 1/4\pi). \tag{15}$$

Hence, $L = 0$ flows possess a dipole moment (11).

Stream function.—We may now evaluate the stream function

$$\Psi(\boldsymbol{r}) = (\Gamma/2\pi) \int \log|\boldsymbol{r} - \boldsymbol{r}'| \rho dV' - (\Omega/2) r^2 \tag{16}$$

and the velocities $u_x = \partial_y \Psi, u_y = -\partial_x \Psi$ outside of B. The stream function consists of the rotational part representing the solid rotation of the squeezed blob

$$y \notin B: \ \Psi(x, y) = \Omega(2\bar{h} y - y^2)\Theta(y) + \psi(x, y) \tag{17}$$

and the irrotational part generated by the double layer

$$\begin{aligned} \psi &= \frac{\Gamma}{2\pi} \partial_y \oint \log|z - x'| d(x') dx' \\ &\approx \frac{\Gamma}{2} [\mathrm{sgn}(y) d + y d_x^H]. \end{aligned} \tag{18}$$

Contrary to the Rankine vortex, the stream function jumps through the double layer. These formulas and (13) yield the boundary values of velocity

$$y = \pm 0: \ u_x = 2U\Theta(y) + \frac{\Gamma}{2} d_x^H, \qquad u_y = \pm \frac{\Gamma}{2} d_x, \tag{19}$$

$$y = h: \quad u_x = \Gamma n + \frac{\Gamma}{2} d_x^H, \qquad u_y = -\frac{\Gamma}{2} d_x. \tag{20}$$

Assuming the relation (14) we express the boundary values of the stream function through the front

$$\Psi|_{\mp 0} = \mp(\Gamma/2) d = \mp(\Omega h^2 - \eta h_x^H), \tag{21}$$

$$\Psi|_h = 2\Omega \bar{h} h + 2\eta h_x^H, \tag{22}$$

where

$$\eta = \Gamma \bar{d}. \tag{23}$$

Balance of forces.—Now we are ready to obtain (14). It follows from the principle of continuity commonly used in weak solutions of hydrodynamics.

We look for the extremum of the free energy

$$E - 2\Omega L = \left[\int_{r<R}\left(\frac{\boldsymbol{u}^2}{2} - 2\Omega\Psi\right) + \int_{r>R}\frac{\boldsymbol{u}^2}{2}\right]dV. \quad (24)$$

with respect to the squeezing $h = R_0 - R$. The extremum condition $\delta(E - 2\Omega L) = 0$ is equivalent to the continuity of the free energy density across the jump. It ensures the balance of the Coriolis and centrifugal forces

$$\text{balance at } y = 0\colon \ -2\Omega\Psi|_{-0} = (1/2)\underset{y=0}{\text{disc}}[u_x^2]. \quad (25)$$

The entries of the balance equation are given by (21) and (19). Plugging them in, we obtain the relation (14)

$$\Omega\Gamma d = 2U^2 + \Gamma U d_x^H. \quad (26)$$

Kinematic boundary conditions.—Having the stream function in terms of the front, we now employ the KBC. In the rotating frame, it reads

$$\text{KBC}\colon \ (\partial_t - U\partial_x)h = -\frac{d}{dx}[\Psi|_{h(x)}]. \quad (27)$$

Combining the KBC with (22), we obtain the dynamics of the front, our main result.

Main result: Benjamin-Davis-Ono equation.—The number of vortices (10) trapped in the line element of B evolves according to the BDO equation

$$\text{BDO}\colon \ (\partial_t + \bar{U}\partial_x)n - (\Gamma/2)\partial_x(n^2 - 4\bar{d}n_x^H) = 0. \quad (28)$$

The linearized BDO has dispersion (1). The BDO equation previously appeared in the description of various fluid interfaces: density [13], vorticity [23], or shear [24]. The BDO also emerged as a hydrodynamic description of the Calogero model [25] and as a theory of edge states in the FQHE [14]. See also [26] on the relation between the Calogero model and 2D hydrodynamics.

The BDO is an integrable equation whose periodic and solitonic solutions are explicit (see, e.g., [27]). A remarkable property that distinguishes BDO is that the first integral of BDO, the "charge," is quantized in units of $8\pi\bar{d}$. For $\bar{d}$ given by (11), the soliton charge is integer

$$\text{quantization}\colon \ \oint n dx \in \mathbb{Z}. \quad (29)$$

Edge solitons with the quantized vorticity is our major result: the edge solitons are vortices trapped in the overshoot. Naturally, their number is integer. Explicitly, the 1-soliton in the frame comoving with the boundary layer, $\xi = x - \bar{U}t$, is

$$\text{soliton}\colon \ n_s(\xi, t) = \frac{8\bar{d}A}{(\xi + v_s t)^2 + A^2}, \qquad v_s = \eta A^{-1}.$$

It carries precisely one vorticity quantum. The constant $|A| \gg l$ determines the width, height, and speed of the soliton. Bumps $A > 0$ travel faster, $v_s > 0$ the overall rotation of the layer, dents travel slower.

In the remaining part of the Letter, we obtain the boundary conditions for vortex matter and identify the coefficient η (23) with the odd viscosity.

Bernoulli equation and pressures.—The vorticity of the flow outside B is uniform: -2Ω in the blob and zero outside. Such flows are governed by the Bernoulli equation. Let us introduce the hydrodynamic potential φ of the irrotational flow generated by the double layer, a harmonic conjugate of ψ, whose boundary value is $\varphi = \text{sgn}(y)\psi^H = (\Gamma/2)d^H$ given by (18). Then

$$y \notin B\colon \ \dot{\varphi} + u^2/2 + p = 0. \quad (30)$$

Since we already know the dynamics of the front, the dynamics of φ allows us to read off the pressure. Omitting the algebra, we present a Bernoulli equation evaluated at both boundaries $y = 0$, $h(x)$

$$y \in \partial B\colon \ (\partial_t + U\partial_x)\varphi = 2\eta\partial_x u_y. \quad (31)$$

Comparing (31) to (30) we find the boundary pressure

$$y \in \partial B\colon \ p + U^2/2 = -2\eta\partial_x u_y. \quad (32)$$

Anomalous stress and odd viscosity.—Now we can find the stress vortex matter exerts on the fluid outside B. It is harmonic. Equation (32) determines the trace $2p + U^2 = -(\tau_{xx} + \tau_{yy})$ and its harmonic extension

$$\tau_{xx} + \tau_{yy} = 4\eta\partial_x u_y. \quad (33)$$

The traceless part of the stress follows from the requirements that the stress is dissipation-free, isotropic, and linear. Such stress was introduced in [11] (independent from vortex matter). It is the spin-2 tensor

$$\tau_{xx} - \tau_{yy} = 2\eta(\partial_x u_y + \partial_y u_x),$$
$$\tau_{xy} = \eta(\partial_y u_y - \partial_x u_x). \quad (34)$$

The coefficient η was called odd viscosity (see [28] for studies of fluids with odd viscosity). Later it was found [12] that the boundless vortex matter features the same stress (there it was called the anomalous stress). The coefficient η computed in [12] was found to be $\eta = \Gamma/8\pi$ as (23). We now see that the odd viscosity has a new interpretation: it is the dipole moment of the overshoot.

Comparing with (34) we observe that the normal stress vanishes. Hence, at the boundary

$$\text{DBC:}\;\; \tau_{yy}|_{\partial B} = 0. \tag{35}$$

This could be used as a boundary condition for vortex matter, which together with (33) determines the stress tensor and the flow outside B: the trace (the pressure) is continuous through the layer, the normal stress vanishes, and the shear and the tangential stress on the boundary are related by the Cauchy-Riemann condition $\tau_{xx} = 2\tau^H_{xy}$. Often the boundary value of the stress is called a dynamic boundary condition (DBC). Our DBC is similar to that of a surface covered by an inextensible film [29].

Action of the Virasoro-Bott group with odd viscosity as a central extension.—We conclude with a brief comment of a natural connection of the edge dynamics to the Virasoro-Bott group, that is, the centrally extended orientation preserving diffeomorphisms of a circle (see, e.g., [30] on Virasoro-Bott group in hydrodynamics). Let us pass to the Lagrangian specification of the irrotational flow outside the blob. We map conformally the outer domain $y > h(x)$ to the upper half-plane Im$z > 0$, and evaluate the dynamics of the holomorphic hydrodynamic potential $\phi = \varphi + i\psi$ ($u_z = \partial_z\phi$, $\partial_{\bar z}\phi = 0$) at the front $\Phi \equiv \phi|_h$. With the help of the transformation $\partial_t\Phi = (\dot\phi + u_y\dot h)|_h = (\dot\phi + u_y^2)|_h$, we obtain

$$\text{Im}\, z = 0:\;\; (\partial_t + U\partial_z)\Phi = T_{zz}. \tag{36}$$

Here T_{zz} is the boundary value of the holomorphic tensor

$$T_{zz} = -(\partial_z\Phi)^2 + 2i\eta\partial_z^2\Phi. \tag{37}$$

Equation (36) is a form of the BDO (28) that is readily analytically continued outside of the blob Im$z > 0$. This form of the BDO equation underlines the geometric meaning of the edge dynamics as the action of the Virasoro-Bott group. The T_{zz} is the stress-energy tensor of conformal field theory, a generator of the Virasoro Poisson-Lie algebra. The brackets for the generators $L_n = (1/2\pi)\oint z^{n-1}T_{zz}dz$ of the Virasoro algebra follows from (37) and canonical hydrodynamics brackets

$$(\pi/16\Omega i)\{L_n, L_m\} = (n-m)L_{n+m} + \eta^2 n^3 \delta_{n+m,0}.$$

This gives yet another interpretation of the odd viscosity. It is the central extension of the Virasoro algebra.

Summing up, we demonstrated that the topological characterization of vortex matter is revealed on the edge in the form of edge waves and quantized solitons. This effect would be lost if the vortex matter was approximated by a uniform vorticity. The edge wave propagates along a novel kind of boundary layer, the overshoot of vorticity. Contrary to the known boundary layers, the vorticity layer offsets the dissipation-free stress of vortex matter, known as anomalous stress characterized by the odd-viscosity kinetic coefficient η. We related it to the dipole moment of the vorticity overshoot and formulated the boundary conditions for vortex matter. We also underline the geometric nature of vortex matter by identifying its flows with the action of the Virasoro-Bott group and the odd viscosity with the central extension.

Apart from fluid mechanics, our results may be relevant for physics of superfluid, atomic Bose-Einstein condensate, and especially for FQHE, since our edge mode is the classical prototype of a FQHE electronic edge state.

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