Complexity of Causal Order Structure in Distributed Quantum Information Processing: More Rounds of Classical Communication Reduce Entanglement Cost

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(Received 22 October 2018; published 16 May 2019)

We prove a trade-off relation between the entanglement cost and classical communication round complexity of a protocol in implementing a class of two-qubit unitary gates by two distant parties, a key subroutine in distributed quantum information processing. The task is analyzed in an information theoretic scenario of asymptotically many input pairs with a small error that is required to vanish sufficiently quickly. The trade-off relation is shown by (i) one ebit of entanglement per pair is necessary for implementing the unitary by any two-round protocol, and (ii) the entanglement cost by a three-round protocol is strictly smaller than one ebit per pair. We also provide an example of bipartite unitary gates for which there is no such trade-off.

DOI: 10.1103/PhysRevLett.122.190502

Introduction.—Quantum information processing achieves its power by composing multiple quantum systems to form a larger quantum system. It is necessary that the components collaborate to behave as a single composite quantum system. In distributed quantum information processing (DQIP), communication channels connecting the components, quantum and/or classical, serve as a resource. Additional correlation shared between the components is another type of resource in DQIP. Shared correlations can be both quantum and classical. The processing power of the individual components and the available communication or correlation resources determine the total information processing capacity of a DQIP system.

Shared entanglement in DQIP is arguably the most resourceful kind of quantum correlation. DQIP protocols exhibit advantages over their classical counterparts by exploiting entanglement, e.g., in communication complexity [1–16], interactive proof systems [17–20], nonlocal games [21–31], measurement-based quantum computation [32–34], and quantum cryptography [35]. Entanglement is distinguished from classical nonlocal DQIP resources such as classical correlation and classical communication in that entanglement can never be generated or increased by the classical resources, however much consumed. In fact, the modern entanglement theory [36,37] *defines* entanglement as the correlation of quantum states that cannot be strengthened under local operations and classical communication (LOCC), where all the intercomponent communications within the DQIP system are restricted to classical.

General LOCC protocols consist of multiple rounds (see Fig. 1). Consider an LOCC task performed by two distant parties, say, Alice and Bob. A round in any protocol for this task consists of one party performing a local operation and communicating a classical message to the other. Protocols with a higher number of rounds are higher in *round complexity*. Every communication must wait a certain minimum amount of time to complete; hence the round complexity of a protocol draws a lower bound on the time required.

The round complexity is a separate resource from the total length of exchanged messages, and the "bandwidth," i.e., the maximum possible length of each message. Entanglement cannot be increased by any LOCC protocol however high in round complexity and long in classical message. The local processing power affects only the set of possible local operations. The resources for DQIP have been extensively investigated [38–58], but known relations between entanglement resource and the round complexity are scarce [57,58].

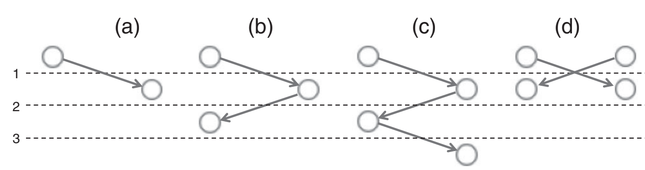

FIG. 1. Schematic description of round complexity. For each of (a)–(d), the horizontal axis represents a configuration of Alice (left) and Bob (right), and the vertical axis for round. Circles and arrows represent local operations and classical communications, respectively. The number of rounds is 1 for (a), 2 for (b), 3 for (c), and 1 for (d). (a) is a protocol with unidirectional communication, while the others are with a bidirectional one. (d) is a protocol with simultaneous message exchange, while the others are not.

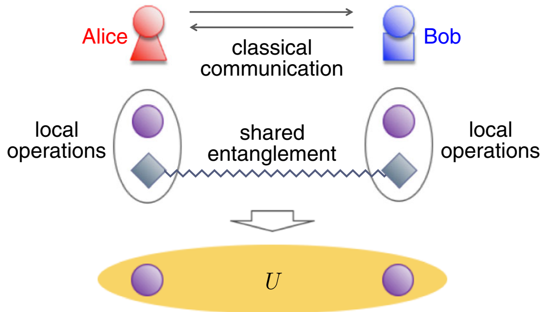

FIG. 2. Implementation of a bipartite unitary gate by LOCC assisted by shared entanglement is depicted. The balls represent physical systems on which the unitary gate is to be implemented, and the diamonds represent parts of the entanglement resource shared in advance.

In this Letter, we report a DQIP task for which the cost of shared entanglement can be reduced by increasing the CC round complexity of protocols. Thus, we jointly analyze entanglement and causal relations, each a fundamental topic of physics, in general, in this single context of quantum information processing. The task is for the two distant parties to implement a class of two-qubit unitary gates by LOCC assisted by shared entanglement (see Fig. 2). The two parties are not allowed to exchange messages simultaneously. We leave the length of each message and the total unrestricted, so that the only source of any reduction in the entanglement cost is in the higher round complexity. We prove that a three-round protocol outperforms all two-round protocols in reducing the entanglement cost. Thereby, we show a clear trade-off relation between the cost of entanglement and complexity in causal order. We also provide a class of bipartite unitary gates for which there is no such trade-off, by proving that a protocol of type (b) and (d) in Fig. 1 achieves the minimum cost of entanglement over all finite-round protocols.

Our result is a more "refined" trade-off between round complexity and other resources in LOCC protocols compared to the previous approaches [38–44] showing the advantages of bidirectional communication over a unidirectional one. Other known results [45–58] analyze zero error-tolerance regimes, while we adopt an information theoretic scenario of infinitely many inputs and a vanishingly small error [59]. The more refined analysis is made possible partly due to the mathematically well-structured tools developed in the quantum Shannon theory [60–62] (see [59] for the details). Outside LOCC, higher-round complexity is known to result in an exponential decrease in computational resources [63].

Setup of our protocol.—Suppose that Alice and Bob, located in two distant laboratories, have n-qubit systems $A^n = A_1 \ldots A_n$ and $B^n = B_1 \ldots B_n$, respectively. They aim to apply a two-qubit unitary gate U on each pair $A_i B_i (i = 1, \ldots, n)$, simultaneously. To accomplish this task, Alice and Bob may perform quantum operations locally in their laboratories, communicate classical messages to each other, and may use copies of a Bell pair $|\Phi_2\rangle \coloneqq (|00\rangle + |11\rangle)/\sqrt{2}$ shared in advance as a resource. They are, however, not allowed to communicate quantum messages or to perform operations that globally act across their laboratories. That is, they accomplish the task by LOCC assisted by entanglement. We assume that they are not allowed to communicate classical messages simultaneously in both directions. The state on system $A^n B^n$ may initially be correlated with an external reference system R, which is inaccessible to Alice and Bob.

Let $E > 0$ be the number of copies of Bell pairs divided by n. Denoting by a and b the quantum registers in which the resource state $|\Phi_2\rangle$ is stored, an LOCC protocol for the above task is represented by a completely positive and trace-preserving (CPTP) map $\mathcal{M}_n$ from $A^n B^n a^{nE} b^{nE}$ to $A^n B^n$. The error of the protocol for a particular initial state $|\psi\rangle^{A^n B^n R}$ is quantified by the fidelity between the target state $U^{\otimes n}|\psi\rangle^{A^n B^n R}$ and the state obtained after the protocol, i.e.,

$$\epsilon(\mathcal{M}_n, \psi) \coloneqq 1 - F(U^{\otimes n}(\psi)U^{\dagger \otimes n}, \mathcal{M}_n(\psi \otimes \Phi_2^{\otimes nE})). \tag{1}$$

We adopted the notation $\psi = |\psi\rangle\langle\psi|$ and $\Phi_2 = |\Phi_2\rangle\langle\Phi_2|$. The fidelity is defined by $F(\rho, \sigma) \coloneqq (\mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2$. The supremum of the above quantity over all ψ is called the *worst-case* error and is denoted by $\epsilon^*(\mathcal{M}_n)$.

An entanglement consumption rate E is said to be achievable by r-round protocols if there exists a sequence $\{\mathcal{M}_n\}_{n=1}^{\infty}$ of r-round protocols such that the worst-case error $\epsilon^*(\mathcal{M}_n)$ vanishes in the limit of n to infinity. For a technical reason, we impose that the convergence of the error is sufficiently fast so that

$$\lim_{n \to \infty} n^4 \epsilon^*(\mathcal{M}_n) = 0. \tag{2}$$

The *entanglement cost* of a two-qubit unitary gate U by r-round protocols is the minimum rate E that is achievable by r-round protocols and is denoted by $E_r(U)$.

In this Letter, we prove that there exists a trade-off relation between the entanglement cost and round complexity for implementing a two-qubit unitary gate. By "trade-off relation," we refer to the fact that the entanglement cost of a unitary gate by the best possible r-round protocol is strictly smaller than any r'-round one, i.e., $E_r(U) < E_{r'}(U)$, for certain $r > r'$.

We consider a class of two-qubit unitary gates of the form

$$U_\theta^{AB} = \cos\left(\frac{\theta}{2}\right) I^A \otimes I^B + i \sin\left(\frac{\theta}{2}\right) \sigma_z^A \otimes \sigma_z^B, \tag{3}$$

where $\theta \in (0, \pi/2]$ and I and σ_z are the identity operator and the Pauli-z operator defined by $I = |0\rangle\langle 0| + |1\rangle\langle 1|$ and

$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, respectively. We prove that the tradeoff relation holds for θ smaller than a constant, by showing that $E_2(U_\theta) > E_3(U_\theta)$. In the following, we describe an outline of the proof of $E_2(U_\theta) \geq 1$ based on our previous work [59] and that of a proof of $E_3(U_\theta) < 1$. A detailed proof of $E_3(U_\theta) < 1$ is provided in Ref. [64].

Conditions for successful protocols.—For a protocol $\mathcal{M}_n$ to be successful, the following conditions must be satisfied. We first analyze a general case where A and B are quantum systems with an arbitrary (but finite) dimension d. We consider a particular initial state $|\Psi_{U^\dagger,n}\rangle := |\Psi_{U^\dagger}\rangle^{\otimes n}$, where $|\Psi_{U^\dagger}\rangle$ is the Choi-Jamiołkowski state corresponding to the inverse of the unitary gate to be implemented. With R_A and R_B denoting d-dimensional reference systems that are inaccessible to Alice and Bob, the Choi-Jamiołkowski state is defined as

$$|\Psi_{U^\dagger}\rangle := U^{\dagger AB}|\Phi_d\rangle^{AR_A}|\Phi_d\rangle^{BR_B}, \tag{4}$$

where Φ_d is the maximally entangled state with Schmidt rank d. The worst-case error $\epsilon^*(\mathcal{M}_n)$ is no smaller than $\epsilon(\mathcal{M}_n, \Psi_{U^\dagger,n})$. Thus, a successful protocol $\mathcal{M}_n$ must satisfy $\epsilon(\mathcal{M}_n, \Psi_{U^\dagger,n}) \approx 0$. Noting that $UU^\dagger = I$, it follows that

$$\mathcal{M}_n(\Psi_{U^\dagger,n}^{A^nB^nR_A^nR_B^n} \otimes \Phi_2^{\otimes nE}) \approx (|\Phi_d\rangle^{AR_A}|\Phi_d\rangle^{BR_B})^{\otimes n}. \tag{5}$$

This condition imposes a restriction on Alice's measurement at the beginning of the protocol. The entanglement consumption rate E in a two-round protocol must be large enough in order that such a measurement by Alice exists for sufficiently large n.

Observe that the initial state in the lhs in Eq. (5) is an entangled state between $A^nR_A^na^{nE}/B^nR_B^nb^{nE}$, while the state in the rhs is a product state in that separation. In addition, both states are pure maximally entangled states between $A^nB^n/R_A^nR_B^n$. Thus, $\mathcal{M}_n$ can be viewed as a protocol that destroys correlation between $A^nR_A^na^{nE}/B^nR_B^nb^{nE}$ in the state $\Psi_{U^\dagger,n}^{A^nB^nR_A^nR_B^n} \otimes \Phi_2^{\otimes nE}$ while maintaining the purity of the whole state as well as the maximal entanglement between $A^nB^n/R_A^nR_B^n$ (Fig. 3). It should be noted that R_A and R_B are reference systems that are inaccessible to Alice and Bob. Thus, the task considered here is different from the transformation of bipartite pure states [68].

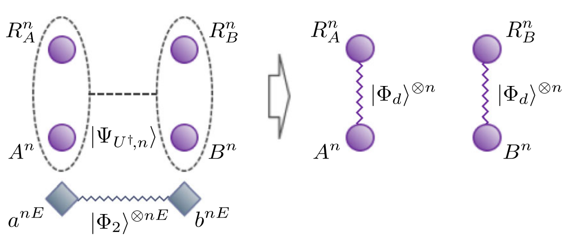

FIG. 3. A graphical representation of a task corresponding to Eq. (5). The task is to destroy the correlation between $A^nR_A^n$ and $B^nR_B^n$ while preserving the maximal entanglement between A^nB^n and $R_A^nR_B^n$ as well as the purity of the whole state.

Let us analyze conditions imposed by Eq. (5) on Alice's measurement at the beginning of a two-round protocol $\mathcal{M}_n$. We denote the output system of the measurement by A'. First, since entanglement between $A^nB^na^{nE}b^{nE}/R_A^nR_B^n$ is nonincreasing under any step in $\mathcal{M}_n$, the reduced state on $R_A^nR_B^n$ must be close to the maximally mixed state for each measurement outcome. We call this condition *obliviousness*, because it is equivalent to the condition that the measurement does not extract any information about the initial state. Second, since the reduced state on $B^nR_B^n$ is not changed by Alice's operation at the end, the maximally entangled state $(\Phi_2^{BR_B})^{\otimes n}$ must be obtained immediately after Bob's measurement. This implies that $A'R_A^n$ and R_B^n must be in a product state on average, after the measurement by Alice. We refer to this condition as *decoupling*.

From decoupling to Markovianization.—Let Ψ_k be the state after Alice's measurement corresponding to the outcome k. The decoupling condition is represented by the quantum mutual information as $I(A'R_A^n:R_B^n)_{\Psi_k} \approx 0$ for each k in a highly probable set, where $I(P:Q)_\rho := S(\rho^P) + S(\rho^Q) - S(\rho^{PQ})$ and S is the von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho\log\rho]$. Since $|\Phi_d\rangle^{AR_A}|\Phi_d\rangle^{BR_B}$ is the maximally entangled state between AB/R_AR_B, there exists a unitary $\hat{U}$ on R_AR_B satisfying $\hat{U}^{R_AR_B}|\Phi_d\rangle^{AR_A}|\Phi_d\rangle^{BR_B} = U^{AB}|\Phi_d\rangle^{AR_A}|\Phi_d\rangle^{BR_B}$. It follows that the conditional quantum mutual information is equal to zero, i.e., $I(A':B^n|R_A^nR_B^n)_{\Psi_k} = 0$, where $I(P:Q|R)_\sigma := S(\sigma^{PR}) + S(\sigma^{QR}) - S(\sigma^R) - S(\sigma^{PQR})$. The chain rule of the quantum mutual information yields $I(A'R_A^n:R_B^n) \geq I(A':B^nR_B^n|R_A^n)$. Consequently, we arrive at

$$I(A':B^nR_B^n|R_A^n)_{\Psi_k} \approx 0, \tag{6}$$

since the conditional quantum mutual information is always non-negative [69].

A tripartite quantum state for which the conditional quantum mutual information is approximately equal to zero, like Eq. (6), is called an *approximate quantum Markov chain* (AQMC) [70]. From condition (6), it follows that Alice's measurement needs to transform the state $|\Psi_{U^\dagger,n}\rangle$ to an AQMC with the assistance of $|\Phi_2\rangle^{\otimes nE}$ while respecting the obliviousness condition. The entanglement consumption rate E must be large enough in order that a measurement by Alice satisfying this condition exists.

Markovianizing cost.—We have proved in Ref. [59] (see Theorem 5 therein) that the entanglement consumption rate E must be no smaller than the Markovianizing cost of $|\Psi_{U^\dagger}\rangle$ in order that there exists a measurement satisfying the condition mentioned above. In general, the Markovianizing cost of a tripartite quantum state ρ^{ABC} is defined as the minimum cost of randomness required for transforming copies of the state to an approximate Markov chain, by a random unitary operation on system A. In the case of pure states, a single-letter formula for the Markovianizing cost is obtained in terms of the Koashi-Imoto decomposition [71], which is used to characterize the structure of quantum Markov chains [72]. In the current context, the relevant Markovianizing cost is that of a "tripartite" pure state $|\Psi_{U^\dagger}\rangle$, with systems B and R_B treated as a single system BR_B.

Outline of proof of $E_2(U_\theta) \geq 1$.—As proved in Refs. [59,73], the Markovianizing cost of $\Psi_{U^\dagger}$ is equal to the von Neumann entropy of a state $\Phi_{U,\infty}^{AR_A} \coloneqq \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \mathcal{E}_U^n(|\Phi_d\rangle\langle\Phi_d|^{AR_A})$, where $\mathcal{E}_U$ is a CPTP map on system A defined by $\mathcal{E}_U(\tau) = \mathrm{Tr}_{BR_B}[U^{AB}(\mathrm{Tr}_B[U^{\dagger AB}(\tau^A \otimes I^B)U^{AB}] \otimes \Phi_d^{BR_B})U^{\dagger AB}]$. For U_θ defined by (3), we have $\mathcal{E}_{U_\theta}(\tau) = \frac{1}{2}[(1+\cos^2\theta)\cdot\tau + \sin^2\theta \cdot \sigma_z \tau \sigma_z]$ and $\Phi_{U_\theta,\infty}^{AR_A} = \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$. Hence, the Markovianizing cost of $\Psi_{U_\theta^\dagger}$ is equal to 1, which completes the proof of $E_2(U_\theta) \geq 1$.

Outline of proof of $E_3(U_\theta) < 1$.—To prove $E_3(U_\theta) < 1$, we first analyze a single-shot protocol proposed in Ref. [49] for implementing U_θ. We will later extend this protocol to the one for implementing $U_\theta^{\otimes n}$ and analyze the total error and the entanglement cost by applying the law of large numbers.

The single-shot protocol consists of a concatenation of two two-round protocols and proceeds as follows: (P1) Alice and Bob implement U_θ by a protocol of type (b) in Fig. 1, using a two-qubit state $|\phi_\theta\rangle^{ab}$ as a shared resource. The protocol succeeds in implementing U_θ with a certain probability p_θ. If it fails, another unitary gate $U_{\theta'}$ is implemented, in which case Alice and Bob continue to the next step. (P2) Alice and Bob implement $U_{\theta-\theta'}$ by a deterministic protocol proposed in Ref. [45], which consumes one Bell pair. The protocol is of type (b), except that the roles of Alice and Bob are exchanged. Noting that $U_{\theta-\theta'}U_{\theta'} = U_\theta$, they succeed in implementing U_θ in total, regardless of the success in (P1). The average entanglement cost of this protocol, measured by the entanglement entropy, is equal to $\bar{E}_\theta = 1 - p_\theta + E(\phi_\theta)$, where $E(\phi_\theta) \coloneqq S(\phi_\theta^a)$ and $\phi_\theta^a \coloneqq \mathrm{Tr}_b[|\phi_\theta\rangle\langle\phi_\theta|^{ab}]$. As we prove in Ref. [64], it holds that $\bar{E}_\theta < 1$ for θ below a strictly positive constant.

Consider the following protocol for implementing $U_\theta^{\otimes n}$: (P0) Alice and Bob obtains n copies of $|\phi_\theta\rangle^{ab}$ from approximately $nE(\phi_\theta)$ copies of Bell pairs, by an entanglement dilution protocol [65] of type (a) in Fig. 1. ($P1'$) They apply (P1) independently on each of n input pairs. Because of the law of large numbers, U_θ is implemented on approximately np_θ pairs of the input. ($P2'$) They apply (P2) to implement $U_{\theta-\theta'}$ on the remaining input pairs, which costs approximately $n(1-p_\theta)$ Bell pairs. In total, the protocol succeeds in implementing $U_\theta^{\otimes n}$ with a high probability by using approximately $n\bar{E}_\theta$ copies of Bell pairs. As depicted in Fig. 4, the three subprotocols are brought together to form a three-round protocol. Thus, it follows that $E_3(U_\theta) \leq \bar{E}_\theta$.

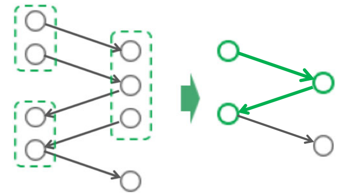

FIG. 4. Transformations of protocols in terms of communication rounds. The figure represents how three protocols are combined to form a three-round protocol.

Unitaries with no trade-off.—So far, we have investigated the case in which there is a difference between $E_2(U)$ and $E_3(U)$. Next, we provide an example of bipartite unitary gates for which there exists no trade-off relation between the entanglement cost and round complexity. Let $\{|t\rangle\}_{t=1}^d$ be a fixed basis of a d-dimensional Hilbert space $\mathcal{H}$. The generalized Pauli operators $\sigma_{pq}(p, q \in \{1, \ldots, d\})$ on $\mathcal{H}$ is defined by $\sigma_{pq} \coloneqq \sum_{t=1}^{d} e^{2\pi i q t/d}|t-p\rangle\langle t|$, where subtraction is taken with mod d. Let A and B be d-dimensional systems. A bipartite unitary gate U_c acting on AB is called a generalized Clifford operator if, for any p, q, r, and s, there exist p', q', r', s', and a phase $\theta_{pqrs} \in \mathbb{R}$ such that

$$U_c(\sigma_{pq} \otimes \sigma_{rs})U_c^\dagger = e^{i\theta_{pqrs}}\sigma_{p'q'} \otimes \sigma_{r's'}. \tag{7}$$

We have proved in Ref. [59] (see Theorem 27 therein) that $E_2(U_c) = \inf_{r\geq 1} E_r(U_c)$ holds. In the following, we prove that the same entanglement cost is achievable by a one-round protocol of type (d) in Fig. 1. This implies that there exists no trade-off relation between the entanglement cost and round complexity for generalized Clifford operators, regardless of whether the two parties are allowed to exchange messages simultaneously.

Consider the following single-shot protocol (see Fig. 1 in Ref. [64]): (i) Alice and Bob initially share a resource state $|\Psi_{U_c}\rangle^{\tilde{A}\tilde{B}ab} \coloneqq (U_c^{\tilde{A}\tilde{B}} \otimes I^{ab})|\Phi_d\rangle^{\tilde{A}a}|\Phi_d\rangle^{\tilde{B}b}$, with a and b being d-dimensional quantum systems; (ii) they perform a projective measurement on system Aa and Bb with respect to bases $\{\sigma_{pq}^{\dagger A}|\Phi_d\rangle^{Aa}\}_{pq}$ and $\{\sigma_{rs}^{\dagger B}|\Phi_d\rangle^{Bb}\}_{rs}$, respectively; (iii) they communicate the measurement outcomes pq and rs to each other; (iv) they perform $\sigma_{p'q'}$ on $\tilde{A}$ and $\sigma_{r's'}$ on $\tilde{B}$, respectively, determined by Eq. (7). This protocol is a one-round protocol of type (d) in Fig. 1. A simple calculation yields that U_c is implemented on the initial state by this protocol.

Let $K(U_c)$ be the entanglement entropy of $|\Psi_{U_c}\rangle^{\tilde{A}\tilde{B}ab}$, i.e., $K(U_c) \coloneqq S(\Psi_{U_c}^{\tilde{A}a})$. Consider the following n-shot protocol, by which the entanglement cost $K(U_c)$ is achievable: (i) Alice and Bob initially share approximately $nK(U_c)$ copies of Bell pairs, which is transformed to n copies of $|\Psi_{U_c}\rangle^{\tilde{A}\tilde{B}ab}$ in one round by entanglement dilution; (ii) they perform the single-shot protocol presented above on each pair. The only source of error of this protocol is in the entanglement dilution, which vanishes exponentially in the limit of n to infinity. The entanglement dilution and the single-shot protocol can be jointly performed as a single one-round protocol.

Beigi and König [74] proved that any bipartite unitary gate can be implemented with arbitrary high precision by a one-round protocol of type (d) in Fig. 1. The protocol proposed therein is universal in the sense that it is applicable to any type of unitary gates. The entanglement cost of the protocol, however, diverges if the total error is required to be vanishingly small. This is in contrast to the protocol presented above, which is specific to the generalized Clifford gates. Gonzales and Chitambar [75] propose a protocol for two-qubit unitary gates that achieves an exponential improvement in the entanglement cost compared to that of Ref. [74]. An exact protocol for a class of two-qubit unitary gates is also reported in Ref. [75], which requires two Bell pairs as a resource.

Conclusion.—We considered the implementation of a bipartite unitary gate by LOCC, assisted by shared entanglement. We proved that a three-round protocol outperforms all two-round LOCC protocols in reducing the entanglement cost for a class of two-qubit unitary gates. Thereby, we provided a first example of a distributed information processing task with nonzero error tolerance for which there exists a clear trade-off relation between the costs of shared entanglement and the round complexity of a protocol. It should be noted that the same trade-off relation holds even if the additional resource of entanglement is available as a catalyst [59]. We also provided an example of unitary gates for which there is no such trade-off. It remains open whether $E_2(U_\theta) \geq 1$ holds when we adopt the condition $\lim_{n\to\infty}\epsilon^*(\mathcal{M}_n) = 0$ instead of condition (2), which is required for applying Theorem 15 in Ref. [76] (see Sec. IX in Ref. [59] for details).

The task represented by Eq. (5) can be regarded as a generalization of quantum state merging [60,77] to a bidirectional communication task (see also [78]). Our result shows that, for this task, protocols with the minimum required communication rounds (i.e., two rounds) cannot necessarily achieve the minimum entanglement cost because of its causal structural complexity. This is in contrast to quantum state merging, in which the optimal entanglement cost is asymptotically achievable by one-way communication.

It was proved in Ref. [57] that a protocol with higher round complexity is more efficient in extracting entanglement from a tripartite quantum state. To compare their result with the trade-off relation presented in this Letter is left as a future work.

This work is supported by the JSPS KAKENHI (Grants No. 15H01677, No. 16H01050, No. 17H01694, No. 18H04286, No. 18J01329, and No. 18K13467) and by the Q-LEAP project of MEXT, Japan.

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