# Quantum State Smoothing for Linear Gaussian Systems

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Quantum state smoothing is a technique for assigning a valid quantum state to a partially observed dynamical system, using measurement records both prior and posterior to an estimation time. We show that the technique is greatly simplified for linear Gaussian quantum systems, which have wide physical applicability. We derive a closed-form solution for the quantum smoothed state, which is more pure than the standard filtered state, while still being described by a physical quantum state, unlike other proposed quantum smoothing techniques. We apply the theory to an on-threshold optical parametric oscillator, exploring optimal conditions for purity recovery by smoothing. The role of quantum efficiency is elucidated, in both low and high efficiency limits.

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Smoothing and filtering are techniques in classical estimation of dynamical systems to calculate probability density functions (PDFs) of quantities of interest at some time $t$, based on available data from noisy observation of such quantities in time. In filtering, the observed data up to time $t$ is used in the calculation. In smoothing, the observed data both before (past) and after (future) $t$ can be used. For dynamical systems where real-time estimation of the unknown parameters is not required, smoothing almost always gives more accurate estimates than filtering. In the quantum realm, numerous formalisms have been introduced which use past and future information [1–7]. Many of these ideas have been applied, theoretically and experimentally, to the estimation of unknown classical parameters affecting quantum systems [8–14], or of hidden results of quantum measurements [15–20]. The optimal improvement obtained by using future information in these applications comes from using classical Bayesian smoothing to obtain the PDF of the variables of interest.

Despite such applications of smoothing to quantum parameter estimation, a quantum analogue for the classical smoothed state (i.e., the PDF) was still missing. As quantum operators for a system at time $t$ do not commute with operators representing the results of later measurements on that system [21], a naïve generalization of the classical smoothing technique would not result in a proper quantum state [4,5,7]. As elucidated by Tsang [4] (see also the Supplemental Material of Ref. [5]), such a procedure would result in a "state" that gives the (typically anomalous) weak value [2] as its expectation value for any observable. Thus, we will refer to this type of smoothed state for a quantum system as the smoothed weak-value (SWV) state. In contrast to this, Guevara and Wiseman [22] recently proposed a theory of quantum state smoothing which also generalizes classical smoothing but which gives a proper smoothed quantum state, i.e., both Hermitian and positive semidefinite.

The quantum state smoothing theory of Ref. [22] considers an open quantum system coupled to two baths (see Ref. [12] for a similar idea). An observer, Alice, monitors one bath and thereby obtains an "observed" measurement record $\mathbf{O}$. Another observer, Bob (who is hidden from Alice), monitors the remaining bath, unobserved by Alice, and thereby obtains an "unobserved" record $\mathbf{U}$. If Alice knew $\overleftarrow{\mathbf{U}}$ as well as $\overleftarrow{\mathbf{O}}$ (the back arrows indicating records in the past), she would have maximum knowledge of the quantum system, i.e., the "true" state $\rho_{\overleftarrow{\mathbf{O}},\overleftarrow{\mathbf{U}}}$ at that time. Thus, Alice's filtered and smoothed states can be defined in the same form of a conditioned state,

$$\rho_C = \sum_{\overleftarrow{\mathbf{U}}} \wp_C(\overleftarrow{\mathbf{U}})\rho_{\overleftarrow{\mathbf{O}},\overleftarrow{\mathbf{U}}}, \tag{1}$$

where the summation is over all possible records unobserved by Alice. For filtering ($\rho_C = \rho_F$), the PDF of unobserved records is $\wp_C(\overleftarrow{\mathbf{U}}) = \wp(\overleftarrow{\mathbf{U}}|\overleftarrow{\mathbf{O}})$ conditioned on her past record $\overleftarrow{\mathbf{O}}$. For smoothing ($\rho_C = \rho_S$), one has $\wp_C(\overleftarrow{\mathbf{U}}) = \wp(\overleftarrow{\mathbf{U}}|\overleftrightarrow{\mathbf{O}})$ conditioned on Alice's past-future record $\overleftrightarrow{\mathbf{O}}$. By construction, Eq. (1) guarantees the positivity of the smoothed quantum state.

In this Letter we present the theory of quantum state smoothing for linear Gaussian quantum (LGQ) systems. This can be applied to a large number of physical systems, e.g., multimodal light fields [23,24], optical, and optomechanical systems [13,20,21,25–35], atomic ensembles [36–38], and Bose-Einstein condensates [39]. Because of the nice properties of LGQ systems, we are able to obtain closed-form solutions for the smoothed LGQ state. This makes them much easier to study even than the two-level system originally considered in Ref. [22], as there is no need to generate numerically the numerous unobserved records appearing in the summation of Eq. (1). LQG smoothing only requires solving a few additional equations compared to classical smoothing for linear Gaussian (LG) systems. The simplicity of our theory will enable easy application to numerous physical systems, and also allows analytical treatment of various measurement efficiency regimes. We give such a treatment here for an optical parametric oscillator (OPO) on threshold [21,25]. As expected, our smoothed quantum state has higher purity than the usual filtered quantum state, while the SWV state is often unphysical, with purity larger than one.

We begin by reviewing the necessary theoretical background of classical LG systems and LGQ systems. We then develop quantum state smoothing for LGQ systems and obtain analytic results in different limits. Finally, we apply LGQ smoothing to the on-threshold OPO.

*LG systems and classical smoothing.*—Consider a classical dynamical system described by a vector of $M$ parameters $\mathbf{x} = \{x_1, x_2, \ldots, x_M\}^\top$. Here $\top$ denotes transpose. This system is regarded as an LG system if and only if it satisfies three conditions [21,40–45]. First, its evolution can be described by a linear Langevin equation,

$$d\mathbf{x} = A\mathbf{x}dt + Ed\mathbf{v}_p. \tag{2}$$

Here $A$ (the drift matrix) and $E$ are constant matrices and $d\mathbf{v}_p$ is the process noise, i.e., a vector of independent Wiener increments satisfying

$$\mathbf{E}[d\mathbf{v}_p] = \mathbf{0}, \qquad d\mathbf{v}_p(d\mathbf{v}_p)^\top = I dt. \tag{3}$$

Here $\mathbf{E}[\ldots]$ represents an ensemble average, and $I$ is the $M \times M$ identity matrix. Second, knowledge about the system is conditioned on a measurement record $\mathbf{y}$ that is linear in $\mathbf{x}$,

$$\mathbf{y}dt = C\mathbf{x}dt + d\mathbf{v}_m, \tag{4}$$

where $C$ is a constant matrix and the measurement noise $d\mathbf{v}_m$ is a vector of independent Wiener increments satisfying similar conditions to Eq. (3). It is possible for the process noise and the measurement noise to be correlated, e.g., from measurement backaction, which is described by a nonzero cross-correlation matrix $\Gamma$, computed from $\Gamma^\top dt = Ed\mathbf{v}_p(d\mathbf{v}_m)^\top$. The third condition is that the initial state of the system [i.e., the initial PDF of $\mathbf{x}$, denoted as $\wp(\mathbf{x})|_{t=0}$] is Gaussian; then the linearity conditions (first and second) guarantee the conditioned state will remain Gaussian:

$$\wp_C(\mathbf{x}) = g(\mathbf{x}; \langle\mathbf{x}\rangle_C, V_C), \tag{5}$$

which is fully described by its mean $\langle\mathbf{x}\rangle_C$ and variance (strictly, covariance matrix) $V_C \equiv \langle\mathbf{x}\mathbf{x}^\top\rangle_C - \langle\mathbf{x}\rangle_C\langle\mathbf{x}\rangle_C^\top$, throughout the entire evolution.

If the above criteria are met, one can compute a filtered LG state conditioned only on the past record (before the estimation time $t$). The filtered mean and variance are given by

$$d\langle\mathbf{x}\rangle_F = A\langle\mathbf{x}\rangle_F dt + \mathcal{K}^+[V_F]d\mathbf{w}_F, \tag{6}$$

$$\frac{dV_F}{dt} = AV_F + V_F A^\top + D - \mathcal{K}^+[V_F]\mathcal{K}^+[V_F]^\top, \tag{7}$$

where $d\mathbf{w}_F \equiv \mathbf{y}dt - C\langle\mathbf{x}\rangle_F dt$ is a vector of innovations, $D = EE^\top$ is the diffusion matrix, and we have defined a "kick" matrix, a function of $V$, via $\mathcal{K}^\pm[V] \equiv VC^\top \pm \Gamma^\top$. Initial conditions for these filtering equations are the mean and variance of the initial Gaussian state.

To solve for a smoothed LG state, one needs to include conditioning on the future record, which can be obtained from the retrofiltering equations

$$-d\langle\mathbf{x}\rangle_R = -A\langle\mathbf{x}\rangle_R dt + \mathcal{K}^-[V_R]d\mathbf{w}_R, \tag{8}$$

$$-\frac{dV_R}{dt} = -AV_R - V_R A^\top + D - \mathcal{K}^-[V_R]\mathcal{K}^-[V_R]^\top, \tag{9}$$

where $\mathcal{K}^-[V]$ was defined above and $d\mathbf{w}_R \equiv \mathbf{y}dt - C\langle\mathbf{x}\rangle_R dt$. As the leading negative signs suggest, these equations are evolved backward in time, from a final condition at $t = T$. This is typically taken to be an uninformative PDF. Combining the filtered and retrofiltered solutions Eqs. (6)–(9), one obtains a smoothed LG state conditioned on the entire measurement record [40–44],

$$\langle\mathbf{x}\rangle_S = V_S(V_F^{-1}\langle\mathbf{x}\rangle_F + V_R^{-1}\langle\mathbf{x}\rangle_R), \tag{10}$$

$$V_S = (V_F^{-1} + V_R^{-1})^{-1}. \tag{11}$$

*LGQ systems.*—For a quantum system analogous to the classical LG one, the system's observables require unbounded spectrums, represented by $N$ bosonic modes. We denote such a system by a vector of $M = 2N$ observable operators $\hat{\mathbf{x}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^\top$, where $\hat{q}_k$ and $\hat{p}_k$ are canonically conjugate position and momentum operators for the $k$th mode, obeying the commutation relation $[\hat{q}_k, \hat{p}_l] = i\hbar\delta_{kl}$. The system is called an LGQ system if its dynamical and measurement equations are isomorphic to those of a classical LG system [21,25,46–49]. For quantum systems there are additional constraints on the system's dynamics [21]. For example, the initial state must satisfy the Schrödinger-Heisenberg uncertainty relation, $V + i\hbar\Sigma/2 \geq 0$. Here $\Sigma_{kl} = -i[\hat{x}_k, \hat{x}_l]$ is the symplectic matrix and $V$ is the covariance matrix $V_{kl} = \langle\hat{x}_k\hat{x}_l + \hat{x}_l\hat{x}_k\rangle/2 - \langle\hat{x}_k\rangle\langle\hat{x}_l\rangle$, for $\hat{x}_k$ being an element of $\hat{\mathbf{x}}$ and $\langle\cdot\rangle$ being the usual quantum expectation value. These let us represent the quantum state of an LGQ system by its Gaussian Wigner function [21] defined as $W(\check{\mathbf{x}}) = g[\check{\mathbf{x}}; \langle\hat{\mathbf{x}}\rangle, V]$, using dummy variable $\check{\mathbf{x}}$.

*Quantum state smoothing for LGQ systems.*—We now apply the quantum state smoothing technique [22] to LGQ systems. Following the Alice-Bob protocol introduced in Eq. (1), a true state of the LGQ system, denoted by the mean $\langle\hat{\mathbf{x}}\rangle_T$ and a variance $V_T$, is obtained given both $\overleftarrow{\mathbf{O}}$ and $\overleftarrow{\mathbf{U}}$ records. That is, the filtering equations (6)–(7) apply, but conditioned both on Alice's observed record [of the form similar to Eq. (4)]

$$\mathbf{y}_o dt = C_o\langle\hat{\mathbf{x}}\rangle_T dt + d\mathbf{w}_o, \tag{12}$$

and on Bob's record, unobserved by Alice, $\mathbf{y}_u dt = C_u\langle\hat{\mathbf{x}}\rangle_T dt + d\mathbf{w}_u$, with independent Wiener noises. The equations for the true state are

$$d\langle\hat{\mathbf{x}}\rangle_T = A\langle\hat{\mathbf{x}}\rangle_T dt + \mathcal{K}_o^+[V_T]d\mathbf{w}_o + \mathcal{K}_u^+[V_T]d\mathbf{w}_u, \tag{13}$$

$$\frac{dV_T}{dt} = AV_T + V_T A^\top + D - \mathcal{K}_o^+[V_T]\mathcal{K}_o^+[V_T]^\top - \mathcal{K}_u^+[V_T]\mathcal{K}_u^+[V_T]^\top, \tag{14}$$

where $\mathcal{K}_r^\pm[V] = VC_r^\top + \Gamma_r^\top$, for $r \in \{o, u\}$.

Since Alice has no access to Bob's record, her conditioned state (filtered or smoothed) is obtained by summing over all possible true states of the system, with probability weights conditional on Alice's observed records ($\overleftarrow{\mathbf{O}}$ or $\overleftrightarrow{\mathbf{O}}$, respectively) as in Eq. (1). For LGQ systems, the state depends on $\overleftarrow{\mathbf{U}}$ only via the mean, Eq. (13). Therefore, we can replace the (symbolic) sum in Eq. (1) by an integral:

$$\rho_C = \int \wp_C(\langle\hat{\mathbf{x}}\rangle_T)\rho_T(\langle\hat{\mathbf{x}}\rangle_T)\mathrm{d}\langle\hat{\mathbf{x}}\rangle_T. \tag{15}$$

Now let us define a "haloed" variable $\mathring{\mathbf{x}} = \langle\hat{\mathbf{x}}\rangle_T$ for notational simplicity. We can replace the conditional state $\rho_C$ and true state $\rho_T$ with their Wigner functions. The latter is Gaussian: $g(\check{\mathbf{x}}; \mathring{\mathbf{x}}, V_T)$. The integral in Eq. (15) convolves this with the PDF $\wp_C(\mathring{\mathbf{x}})$ conditioned on the observed records. This PDF is a conditioned (filtered or smoothed) LG distribution for $\mathring{\mathbf{x}}$, based on the observed data, $\wp_C(\mathring{\mathbf{x}}) = g(\mathring{\mathbf{x}}; \langle\mathring{\mathbf{x}}\rangle_C, \mathring{V}_C)$, where $\mathring{V}_C$ is the conditional variance for the variable $\mathring{\mathbf{x}}$ [50]. As both functions inside the integral Eq. (15) are Gaussian, the Wigner function for $\rho_C$ is also Gaussian:

$$g(\check{\mathbf{x}}; \langle\hat{\mathbf{x}}\rangle_C, V_C) = \int g(\mathring{\mathbf{x}}; \langle\mathring{\mathbf{x}}\rangle_C, \mathring{V}_C) g(\check{\mathbf{x}}; \mathring{\mathbf{x}}, V_T) d\mathring{\mathbf{x}}. \tag{16}$$

By elementary properties of convolutions, we get the conditioned mean $\langle\hat{\mathbf{x}}\rangle_C = \langle\mathring{\mathbf{x}}\rangle_C$ and the conditioned variance $V_C = \mathring{V}_C + V_T$. This will allow us to solve for the filtered and smoothed quantum states for LGQ systems.

Now, all that remains is to apply classical LG estimation theory (filtering or smoothing) to determine $\langle\mathring{\mathbf{x}}\rangle_C$ and $\mathring{V}_C$. We first obtain [50] filtering equations for $\mathring{\mathbf{x}}$, using the past observed record Eq. (12),

$$d\langle\mathring{\mathbf{x}}\rangle_F = A\langle\mathring{\mathbf{x}}\rangle_F dt + \mathcal{K}_o^+[\mathring{V}_F + V_T]d\mathring{\mathbf{w}}_F, \tag{17}$$

$$\frac{d\mathring{V}_F}{\mathrm{d}t} = A\mathring{V}_F + \mathring{V}_F A^\top + \mathring{D} - \mathcal{K}_o^+[\mathring{V}_F + V_T]\mathcal{K}_o^+[\mathring{V}_F + V_T]^\top, \tag{18}$$

where we have defined $\mathring{D} = \sum_{\mathrm{r}\in\{\mathrm{o,u}\}}\mathcal{K}_\mathrm{r}^+[V_T]\mathcal{K}_\mathrm{r}^+[V_T]^\top$, and $d\mathring{\mathbf{w}}_F = \mathbf{y}_o dt - C_o\langle\mathring{\mathbf{x}}\rangle_F dt$. We also show in Ref. [50] that this haloed filtered variance is related to the variance of the usual quantum filtered state $V_F$ (computed without invoking the unobserved record) via $V_F = \mathring{V}_F + V_T$ with the same mean $\langle\hat{\mathbf{x}}\rangle_F = \langle\mathring{\mathbf{x}}\rangle_F$, consistent with the convolution (16). For the retrofiltering equations for $\mathring{\mathbf{x}}$, using the future record, we have

$$-d\langle\mathring{\mathbf{x}}\rangle_R = -A\langle\mathring{\mathbf{x}}\rangle_R dt + \mathcal{K}_o^-[\mathring{V}_R - V_T]d\mathring{\mathbf{w}}_R, \tag{19}$$

$$-\frac{d\mathring{V}_R}{dt} = -A\mathring{V}_R - \mathring{V}_R A^\top + \mathring{D} - \mathcal{K}_o^-[\mathring{V}_R - V_T]\mathcal{K}_o^-[\mathring{V}_R - V_T], \tag{20}$$

which lead to a similar variance relation $V_R = \mathring{V}_R - V_T$ [50]. However, the minus sign in the $\mathring{V}_R$ relation indicates that the convolution (16) does not apply for retrofiltering, which propagates in the backward direction in time.

We then combine the haloed filtering and retrofiltering equations, as in Eqs. (10) and (11), to obtain the haloed smoothing equations, and using Eq. (16), we arrive at the LGQ state smoothing equations

$$\langle\hat{\mathbf{x}}\rangle_S = (V_S - V_T)[(V_F - V_T)^{-1}\langle\hat{\mathbf{x}}\rangle_F + (V_R + V_T)^{-1}\langle\hat{\mathbf{x}}\rangle_R], \tag{21}$$

$$V_S = [(V_F - V_T)^{-1} + (V_R + V_T)^{-1}]^{-1} + V_T, \tag{22}$$

as the main result of this Letter. In the classical limit, where there is no uncertainty relation for $V_T$ and we can let $V_T \to 0$, these reproduce classical LG smoothing, Eqs. (10) and (11), as expected.

The advantages LGQ state smoothing offers over filtering are readily seen in Fig. 1, where we note that the purity for a Gaussian state is defined as $P = (\hbar/2)\sqrt{|V|^{-1}}$ [21] for a variance $V$. The smoothed state has a smaller variance (higher purity) than the filtered state, but has a larger variance than a pure state (purity less than unity). In

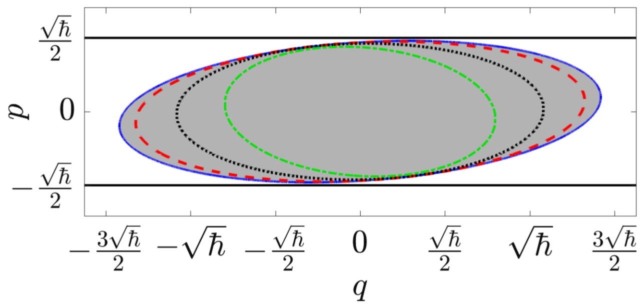

FIG. 1. Various long-time states of the on-threshold OPO system in Eq. (24), represented by their 1-SD Wigner function contours in phase space, centered at the origin. The homodyne angles used by Alice and Bob ($\theta_o$, $\theta_u$) are at the black dot in Fig. 2. The unconditional state (solid black) shows infinite and finite variances in $q$ and $p$, respectively, as a result of the damping and squeezing. Alice's filtered and smoothed states, are blue (filled gray) and dashed-red ellipses, respectively. The dotted-black ellipse shows the (pure) true state, conditioned on both Alice's and Bob's results, while the dot-dashed green ellipse shows the SWV state.

contrast, the SWV state for the same system [i.e., using Eqs. (10) and (11)] is unphysical (its ellipse is smaller than that of a pure state).

Now that we have the closed-form expression for the smoothed LGQ state, we can investigate, in the steady state, some interesting limits in Alice's measurement efficiency $\eta_o$, the fraction of the system output which is observed by Alice.

If, as in the OPO system we will consider later, the unconditioned ($\eta_o = 0$) variance diverges, then Alice's conditioned (filtered and retrofiltered) variances, if finite, must grow as $\eta_o \to 0$. From Eqs. (21) and (22), when $V_F$ and $V_R$ are large, compared to $V_T$, the smoothed LGQ state reduces to the SWV state Eqs. (10) and (11). The SWV state has the same form as classical smoothed states, which often have the same scaling as filtered states, but with a multiplicative constant improvement [8,14,51]. Consequently, in the limit $\eta_o \to 0$, we expect $P_{\rm SWV} = P_S \propto P_F$ as functions of $\eta_o$.

In the opposite limit, $\eta_o \to 1$, we analytically show [50] that the relative purity recovery (RPR),

$$\mathcal{R} = \frac{P_S - P_F}{1 - P_F}, \qquad (23)$$

a measure of how much the purity is recovered from smoothing over filtering relative to the maximum recovery possible, usually scales with the unobserved efficiency. That is, $\mathcal{R} \propto \eta_u \equiv 1 - \eta_o$.

*Example of the on-threshold OPO system.*—We now apply quantum state smoothing to the on-threshold OPO [21,25], an LGQ system with $N = 1$ described by the master equation

$$\hbar\dot{\rho} = -i[(\hat{q}\,\hat{p} + \hat{p}\,\hat{q})/2, \rho] + \mathcal{D}[\hat{q} + i\hat{p}]\rho. \qquad (24)$$

The first term defines a Hamiltonian giving squeezing along the $p$ quadrature, while the second term describes the oscillator damping. Here, the drift and diffusion matrices are $A = \mathrm{diag}(0, -2)$ and $D = \hbar I$. Let us assume that Alice observes the damping channel via homodyne detection. Therefore, the matrix $C_o$ in Eq. (12) is $C_o = 2\sqrt{\eta_o/\hbar}(\cos\theta_o, \sin\theta_o)$, where $\theta_o$ is the homodyne phase [21,25]. For simplicity, we assume Bob also performs a homodyne measurement, with a different phase $\theta_u$, so that $C_u = 2\sqrt{\eta_u/\hbar}(\cos\theta_u, \sin\theta_u)$. The measurement backactions are described by matrices $\Gamma_r = -\hbar C_r/2$, for $r \in \{o, u\}$.

We now solve for filtered and smoothed states for the OPO in steady state. We are particularly interested in the RPR (23) of smoothing over filtering, and in the combinations of homodyne phases that result in the largest RPR. The RPR is always positive (see Fig. 2), meaning that the smoothed quantum state always has higher purity than the corresponding filtered one. If Alice's phase $\theta_o$ is fixed, one might guess that Bob's phase giving the best purity improvement should be the same, $\theta_u = \theta_o$. However, that is not at all true (see Fig. 2). The optimal $\theta_u^{\rm opt}$ is not a trivial function of $\theta_o$. Rather, $\theta_u^{\rm opt} \approx 0$, i.e., Bob should measure

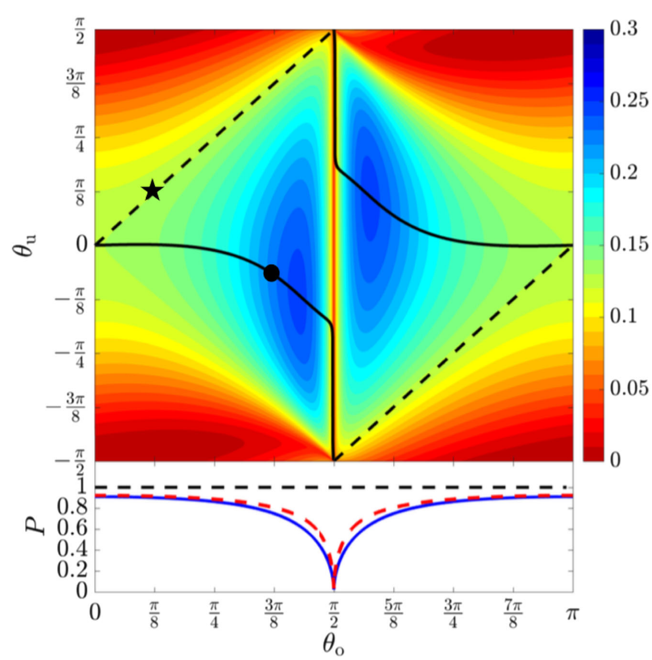

FIG. 2. (Top) Contour plots of the RPR, Eq. (23), for the OPO system for different values of observed and unobserved homodyne phases using $\eta_o = 0.5$. The dashed line represents $\theta_o = \theta_u$ and the solid line is the optimal $\theta_u$ (that giving the highest RPR for each value of $\theta_o$). The circle and the star relate to Figs. 1 and 3, respectively. (Bottom) Purity for the OPO's filtered (solid blue) and smoothed (dashed red) states, choosing the optimal $\theta_u^{\rm opt}$ for each $\theta_o$.

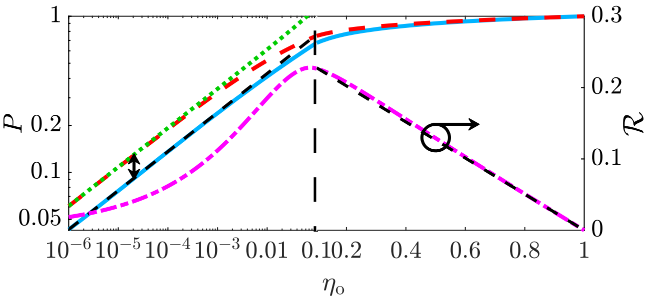


FIG. 3. Purities, and the RPR, Eq. (23), at the starred point in Fig. 2, for the full range of Alice's measurement efficiency $\eta_o$, with the lower efficiencies plotted on a log scale and the higher efficiencies on a linear scale, where the dashed vertical line at $\eta_o = 0.1$ indicates the split. On both sides, we plot the purities of the filtered (solid blue), smoothed (dashed red), and the SWV (dotted green) states, all on a log scale (left-hand-side axis); and the RPR ($\mathcal{R}$) (dot-dashed magenta), on a linear scale (the right-hand-side axis). For $\eta_o \to 0$, $P_F$ matches the simple analytic expression [50] $\sqrt{2|\cos\theta_o|}\eta_o^{1/4}$ (dashed black on left), and smoothing gives a factor of $\sqrt{2}$ improvement [50], as shown by the small $\updownarrow$ symbol. For $\eta_o \to 1$, the RPR is $\propto (1-\eta_o)$ (dashed black on right).

the $q$ quadrature, which is presumably related to the fact that, without measurement in, the variance in $q$ diverges.

We then examine, in Fig. 3, the low and high efficiency limits for the OPO system at the starred point in Fig. 2. As predicted earlier, in the limit $\eta_o \to 0$ (left), the purities of the smoothed LGQ state and the SWV state are almost identical, and have a constant factor of improvement over that for filtering, as can be verified analytically [50]. However, $P_{\rm SWV}$ begins to separate from $P_S$ when the purities are no longer small, as the former proceeds to have purity greater than 1 when $\eta_o > 0.06$. In the limit $\eta_o \to 1$ (right), we see that the RPR has linear scaling in $\eta_u = 1 - \eta_o$, as expected. The approximation holds surprisingly well even when $\eta_u$ is not small.

To conclude, we have developed the theory of quantum state smoothing, which gives valid smoothed quantum states, for LGQ systems, a class of systems with wide physical applicability. By utilizing the Gaussian properties, we obtained closed-form smoothing solutions that do not require simulations of ensembles of unobserved measurement records and corresponding true states. This enabled us to perform detailed analysis of the smoothed quantum state for various measurement regimes. A question for future work is to understand the (numerically found) optimal strategy for greatest improvement in the purity. There are also interesting questions regarding how the smoothed LGQ variance (22) would react to inserting an invalid true state [i.e., one that does not solve Eq. (14)]. Finally, we could compare the smoothed LGQ state to other state estimation techniques using future information, such as the most likely path approach in Refs. [6,52].

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