# Metric-Torsion Duality of Optically Chiral Structures

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We develop a metric-torsion theory for chiral structures by using a generalized framework of transformation optics. We show that the chirality is uniquely determined by a metric with the local rotational degree of freedom. In analogy to the dislocation continuum, the chirality can be alternatively interpreted as the torsion tensor of a Riemann-Cartan space, which is mimicked by the anholonomy of the orthonormal basis. As a demonstration, we reveal the equivalence of typical three-dimensional chiral metamaterials in the continuum limit. Our theory provides an analytical recipe to design optical chirality.

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*Introduction.*—The analog space-time where a transparent electromagnetic medium is fully equivalent to a pseudo-Riemannian metric has been the subject of great interest for fundamental reasons dating back to the early days of general relativity [1–7]. In 2006, Pendry *et al.* [8] and Leonhardt [9] independently proposed the scheme of transformation optics [10–12] by reversely applying this material-geometry correspondence to practical optical design, opening fascinating opportunities to manipulate the propagation of electromagnetic fields. Well-known examples include invisible cloaks [13], optical illusion [14], wide-angle lenses [15], flat antennas [16], extreme plasmonics [17,18], and advanced near-zero permittivity materials [19–21]. In principle, the active coordinate transformation creates an optical Riemannian space where the permittivity $\boldsymbol{\epsilon}$ and permeability $\boldsymbol{\mu}$ measure the local optical length and fix the causal structure of the space-time as the metric in general relativity.

While significant effort has been made concerning the paradigms [22–26] and extensions to time domain [27], nonlinearity [28], and $\mathcal{PT}$ symmetry [29], how to incorporate the chiral structures (or the spin degree of freedom (d.o.f.) of fields) with transformation optics remains elusive. Modeling the chiral structure as a curved space is an essential and rather nontrivial problem. In macroscopic electrodynamics, the chiral effect manifests as the reciprocal magnetoelectric coupling $\boldsymbol{\kappa}$ in the constitutive relations of naturally occurring or artificial chiral materials [30,31], leading to the natural optical activity, circular dichroism, negative index [32], and repulsive Casimir force [33]. The helical building blocks indicate a geometrical origin from the inhomogeneous $(\boldsymbol{\epsilon},\boldsymbol{\mu})$, or equivalently from the metric. In covariant vacuum electrodynamics, however, the constituent tensors of a curved Riemannian space are solely determined by the metric [3–5], and only a gyration vector $w_i \propto g_{0i}/g_{00}$ appears from the nonreciprocal magnetoelectric effect in moving medium [3,34]. To describe more generic constitutive relations which are generally believed to have no geometric origins, Maxwell's equations are formulated to a metric-free form in the premetric electrodynamics [35]. Although it has been proposed to interpret the isotropic chiral effect as the torsion of a Riemann-Cartan space [36], there is not yet a direct connection to real chiral structures. A satisfactory theory of the chiral structures which captures the geometrical nature is still lacking.

In this Letter, we present a metric-torsion theory for the chiral structures with a generalized scheme of transformation optics beyond coordinate transformation. As a key insight, we show that the chiral structure indeed exhibits a pure metric description associated with the anholonomic transformation, which bridges the gap between transformation optics and covariant vacuum electrodynamics. By applying the formal invariance to the local orthonormal basis [23], the structural chirality has a dual Riemann-Cartan description [37] where the anisotropic chiral term is interpreted as the torsion, mimicked by the object of anholonomity. Different from previous studies [36–39], our theory not only reveals the underlying geometrical and topological structures of chirality but also provides an analytically closed form for the chiral constitutive relations, paving the way towards the quantitative design of chiral materials from the achiral background.

*Limitation of transformation optics.*—We begin with a brief review of the limitation of conventional transformation optics in three-dimensional (3D) covariant formalism [40]. The source free Maxwell's equations of monochromatic waves read $\epsilon^{abc}\partial_b E_c - i\omega\mathcal{B}^a = 0$, $\epsilon^{abc}\partial_b H_c + i\omega\mathcal{D}^a = 0$, where $\omega$ is the angular frequency, $\epsilon^{abc} = e^{abc}/\sqrt{g}$ is the Levi-Civita symbol with $e^{abc} = 0, \pm 1$ the permutation symbol, $g = \det(g_{\alpha\beta})$, both $\epsilon^{abc}$ and the excitations $(\mathcal{B}^a, \mathcal{D}^a)$ are tensor densities of weight $+1$. Without magnetoelectric coupling, the Maxwell's equations are supplemented with the constitutive relations $\mathcal{D}^a = \epsilon_0\epsilon^{ab}E_b$, $\mathcal{B}^a = \mu_0\mu^{ab}H_b$. Under coordinate transformation, $\epsilon^{ab}$ and $\mu^{ab}$ obey the tensorial law [25]

$$\epsilon'^{\alpha\beta} = J^{-1}J^\alpha_a J^\beta_b \epsilon^{ab}, \qquad \mu'^{\alpha\beta} = J^{-1}J^\alpha_a J^\beta_b \mu^{ab}, \qquad (1)$$

where $J^\alpha_a = \partial x'^\alpha/\partial x^a$ is the Jacobi matrix and $J = \det(J^a_\alpha)$. Unless otherwise specified, we refer to the curved geometry only for the transformed space to focus on the effect of transformation. From the passive viewpoint, the coordinate transformation is equivalent to the transformation between the holonomic *coordinate bases* in the tangent space according to $\boldsymbol{e}_\alpha = J^a_\alpha \boldsymbol{e}_a$, whereas $\boldsymbol{e}_a = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ is the Cartesian basis. The problem is that the three independent variables for an arbitrary coordinate transformation is fewer than the six d.o.f. for each of the real symmetric $(\boldsymbol{\epsilon}, \boldsymbol{\mu})$ of a lossless medium [34]. Thus, not all metric transformation media can be diagonalized to a background material in the flat space by a similarity transformation with the Jacobi matrix, which indicates that the metric contains more information than originally believed. Here, the metric transformation media refer to the transformation media which relate a metric space created by spatial transformation. In particular, the material tensor created from a flat vacuum $\epsilon^{\alpha\beta} = \mu^{\alpha\beta} = \sqrt{g}g^{\alpha\beta}$, where $g^{\alpha\beta} = J^\alpha_a J^\beta_b \delta^{ab}$, is a subset of the general metric medium [3]. The d.o.f. redundance is consistent with the fact that the chirality of real chiral materials emerges from the handed inhomogeneity of $(\boldsymbol{\epsilon}, \boldsymbol{\mu})$ without inversion symmetry.

*Chirality from metric.*—To describe the chiral continuum with spatial transformation, we note a general metric can be locally diagonalized to the Euclidean form by $g_{\alpha\beta} = e^a_\alpha e^b_\beta \delta_{ab}$, where $e^a_\alpha$ is usually not a Jacobi matrix [41]. Inspired by this and following [23], we propose the generalized metric transformation media whose parameters $(\boldsymbol{\epsilon}', \boldsymbol{\mu}')$ are created by replacing $J^\alpha_a$ with $e^\alpha_a$ in Eq. (1), where $e^\alpha_a$ is reciprocal to $e^a_\alpha$. We further propose that the materials associated with the *anholonomic* $e^\alpha_a$, which are not Jacobi matrices, are intrinsic chiral media with symmetric chiral tensor $\boldsymbol{\kappa}$. In this context, the chiral materials depend entirely on the metric of the transformed space, and we refer to it as the metric description in consistent with the covariant vacuum electrodynamics. In the dual Riemann-Cartan description below, we compute $\boldsymbol{\kappa}$ directly as the torsion of the space.

*Chirality from anholonomy.*—We now solve $\boldsymbol{\kappa}$ explicitly by considering a thought experiment to create ideally chiral metamaterials [42–44]. We start with a virtually achiral continuum where the anisotropic $(\boldsymbol{\epsilon}, \boldsymbol{\mu})$ at two nearby points $p_1$ and $p_2$ are shown in Fig. 1(a). In practice, the anisotropic continuum can be regarded as the homogenized nanostructured metamaterial fabricated by either top-down or bottom-up method. At each point, it is attached with a basis $\boldsymbol{e}_a$ representing by the triad aligned with the Cartesian axes. To create structural chirality, we rotate the triads locally to construct a basis field $\hat{\boldsymbol{e}}_\alpha(x)$ [Fig. 1(b)], which relates with the Cartesian basis by $\hat{\boldsymbol{e}}_\alpha(x) = e^a_\alpha(x)\boldsymbol{e}_a$. For simplicity, we consider only the orthonormal frame $\hat{\boldsymbol{e}}_\alpha \cdot \hat{\boldsymbol{e}}_\beta = \delta_{\alpha\beta}$. It is reasonable to assume that the triads for infinitesimally apart distances connect smoothly. If the local anisotropy of the medium follows the triad during rotation, i.e., the material parameters $(\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\mu}})$ in the orthonormal basis take the same values as $(\boldsymbol{\epsilon}, \boldsymbol{\mu})$, we obtain a chiral medium. As a metric medium, the material $(\boldsymbol{\epsilon}', \boldsymbol{\mu}')$ are described by Eq. (1) with $e^a_\alpha$ replacing $J^a_\alpha$. In the $\hat{\boldsymbol{e}}_\alpha$ basis, however, the permittivity and permeability are $(\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\mu}})$, and the information of structural chirality is encoded in the object of anholonomity $C^\gamma_{\alpha\beta}$, which is defined as the structure constant of the Lie bracket: $[\hat{\boldsymbol{e}}_\alpha, \hat{\boldsymbol{e}}_\beta] = C^\gamma_{\alpha\beta}\hat{\boldsymbol{e}}_\gamma$ [41]. By construction, $C^\gamma_{\alpha\beta}$ is antisymmetric in its lower indices. In components, it reads $C^\gamma_{\alpha\beta} = e^\gamma_a(\partial_\alpha e^a_\beta - \partial_\beta e^a_\alpha)$. Formally, the anholonomic Maxwell's equations read [45]

$$\epsilon^{\alpha\beta\gamma}[\partial_\beta E_\gamma + C^\delta_{\beta\gamma}E_\delta] - i\omega\mathcal{B}^\alpha = 0,$$
$$\epsilon^{\alpha\beta\gamma}[\partial_\beta H_\gamma + C^\delta_{\beta\gamma}H_\delta] + i\omega\mathcal{D}^\alpha = 0, \qquad (2)$$

where $(E,H)_\alpha = e^a_\alpha(E,H)_a$ and $\partial_\alpha = e^a_\alpha\partial_a$ is the Pfaffian derivative. In the coordinate basis, Eq. (2) reproduce the usual forms because $C^\gamma_{\alpha\beta}$ vanishes identically for nonsingular coordinate transformation.

The presence of $C^\gamma_{\alpha\beta}$ is crucial for the geometrical interpretation of the chiral effect. In the tetrad formalism of relativity, the orthonormal basis defines a local frame carried by an accelerated observer along its world line [41], and Eq. (2) describe the electromagnetism observed in the noninertial frame with $C^\gamma_{\alpha\beta}$ a Coriolis term [6]. In this Letter, however, we generalize the formal invariance of

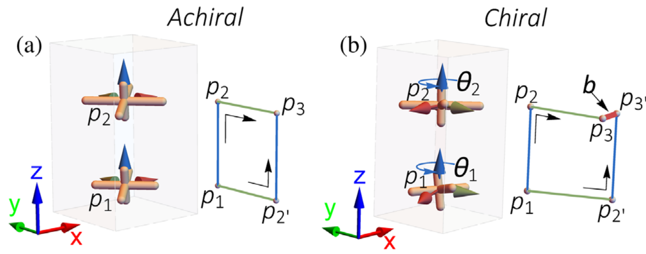


FIG. 1. Schematics of the local anisotropy of $(\boldsymbol{\epsilon}, \boldsymbol{\mu})$ (the crossed rods) and the local frames (the triads) at two nearby points in (a) the achiral background and (b) the chiral continuum. The Cartan's circuits are shown in the right panels.

Maxwell's equations to the orthonormal basis and alternatively regard Eq. (2) as the Maxwell's equations in the Cartesian system by recasting to $e^{\alpha\beta\gamma}\partial_\beta E_\gamma - i\omega\bar{\mathcal{B}}^\alpha = 0$, $e^{\alpha\beta\gamma}\partial_\beta H_\gamma + i\omega\bar{\mathcal{D}}^\alpha = 0$. Here, we have introduced new excitations

$$\bar{\mathcal{D}}^\alpha = \bar{\epsilon}^{\alpha\beta}E_\beta - \frac{i}{c}\kappa^{\alpha\beta}H_\beta,$$
$$\bar{\mathcal{B}}^\alpha = \frac{i}{c}\kappa^{\alpha\beta}E_\beta + \bar{\mu}^{\alpha\beta}H_\beta, \tag{3}$$

where

$$\kappa^{\alpha\beta} = \frac{\lambda_0}{4\pi}e^{\alpha\gamma\delta}C^\beta_{\gamma\delta}, \tag{4}$$

with $\lambda_0 = 2\pi c/\omega$ the vacuum wavelength. In this way, we obtain the *homogenized* chiral constitutive relation for the chiral material with the chiral tensor being interpreted as the anholonomy of the orthonormal basis. We refer this approach as the Riemann-Cartan description which will be elaborated below. The dimension of $C^\gamma_{\alpha\beta}$ is [rad m$^{-1}$] since it is weighted by $\lambda_0/4\pi$. Comparing Eq. (3) with the standard bianisotropic constitutive relations [30], the Lorentz reciprocity restricts the chiral tensor to be symmetric $\boldsymbol{\kappa} = \boldsymbol{\kappa}^T$. Therefore, Eq. (3) describe intrinsic chiral media instead of the pseudochiral omega media with antisymmetric $\boldsymbol{\kappa}$. From Eq. (4), it is useful to decompose $\boldsymbol{\kappa}$ into two irreducible parts with distinct geometric origins: a trace part consisting of diag$(C^1_{23}, C^2_{31}, C^3_{12})$, and an off diagonal part taking values from $C^\alpha_{\alpha\beta}(\alpha \neq \beta)$. In addition, $\boldsymbol{\kappa}$ is even under time reversal and odd under spatial inversion [30]. It also satisfies with the Post traceless constraint [30] and electromagnetic duality symmetry [46].

*Torsion.*—Now we illustrate how $C^\gamma_{\alpha\beta}$ is interpreted as the torsion tensor of a Riemann-Cartan space. In the coordinate base, torsion breaks the infinitesimal Cartan's circuits [6]. Here, the nontrivial chiral metric is locally diagonalized to the Euclidean form in the $\hat{\boldsymbol{e}}_\alpha$ basis. Applying the first Cartan's structure equation $\nabla_X Y - \nabla_Y X = [X, Y]$ to $\hat{\boldsymbol{e}}_\alpha$ gives rise to $C^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}$. As a nontensorial object, the object of anholonomity breaks the quadrilateral defined on the integral curves of the bases. Consider the integral curves of $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ originating from an arbitrary reference point $p_1$ in the right panel of Fig. 1(b), the misfit $p_3 p_{3'}$ of the infinitesimal quadrilateral along two different paths by swapping the order along the integral curves is $\boldsymbol{b} = C^1_{\alpha\beta}ds^\alpha ds^\beta \hat{\boldsymbol{e}}_1$, where $ds^\alpha$ ($\alpha = 2, 3$) denotes the distance along the $\hat{\boldsymbol{e}}_\alpha$ direction [45]. Central to our theory is that the formal invariance of Maxwell's equations identifies the isomorphism between Cartan's circuit and the nonclosed quadrilateral, thereby enabling the simulation of torsion in the coordinate basis with $C^\gamma_{\alpha\beta}$ in $\hat{\boldsymbol{e}}_\alpha$ as defining the metric in conventional transformation optics. Moreover, a full Riemann-Cartan structure is achieved when we consider the independent metric from either a nontrivial background or a successive coordinate transformation. The dictionary of related quantities is summarized in Table I. It is interesting that the presence of torsion in the constitutive relation is in agreement with the other light-torsion coupling approaches such as through vacuum polarization in the quantized level [39], the premetric electrodynamics [6], and the nonminimal coupling [36], without modifying the Maxwell's equations [37]. Different from [36] where the formal theory of Maxwell's equations in the Riemann-Cartan space is considered, our program starting from the anholonomic frame not only captures the geometrical nature of the structural chirality but also provides an alternative Coriolis coupling between light and torsion without assuming the admissible nonminimal coupling [36–39].

TABLE I. The duality dictionary of the chiral continuum.

| Expression | Background | Metric | Riemann-Cartan |
|---|---|---|---|
| Basis | Coordinate | Coordinate | Noncoordinate |
| Geometry | $\delta_{ab}$ | $g_{\alpha\beta}$ | $\delta_{\alpha\beta}$, $C^\gamma_{\alpha\beta}$ |
| Optics | $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ | $\boldsymbol{\epsilon}'$, $\boldsymbol{\mu}'$ | $\bar{\boldsymbol{\epsilon}}$, $\bar{\boldsymbol{\mu}}$, $\boldsymbol{\kappa}$ |

The Riemann-Cartan structure is reminiscent of the geometrical theory of distributed dislocations in elastic and crystalline solids [47–53], where torsion is interpreted as the local density of dislocations. Since the lower indices $\alpha$ and $\beta$ denote the area element of the circuit, and the upper index $\gamma$ denotes the direction of the Burgers vector, we deduce that the diagonal $\boldsymbol{\kappa}$ is associated with the screw dislocation continua whose local Burgers vector is perpendicular to the area element, while the off-diagonal terms are associated with the edge dislocation continua whose Burgers vector is parallel to the area element. The Burgers circuit indicates the nontrivial topological structure of electromagnetism in the integral form. Specifically, the Burgers vector leads to a correction term in Faraday's law: $\oint_\Gamma \mathbf{E}\cdot d\boldsymbol{x} = i\omega\int_S \mathbf{B}\cdot d\boldsymbol{\sigma} + \boldsymbol{b}\cdot\mathbf{E}$, where $\mathbf{B} = \mu_0\mu\mathbf{H}$ and $\Gamma$ denotes the loop enclosing an infinitesimal area $\sigma$. A similar expression applies to Ampère's law.

*Application.*—To validate our theory, we examine two minimal models of 3D chiral metamaterials [42] which have a previously unrecognized connection. The first model originates from the cholesteric liquid crystals (CLCs) which exhibit optical activity for on axis propagation [54–57]. As depicted in Fig. 2(a), CLC locally behaves like a uniaxial material in the homogeneous limit, where the in-plane material tensors are given by $\chi'_{ab} = \chi_\perp\delta_{ab} + (\chi_\parallel - \chi_\perp)n_a n_b$, $(a, b = x, y)$ with $\chi$ representing $\epsilon$ or $\mu$. The local anisotropic director $\mathbf{n}(z) = (\cos\theta(z), \sin\theta(z), 0)$ rotates with a local *spatial* angular velocity $\omega_s(z) = \partial\theta/\partial z$ about the local $z$ axis. The original CLC is a special case of $\omega_s = \pi/a$ with constant pitch $a$.

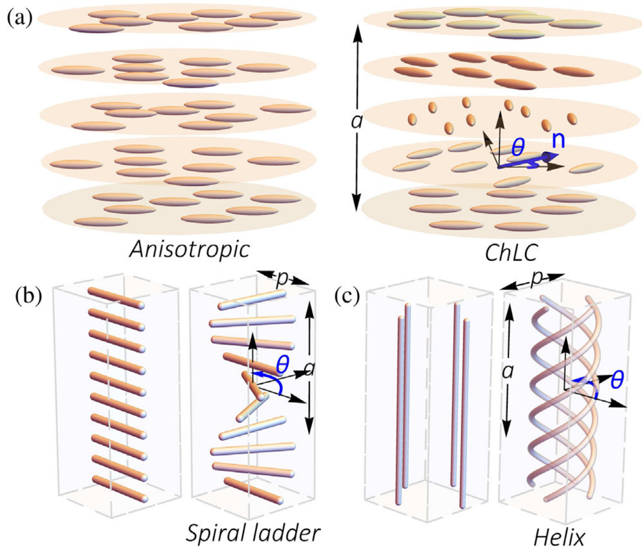


FIG. 2. (a) Schematics of a CLC (right) and the anisotropic background (left). (b),(c) Unit cells of typical 3D chiral metamaterials consisting of spiral ladders and helices (right) and the anisotropic backgrounds (left). The bars are real nanostructures.

In the metric description, it can be modeled as a generalized transformation medium constructed from an achiral background: $\boldsymbol{\chi}' = R_z(\theta)\boldsymbol{\chi}R_z^T(\theta)$, where $\boldsymbol{\chi} = \mathrm{diag}(\chi_\parallel, \chi_\perp, \chi_z)$ represents the homogeneous Cartesian anisotropy and $R_z = R_2 \oplus I_1$ is the rotation matrix about $z$ axis with $R_2$ the 2D rotation matrix, $I_1 = 1$. Instead of directly solving the wave equation from $\boldsymbol{\chi}'$ [54,55], we note that $\mathbf{n}(z)$ introduces naturally an orthonormal basis by $\hat{\boldsymbol{e}}_\alpha = [R_z]^a_\alpha \boldsymbol{e}_a$, in which the chiral constitutive relation can be extracted. The propagation equation in $\hat{\boldsymbol{e}}_\alpha$ basis reads [45,57]

$$\frac{d\psi}{dz} = i\frac{\omega}{c}(\mathbb{G} + i\mathbb{K})\psi. \tag{5}$$

In Eq. (5), we have split the physical effect into a local anisotropic material part plus a geometric chiral part

$$\mathbb{G} = \begin{pmatrix} 0 & 0 & \mu_\parallel & 0 \\ 0 & 0 & 0 & \mu_\perp \\ \epsilon_\perp & 0 & 0 & 0 \\ 0 & \epsilon_\parallel & 0 & 0 \end{pmatrix}, \qquad \mathbb{K} = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & -\kappa & 0 \end{pmatrix},$$

where $\kappa = -k_0\omega_s$, $\psi = [R_2 \otimes I_2]\psi_0$ with $I_2$ the $2\times 2$ identity matrix and $\psi_0 = (E_x, E_y, Z_0H_y, -Z_0H_x)^T$ with $Z_0 = \sqrt{\mu_0/\epsilon_0}$. It can be shown that the spatial angular velocity is the structure constant

$$[\hat{\boldsymbol{e}}_\alpha, \hat{\boldsymbol{e}}_z] = -\epsilon_{\alpha\beta z}\omega_s\hat{\boldsymbol{e}}_\beta, \qquad (\alpha, \beta = x, y). \tag{6}$$

Therefore, we find that $\mathbb{K}$ is determined by the object of anholonomity, which is further identified with torsion in the coordinate basis, and Eq. (5) is formally identical to the propagation equation in a chiral material in the Cartesian system with $\bar{\boldsymbol{\epsilon}} = \mathrm{diag}(\epsilon_\parallel, \epsilon_\perp, \epsilon_z)$, $\bar{\boldsymbol{\mu}} = \mathrm{diag}(\mu_\parallel, \mu_\perp, \mu_z)$ and $\boldsymbol{\kappa} = \mathrm{diag}(\kappa, \kappa, \kappa_z)$ [45]. Notably, $\boldsymbol{\kappa}$ decouples from the anisotropic background as a purely geometric effect and appears in both $\bar{\mathbf{D}}$ and $\bar{\mathbf{B}}$ even for the nonmagnetic system. Besides providing a simple example for our theory, the CLC model can be used to analytically design chiral metamaterials.

We compare the CLC model with 3D chiral metamaterials consisting of arrays of subwavelength helical inclusions such as spiral ladders or helices [42–44]. As shown in Figs. 2(b) and 2(c), the chiral inclusions can be obtained from an anisotropic background by the helical coordinate transformation $x'^\alpha = [R_z]^\alpha_\beta x^\beta$ [58]. This transformation also models other helical systems including discrete screw dislocations [59] and helical waveguides [60,61]. The corresponding transformation medium is given by $(\boldsymbol{\epsilon}', \boldsymbol{\mu}') = J^{-1}J_z(\boldsymbol{\epsilon}, \boldsymbol{\mu})J_z^T$ with

$$J_z = \begin{pmatrix} \cos\theta & -\sin\theta & -\omega_s y \\ \sin\theta & \cos\theta & \omega_s x \\ 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

here $(\boldsymbol{\epsilon}, \boldsymbol{\mu})$ are the same as $\boldsymbol{\chi}$ in CLCs. In contrast to the CLC model where each point twists locally, the helical coordinate transformation twists globally, leading to the nonvanishing radius dependent $iz$ components in $J_z$. Consequently, the volume preserving Eq. (7) describes a screw dislocation whose Burgers vector can be defined on arbitrary loops encompassing the screw axis. Despite the difference, the two models are equivalent with each other in the infinitesimal limit. Consider a 2D array of meta-atoms made from Eq. (7), the lattice constant $p$ defines a cutoff length for the meta-atoms. If $p$ cannot be neglected compared to the wavelength $\lambda_0$, the response may lead to additional resonances depending on $p$ and the constituent composition. When $p/\lambda_0 \ll 1$, the $iz$ components of $J_z$ are practically negligible, and the lattice of meta-atoms converges to CLC. In the homogenization language, the Riemann-Cartan description is the effective medium of the chiral metamaterial consisting of screw dislocated meta-atoms, and torsion emerges as the homogenization limit of the locally Riemannian spaces of isolated dislocations. To our best knowledge, our theory provides the first concrete illustration for the homogenization picture of torsion, which has been proved with rigorous mathematics for the edge dislocations only recently [62].

In addition to anisotropy, chiral materials can be constructed from isotropic materials with local inhomogeneity. The metric description fails for $\chi_\parallel = \chi_\perp$ in the CLC model when $\boldsymbol{\chi}' = \boldsymbol{\chi} \propto \mathrm{diag}(1,1,1)$, while the Riemann-Cartan description remains valid. It follows that there exist no chiral medium with $\epsilon = \mu = 1$, $\kappa \neq 0$. Furthermore, more complex chiral structures can be realized for applications by (pieced) continuously or randomly distributed [63] or oriented bases.

*Discussion and conclusion.*—We finally comment on the basic feature of the transformations of coordinate and local orthonormal basis. The coordinate transformation defines an elastic deformation where the displacement vector specifies the translational d.o.f. In contrast, utilizing orthonormal bases violates the Frobenius integrability condition, and it is impossible to find a coordinate transformation to generate the orthonormal bases $\hat{\boldsymbol{e}}_{\alpha}$, which defines the independently rotational d.o.f. Physically speaking, the cross-coupling of fields produces a phase difference between states of opposite helicities, leading to optical activity. Geometrically, the polarization rotation can be interpreted as the parallel transport of the polarization vector in the local frame, whereas $C^{\gamma}_{\alpha\beta}$ plays the role of Berry curvature [64]. In this sense, the chiral materials can be regarded as the optical Cosserat continua which are elastic micropolar media with the internal torque [47].

In summary, we have developed a metric-torsion theory for light propagation in chiral structures. Our theory not only provides a unified geometrical insight for chiral materials based on frame transformation but also introduces an analytical approach for the practical device design with chiral materials. It may offer a platform to simulate wave propagation in the torsional space-time which remains inaccessible in gravitational physics.

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