Flemish Strings of Magnetic Solitons and a Nonthermal Fixed Point in a One-Dimensional Antiferromagnetic Spin-1 Bose Gas

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Thermalization in a quenched one-dimensional antiferromagnetic spin-1 Bose gas is shown to proceed via a nonthermal fixed point through annihilation of Flemish-string bound states of magnetic solitons. A possible experimental situation is discussed.

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Introduction.—Ultracold atomic gases offer an ideal playground for studying universal nonequilibrium dynamics due to their high controllability and isolation from the environment [1–3]. Indeed, fundamental aspects of universal thermalization dynamics near critical points at equilibrium phase transitions have been studied in terms of the Kibble-Zurek mechanism [4–9] and dynamical critical phenomena [10]. Even far from critical points, isolated systems are found to exhibit universal thermalization in the decay of turbulence [11,12] and coarsening dynamics [13–31]. However, unlike the dynamics near critical points [32,33], a unified framework for understanding the universal behavior is yet to be established.

Recently, a nonthermal fixed point (NTFP) [11,12,34,35] has attracted growing interest as a universal thermalization scenario in isolated quantum systems. It is expected to provide a unified framework of different nonequilibrium phenomena in diverse systems ranging from cosmology to cold atoms [34,35]. As illustrated in Fig. 1(a), a system with a strongly quenched initial state undergoes relaxation via a NTFP at which the scale-invariant thermalization dynamics emerges transiently. The NTFP is characterized by a universal scaling of an equal-time correlation function $C(\boldsymbol{x}, t)$ as $C(\boldsymbol{x}, t) = s^{-\gamma} C(\boldsymbol{x} s^{\beta}, ts)$ with an arbitrary parameter s and scaling exponents β and γ. Setting $s = t^{-1}$, we obtain the dynamical scaling

$$C(\boldsymbol{x}, t) = t^{\gamma} f(\boldsymbol{x} t^{-\beta}), \qquad (1)$$

where $f(\boldsymbol{y}) = C(\boldsymbol{y}, 1)$ is a universal function. After passing through the NTFP, the system relaxes toward a thermalized or stationary state. This scenario has been studied theoretically in two- and three-dimensional Bose gases [36–42] and a relativistic $O(N)$ model [43–45] in the context of wave turbulence [46–48] and coarsening dynamics [49,50]. Here weakly interacting waves or topological objects such as vortices and domain walls play essential roles in thermalization processes. Very recently, the universal scaling in Eq. (1) in momentum space was observed in one-dimensional (1D) ultracold atomic gases [51,52]. However, it remains to be fully understood under what conditions a system shows universal behavior especially in an experimentally controllable manner. For example, a weak quench in Fig. 1(a) generates a small number of elementary excitations and may become thermalized without passing through a NTFP.

In this Letter, we theoretically study quench dynamics in a 1D spin-1 Bose gas with an antiferromagnetic (AFM) interaction, unveiling universal thermalization dynamics through a NTFP caused by exotic bound states of twisted magnetic solitons, which we refer to as Flemish strings [see Figs. 1(b) and 1(c)]. Since the spin-exchange interaction is AFM, the relaxation dynamics after the quench in Fig. 2(a)

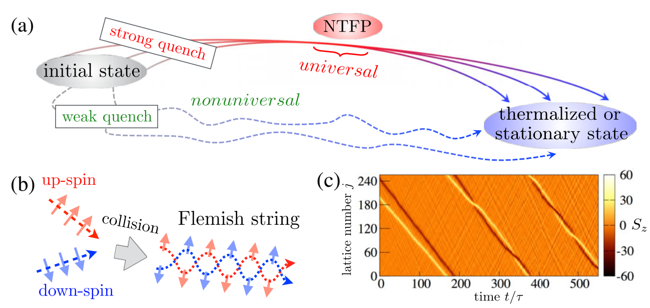

FIG. 1. (a) Schematic illustration of a NTFP. An isolated strongly quenched quantum system undergoes thermalization via a NTFP, where the universal scaling relation (1) emerges en route to a thermalized or stationary state. However, a weakly quenched system may undergo nonuniversal thermalization. (b) Schematic illustration of formation of a Flemish string. A collision between two magnetic solitons with opposite signs of magnetization creates a Flemish string, which has a twisted magnetic structure. The solid (dashed) arrows show the direction of magnetization (trajectories of magnetization). (c) Spatiotemporal distribution of $\hat{S}_{z,j}$ showing formation of a Flemish string, which is obtained from the multicomponent Gross-Pitaevskii equation for the Hamiltonian in Eq. (2) subject to the periodic boundary condition.

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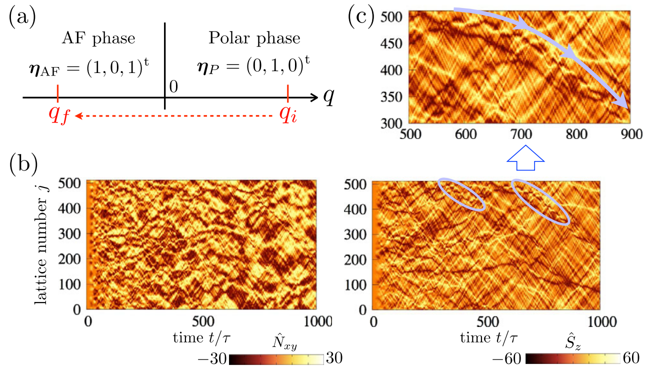

FIG. 2. (a) Quench protocol. A spin-1 antiferromagnetic Bose gas with a quadratic Zeeman term has two mean-field ground states: AFM and polar, where $\boldsymbol{\eta}_{\rm AFM}$ and $\boldsymbol{\eta}_P$ are the corresponding spinor wave functions. We suddenly quench the strength q of the quadratic Zeeman term from q_i to q_f. (b) Spatiotemporal distributions of nematicity $\hat{N}_{xy,j}$ (left) and spin $\hat{S}_{z,j}$ (right) calculated from a single trajectory in the TWA with $U_2/U_0 = 0.050$, $\Gamma = 2$, and $\tau = 4\hbar/J$. The left panel shows the growth of nematic domains. The bright and dark lines in the right panel show positively and negatively charged magnetic solitons, respectively. The number of solitons decreases in time [see also Fig. 4(c)] as Flemish strings enclosed by solid circles are annihilated. (c) Enlarged figure of $\hat{S}_{z,j}$ in (b). The arrow shows the direction of motion of a Flemish string.

is characterized by nematicity [53–56]. We find that a nematic correlation function transiently exhibits the dynamical scaling (1) with $\beta \simeq 0.32$ and $\gamma \simeq 0.11$. The underlying physics of this universal thermalization through a NTFP is highly nontrivial cooperative soliton dynamics, where two oppositely charged magnetic solitons, which are stable individually [57–60], undergo pair annhilation by forming a Flemish string. While there are a number of studies for quench dynamics of an AFM spinor Bose gas [26,61–71], universal thermalization through Flemish strings has never been reported.

Our results reported here present the first clear evidence of the soliton-induced NTFP. In Ref. [72], a NTFP in a 1D Bose gas is theoretically discussed, but the dynamical scaling (1), which is regarded as the hallmark of a NTFP [34], is not confirmed. In Ref. [52], a relation between the observed universal thermalization dynamics and a random gray soliton model in a 1D repulsive Bose gas [72] is discussed. However, the observed exponent β deviates by 50% from the exponent obtained by assuming the theoretical model, and the other key exponent $\alpha = \beta + \gamma$ is not explained from the perspective of solitons. Thus, the relation between the NTFP and solitons has remained unclear.

Furthermore, we address an unsolved issue raised in Refs. [73,74], where thermalization dynamics after the same quench as ours in Fig. 2(a) was experimentally studied in trapped 1D AFM Bose gases of ^{23}Na. The experiments investigate the observed dynamics in terms of coarsening [49,50], but universal aspects of thermalization have remained an unsolved issue. We argue that the universal thermalization dynamics characterized by the NTFP did not emerge in the experiments of Refs. [73,74] because the quench was so weak that only a few solitons were created. We will show that the NTFP due to Flemish strings should emerge for stronger quench.

Universal thermalization dynamics in a uniform system.—We consider a translation-invariant spin-1 Bose-Hubbard model [75] subject to the periodic boundary condition with the lattice constant a and the number of lattices M. We denote the annihilation (creation) operator of spin-1 bosons with magnetic quantum number m at site j as $\hat{b}_{m,j}$ ($\hat{b}^\dagger_{m,j}$). Then, the Hamiltonian is given by

$$\hat{H}_{\rm BH} = -J\sum_{m,j}(\hat{b}^\dagger_{m,j+1}\hat{b}_{m,j} + {\rm H.c.}) + q\sum_{m,j} m^2 \hat{b}^\dagger_{m,j}\hat{b}_{m,j} + \frac{U_0}{2}\sum_j \hat{\rho}_j(\hat{\rho}_j - 1) + \frac{U_2}{2}\sum_j(\hat{\boldsymbol{S}}_j^2 - 2\hat{\rho}_j), \tag{2}$$

with the hopping parameter J, the quadratic Zeeman coefficient q, the density interaction coefficient U_0, and the spin interaction coefficient U_2. Here, $\hat{\rho}_j \coloneqq \sum_m \hat{b}^\dagger_{m,j}\hat{b}_{m,j}$ is the total particle-number operator and $\hat{S}_{\alpha,j} \coloneqq \sum_{m,n}\hat{b}^\dagger_{m,j}(s_\alpha)_{mn}\hat{b}_{n,j}(\alpha = x, y, z)$ with the spin-1 spin matrices $(s_\alpha)_{mn}$ is the spin-vector operator.

We focus on the AFM interaction ($U_2 > 0$), so that the mean-field ground-state phase is either polar ($q > 0$) or AFM ($q < 0$) [see Fig. 2(a)]. Since both phases are nonmagnetic, the spinor order parameter is a second-rank tensor of the spin matrices [53–55] described by the nematic tensor $\mathcal{N}_{\mu\nu,j} \coloneqq \langle \hat{N}_{\mu\nu,j}\rangle$ with $\hat{N}_{\mu\nu,j} \coloneqq \frac{1}{2}\sum_{l,m,n}\hat{b}^\dagger_{m,j}[(s_\mu)_{mn}(s_\nu)_{nl} + (\mu \leftrightarrow \nu)]\hat{b}_{l,j}$. Here, $\langle\cdots\rangle$ means the average over the state vector $|\psi(t)\rangle$. In the AFM phase, the nematic tensor becomes

$$\mathcal{N}^{\rm AFM}_{\mu\nu} = \frac{N}{2M}\begin{pmatrix} 1-\cos(2\alpha) & -\sin(2\alpha) & 0 \\ -\sin(2\alpha) & 1-\cos(2\alpha) & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{3}$$

where α is the azimuthal angle in spin space and N is the number of condensed bosons. When we quench q from the polar phase to the AFM phase as shown in Fig. 2(a), the system gets highly excited and starts relaxation.

We employ the truncated Wigner approximation (TWA) [76–78] to numerically study the thermalization dynamics of a quenched AFM Bose gas with $J/U_0 = 40$, $M = 512$, and $N = 2\times 10^4$, where the system is in a deep superfluid regime and the TWA should work well [77,78]. The initial state is chosen to be the Bogoliubov vacuum for the polar phase [29], and we suddenly quench q from $q_i = NU_2/M$ to $q_f = (1-\Gamma)NU_2/M$, where Γ controls the strength of the quench.

Figure 2(b) shows the spatiotemporal distributions of nematicity $\hat{N}_{xy,j}$ (left) and magnetization $\hat{S}_{z,j}$ (right) obtained from a single trajectory. The typical domain size of $\hat{N}_{xy,j}$ grows in time, while the number of locally magnetized domains, which are magnetic solitons [57–60], decreases in time [see Fig. 4(c)]. As shown in Fig. 2(c), magnetic solitons can form a Flemish string in which two magnetic solitons form a virtual bound state and evolve in time in a twisted manner. The formation of a Flemish string is essential for thermalization as discussed later.

To investigate the universal scaling in Eq. (1), we calculate the nematic correlation function defined by

$$C_N(j,t) \coloneqq \frac{1}{M}\sum_{k=0}^{M-1}\langle \hat{N}_{xy,j+k}\hat{N}_{xy,k}\rangle(t). \tag{4}$$

Here, we consider the xy component of $\hat{N}_{\mu\nu,j}$ because Eq. (3) shows that the order parameter of the AFM phase can be characterized by α alone. The time evolution of $C_N(j,t)$ shows that the correlation length $L_c(t)$ monotonically increases in time as shown in Fig. 3(a). Here, $L_c(t)$ is determined from $C_N(j=L_c(t)/a,t)=0.5C_N(j=0,t)$. When the abscissa and the ordinate are normalized by $L_c(t)/a$ and $C_N(0,t)$, respectively, the correlation functions at different times collapse into a single universal curve as shown in Fig. 3(b). This dynamical scaling indicates the emergence of a NTFP.

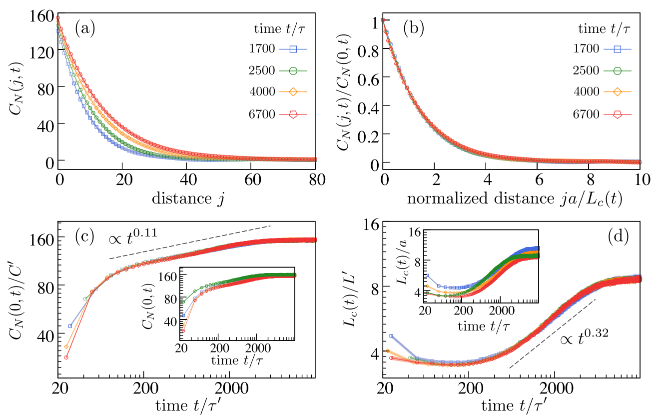

FIG. 3. (a) Time evolution of a correlation function $C_N(j,t)$ for $U_2/U_0=0.05$ and $\Gamma=2$, and (b) the same quantity with the abscissa and the ordinate normalized by $L_c(t)/a$ and $C_N(0,t)$, respectively. In (b), all curves collapse to a single universal curve consistent with the dynamical scaling described by Eq. (1). (c),(d) Time evolution of (c) $C_N(0,t)$ and (d) $L_c(t)$ for four sets of spin interaction and quench strength parameters: $(U_2/U_0,\Gamma)=(0.075,\ 2)$ (green circle), (0.05, 2) (blue square), (0.05, 2.2) (yellow diamond), and (0.05, 2.4) (red pentagon). The insets show the same results for the unnormalized axes. For comparison, all axes in the main panels of (c),(d) are multiplied by different constants L', C', τ' for each run. These results show that the exponents $\beta\simeq 0.32$ and $\gamma\simeq 0.11$ are universal and independent of system's parameters. The power laws transiently appear and finally disappear when the system is close to a stationary state, which is consistent with Fig. 1(a). Color bands show 3σ error bars in the TWA calculation.

The universality class of the NTFP can be classified by the exponents β and γ in Eq. (1), which are derived from power laws $L_c(t)\propto t^{\beta}$ and $C_N(0,t)\propto t^{\gamma}$ [79]. Figures 3(c) and 3(d) show $L_c(t)$ and $C_N(0,t)$ for four sets of parameters U_2 and Γ, from which we obtain $\beta\simeq 0.32$ and $\gamma\simeq 0.11$ by the least-squares fit. These exponents are independent of the parameters and thus universal.

These exponents have never been reported in literature. To compare them with the wave-number representation of previous studies [36–42,51,52], we perform the Fourier transformation of Eq. (1) in 1D systems, obtaining $\bar{C}(k,t)=t^{\alpha}g(kt^{\beta})$ with some function $g(x)$. Then, we find an exponent $\alpha=\gamma+\beta$ with $\alpha\simeq 0.43$, which does not agree with the results in Refs. [36–42,51,52].

The universality of the thermalization dynamics emerges from two different types of soliton solutions: i.e., magnetic solitons and Flemish strings. The magnetic soliton is a solution of the integrable Gardner equation [80], which can be derived from application of singular perturbation theory to the continuous classical model of Eq. (2) [57] (see Supplemental Material [81]). As shown in Fig. 4(a), it has a locally magnetized stable object and separates the two different nematic orders, thereby decreasing the nematic correlation length. The thermalization dynamics would be promoted if a single magnetic soliton could be annihilated. However, it is, in fact, stable and cannot disappear spontaneously.

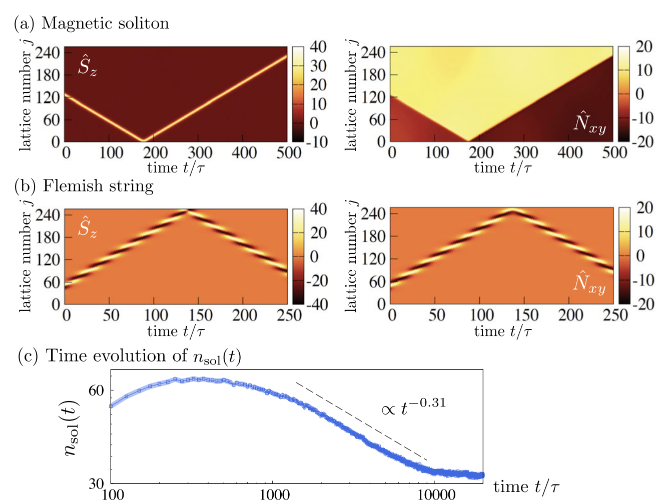

FIG. 4. (a) Magnetic soliton and (b) Flemish string. The left and right panels show the spatiotemporal distributions of $\hat{S}_{z,j}$ and $\hat{N}_{xy,j}$ obtained from the mean-field solution to Eq. (2) subject to the Neumann boundary condition. (a) A magnetic soliton is a locally magnetized stable object and separates the nematic order into two regions. (b) A Flemish string is a bound state of two magnetic solitons which periodically change their relative positions. In contrast to (a), it does not divide the nematic order into two regions having different nematicity. (c) Time evolution of a measure of the magnetic-soliton number $n_{\rm sol}(t)$ for $U_2/U_0=0.05$ and $\Gamma=2$, showing $n_{\rm sol}\propto t^{-0.31}$, which is almost the inverse of $L_c(t)\propto t^{\beta}$ found in Fig. 3(d). Color bands show 3σ error bars in the TWA calculations.

The annihilation of solitons proceeds through that of a Flemish string. This is a soliton solution of the integrable Manakov equation [92–95], which is a continuous classical model of Eq. (2) with $U_2 = 0$ (see Supplemental Material [81]). In contrast to magnetic solitons, it does not separate the nematic order as shown in Fig. 4(b). This string is unstable for $U_2 \neq 0$, so that it eventually vanishes and nematic domains can grow. We have confirmed that Flemish strings are formed through collisions of magnetic solitons, as indicated by the solid curves in Fig. 2(b). The formation of Flemish strings and their subsequent decay constitute the main mechanism for soliton annihilation. The details including the lifetime and the number of Flemish strings are discussed in the Supplemental Material [81].

Another numerical evidence for the nematic domain growth due to solitons is the number of magnetic solitons. We calculate $n_{\rm sol} \coloneqq \sum_{j=0}^{M-1} \langle \theta(\hat{S}_{z,j}^2 - S_{\rm th}^2) \rangle$ with $S_{\rm th} = 0.7\rho_b$ for the bulk particle number ρ_b [96], where $\theta(x)$ is the unit-step function. This should be proportional to $t^{-\beta}$ if $L_c(t)$ obeys the power law t^β. We find $n_{\rm sol} \propto t^{-0.31}$ as shown in Fig. 4(c), which is consistent with $\beta \simeq 0.32$ obtained from $L_c(t)$. This confirms the fact that annihilation of magnetic solitons is essential in the universal thermalization dynamics through the NTFP. This universal behavior appears when many solitons are generated. However, for longer times, the universal relaxation stops and $L_c(t)$ and $C_N(0, t)$ saturate because the number of solitons becomes small.

Analysis of the experimental thermalization dynamics.—Let us examine the experiments [73,74] where the thermalization dynamics in trapped 1D AFM spinor Bose gases was investigated using density correlation functions. One is naturally led to ask whether or not the observed thermalization dynamics is universal. By numerically solving the spin-1 Gross-Pitaevskii (GP) equation with initial noises in the finite-temperature TWA method [77,97–99], we obtain results in good agreement with the experiments [81]. We calculate the correlation function $C_N(z, t)$ with the parameters used in the experiments as shown in Fig. 5(a) [100]. Although $C_N(0, t)$ shows the power-law behavior in a short time (200–350 ms), we find neither the dynamical scaling of $C_N(z, t)$ nor the power law of $L_c(t)$. We therefore conclude that the universal thermalization dynamics does not emerge in the experiments.

The absence of the universal scaling in Eq. (1) is attributed to a weak quench in the experiments, where $q(t)$ is quenched from 2.8 to -4.2 Hz [101]. The energy scale of this change is given in terms of the spin interaction coefficient c_2' of the GP model and the bulk density $\rho_{\rm b,exp}$ by $0.06 c_2' \rho_{\rm b,exp}$, which is too small to excite a sufficient number of magnetic solitons.

To investigate the universal thermalization dynamics, we need a strong quench whose energy scale is of the order of the spin interaction energy $c_2' \rho_{\rm b,exp}$. We numerically quench q from 2.8 to -117 Hz. Figure 5(b) shows the dynamical scaling in Eq. (1) and the power laws with the same exponents as those of the uniform system. Hence, we can confirm the universal thermalization dynamics through the NTFP. These results are consistent with the NTFP scenario in Fig. 1(a), where the strong quench leads to the NTFP but the weak one does not. While the spatial distribution of $\hat{N}_{xy,j}$ has not been observed, $\hat{S}_{z,j}$ can be measured experimentally. Thus, it is possible to estimate the exponent β from $n_{\rm sol}$. The inset of Fig. 5(f) confirms $n_{\rm sol} \propto t^{-\beta}$, which is an experimentally observable signature of the NTFP.

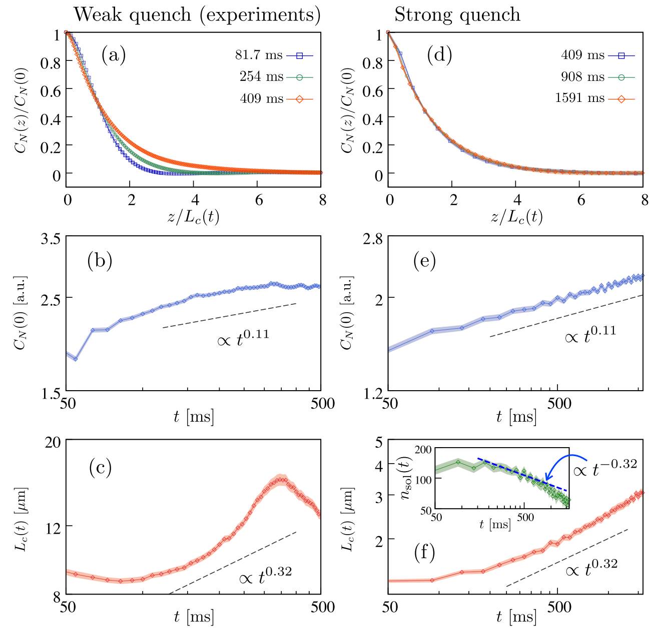

FIG. 5. Numerical results for (a) $C_N(z, t)$, (b) $C_N(0, t)$, and (c) $L_c(t)$ following weak quench with the experimental parameters of Refs. [73,74]. They show neither the universal thermalization dynamics characterized by the dynamical scaling for $C_N(z, t)$ nor the power law for $L_c(t)$. (e)–(f) show the corresponding results for strong quench. We confirm the dynamical scaling and the power laws with the exponents found in the uniform system (see Fig. 3). The inset in (f) shows $n_{\rm sol} \propto t^{-0.32} \propto L_c(t)^{-1}$ which is a measure of the number of magnetic solitons. Color bands show 3σ error bars in the TWA calculations.

Conclusion.—By numerically studying the 1D thermalization in an antiferromagnetic spin-1 Bose gas, we find the universal nematic thermalization dynamics through a NTFP with scaling exponents $\beta \simeq 0.32$ and $\gamma \simeq 0.11$ in Eq. (1). We identify the thermalization mechanism to be annihilation of Flemish strings of magnetic soliton pairs. The universal thermalization dynamics is discussed in the experimental settings in Refs. [73,74]. In 1D ultracold atomic gases, several different types of solitons other than magnetic solitons have experimentally been created [102–110] and their dynamics has extensively been investigated [111–127]. It is interesting to investigate other NTFP universality classes with these solitons.

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[1] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[2] J. Eisert, M. Friesdorf, and C. Gogolin, Nat. Phys. **11**, 124 (2015).
[3] E. Altman, arXiv:1512.00870.
[4] L. E. Sadler, J. M. Higbie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, Nature (London) **443**, 312 (2006).
[5] C. N. Weiler, T. W. Neely, D. R. Scherer, A. S. Bradley, M. J. Davis, and B. P. Anderson, Nature (London) **455**, 948 (2008).
[6] G. Lamporesi, S. Donadello, S. Serafini, F. Dalfovo, and G. Ferrari, Nat. Phys. **9**, 656 (2013).
[7] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, Science **347**, 167 (2015).
[8] S. Brauna, M. Friesdorfc, S. S. Hodgmana, M. Schreibera, J. P. Ronzheimera, A. Rierac, M. d. Reye, I. Bloch, J. Eisertc, and U. Schneidera, Proc. Natl. Acad. Sci. U.S.A. **112**, 3641 (2015).
[9] M. Anquez, B. A. Robbins, H. M Bharath, M. Boguslawski, T. M. Hoang, and M. S. Chapman, Phys. Rev. Lett. **116**, 155301 (2016).
[10] E. Nicklas, M. Karl, M. Höfer, A. Johnson, W. Muessel, H. Strobel, J. Tomkovič, 1 T. Gasenzer, and M. K. Oberthaler, Phys. Rev. Lett. **115**, 245301 (2015).
[11] J. Berges, A. Rothkopf, and J. Schmidt, Phys. Rev. Lett. **101**, 041603 (2008).
[12] J. Berges, K. Boguslavski, S. Schlichting, and R. Venugopalan, Phys. Rev. Lett. **114**, 061601 (2015).
[13] K. Kudo and Y. Kawaguchi, Phys. Rev. A **84**, 043607 (2011).
[14] H. Takeuchi, K. Kasamatsu, M. Tsubota, and M. Nitta, Phys. Rev. Lett. **109**, 245301 (2012).
[15] K. Kudo and Y. Kawaguchi, Phys. Rev. A **88**, 013630 (2013).
[16] J. Hofmann, S. S. Natu, and S. Das Sarma, Phys. Rev. Lett. **113**, 095702 (2014).
[17] S. De, D. L. Campbell, R. M. Price, A. Putra, B. M. Anderson, and I. B. Spielman, Phys. Rev. A **89**, 033631 (2014).
[18] K. Kudo and Y. Kawaguchi, Phys. Rev. A **91**, 053609 (2015).
[19] H. Takeuchi, Y. Mizuno, and K. Dehara, Phys. Rev. A **92**, 043608 (2015).
[20] H. Takeuchi, J. Low Temp. Phys. **183**, 169 (2016).
[21] L. A. Williamson and P. B. Blakie, Phys. Rev. Lett. **116**, 025301 (2016).
[22] I.-K. Liu, R. W. Pattinson, T. P. Billam, S. A. Gardiner, S. L. Cornish, T.-M. Huang, W.-W. Lin, S.-C. Gou, N. G. Parker, and N. P. Proukakis, Phys. Rev. A **93**, 023628 (2016).
[23] L. A. Williamson and P. B. Blakie, Phys. Rev. A **94**, 023608 (2016).
[24] N. T. Phuc, T. Momoi, S. Furukawa, Y. Kawaguchi, T. Fukuhara, and M. Ueda, Phys. Rev. A **95**, 013620 (2017).
[25] A. Bourges and P. B. Blakie, Phys. Rev. A **95**, 023616 (2017).
[26] L. M. Symes and P. B. Blakie, Phys. Rev. A **96**, 013602 (2017).
[27] L. A. Williamson and P. B. Blakie, Phys. Rev. Lett. **119**, 255301 (2017).
[28] N. Shitara, S. Bir, and P. B. Blakie, New J. Phys. **19**, 095003 (2017).
[29] K. Fujimoto, R. Hamazaki, and M. Ueda, Phys. Rev. Lett. **120**, 073002 (2018).
[30] H. Takeuchi, Phys. Rev. A **97**, 013617 (2018).
[31] L. A. Williamson and P. B. Blakie, arXiv:1902.10792.
[32] J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, Cambridge, England, 1996).
[33] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[34] J. Berges, AIP Conf. Proc. **739**, 3 (2005).
[35] J. Berges, arXiv:1503.02907.
[36] B. Nowak, D. Sexty, and T. Gasenzer, Phys. Rev. B **84**, 020506(R) (2011).
[37] B. Nowak, J. Schole, D. Sexty, and T. Gasenzer, Phys. Rev. A **85**, 043627 (2012).
[38] J. Schole, B. Nowak, and T. Gasenzer, Phys. Rev. A **86**, 013624 (2012).
[39] M. Karl, B. Nowak, and T. Gasenzer, Sci. Rep. **3**, 2394 (2013).
[40] M. Karl, B. Nowak, and T. Gasenzer, Phys. Rev. A **88**, 063615 (2013).
[41] M. Karl and T. Gasenzer, New J. Phys. **19**, 093014 (2017).
[42] A. N. Mikheev, C.-M. Schmied, and T. Gasenzer, arXiv:1807.10228.
[43] J. Berges and D. Sexty, Phys. Rev. D **83**, 085004 (2011).
[44] A. Pineiro Orioli, K. Boguslavski, and J. Berges, Phys. Rev. D **92**, 025041 (2015).
[45] J. Berges and B. Wallisch, Phys. Rev. D **95**, 036016 (2017).
[46] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I: Wave Turbulence* (Springer, Berlin, 1992).
[47] S. Nazarenko, *Wave Turbulence*, Lecture Notes in Physics (Springer, Heidelberg, 2011), Vol. 825.
[48] A. Newell, Annu. Rev. Fluid Mech. **43**, 59 (2011).
[49] A. Bray, Adv. Phys. **43**, 357 (1994).
[50] U. Täuber, *Critical Dynamics* (Cambridge University Press, Cambridge, England, 2014).
[51] M. Prüfer, P. Kunkel, H. Strobel, S. Lannig, D. Linnemann, C.-M. Schmied, J. Berges, T. Gasenzer, and M. K. Oberthaler, Nature (London) **563**, 217 (2018).

[52] S. Erne, R. Buecker, T. Gasenzer, J. Berges, and J. Schmiedmayer, Nature (London) **563**, 225 (2018).
[53] I. Carusotto and E. J. Mueller, J. Phys. B **37**, S115 (2004).
[54] C. D. Hamley, C. S. Gerving, T. M. Hoang, E. M. Bookjans, and M. S. Chapman, Nat. Phys. **8**, 305 (2012).
[55] L. D. Sarlo, L. Shao, V. Corre, T. Zibold, D. Jacob, J. Dalibard, and F. Gerbier, New J. Phys. **15**, 113039 (2013).
[56] T. Zibold, V. Corre, C. Frapolli, A. Invernizzi, J. Dalibard, and F. Gerbier, Phys. Rev. A **93**, 023614 (2016).
[57] A. M. Kamchatnov, Y. V. Kartashov, P.-É. Larré, and N. Pavloff, Phys. Rev. A **89**, 033618 (2014).
[58] C. Qu, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **116**, 160402 (2016).
[59] C. Qu, M. Tylutki, S. Stringari, and L. P. Pitaevskii, Phys. Rev. A **95**, 033614 (2017).
[60] A. Gallemí, L. P. Pitaevskii, S. Stringari, and A. Recati, Phys. Rev. A **97**, 063615 (2018).
[61] H. Schmaljohann, M. Erhard, J. Kronjäger, M. Kottke, S. van Staa, L. Cacciapuoti, J. J. Arlt, K. Bongs, and K. Sengstock, Phys. Rev. Lett. **92**, 040402 (2004).
[62] J. Kronjäger, C. Becker, P. Soltan-Panahi, K. Bongs, and K. Sengstock, Phys. Rev. Lett. **105**, 090402 (2010).
[63] T.Świsłocki, E. Witkowska, J. Dziarmaga, and M. Matuszewski, Phys. Rev. Lett. **110**, 045303 (2013).
[64] E. Witkowska, J. Dziarmaga, T.Świsłocki, and M. Matuszewski, Phys. Rev. B **88**, 054508 (2013).
[65] E. Witkowska, T. Świsłocki, and M. Matuszewski, Phys. Rev. A **90**, 033604 (2014).
[66] K. Jiménez-García, A. Invernizzi, B. Evrard, C. Frapolli, J. Dalibard, and F. Gerbier, arXiv:1808.01015.
[67] S. W. Seo, W. J. Kwon, S. Kang, and Y. Shin, Phys. Rev. Lett. **116**, 185301 (2016).
[68] T. Świsłocki, E. Witkowska, and M. Matuszewski, Phys. Rev. A **94**, 043635 (2016).
[69] S. Kang, S. W. Seo, J. H. Kim, and Y. Shin, Phys. Rev. A **95**, 053638 (2017).
[70] T. Świsłocki, A. Zembrzuski, M. Matuszewski, and E. Witkowska, Phys. Rev. A **97**, 033629 (2018).
[71] L. M. Symes, D. Baillie, and P. B. Blakie, Phys. Rev. A **98**, 063618 (2018).
[72] M. Schmidt, S. Erne, B. Nowak, D. Sexty, and T. Gasenzer, New J. Phys. **14**, 075005 (2012).
[73] A. Vinit, E. M. Bookjans, C. A. R. Sá de Melo, and C. Raman, Phys. Rev. Lett. **110**, 165301 (2013).
[74] A. Vinit and C. Raman, New J. Phys. **20**, 095003 (2018).
[75] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen(De), and U. Sen, Adv. Phys. **56**, 243 (2007).
[76] A. Sinatra, C. Lobo, and Y. Castin, J. Phys. B **35**, 3599 (2002).
[77] P. B. Blakie, A. S. Bradley, M. J. Davis, R. J. Ballagh, and C. W. Gardiner, Adv. Phys. **57**, 363 (2008).
[78] A. Polkovnikov, Ann. Phys. (Amsterdam) **325**, 1790 (2010).
[79] From Eq. (1) and the definition of $L_c(t)$, we obtain a relation $0.5f(0) = f[L_c(t)t^{-\beta}/a]$. As the left-hand side is independent of t, we obtain $L_c(t) \propto t^{\beta}$.
[80] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction* (New York, Wiley, 1989).
[81] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.173001 for integrable models of magnetic solitons and Flemish strings, the main mechanism for magnetic-soliton annihilation, and our numerical method for the experiments of Refs. [73,74]. References [82–91] are included.
[82] Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[83] V. Zakharov and S. Nazarenko, Physica (Amsterdam) **201D**, 203 (2005).
[84] S. Nazarenko and M. Onorato, Physica (Amsterdam) **219D**, 1 (2006).
[85] S. Nazarenko and M. Onorato, J. Low Temp. Phys. **146**, 31 (2007).
[86] D. Proment, S. Nazarenko, and M. Onorato, Phys. Rev. A **80**, 051603(R) (2009).
[87] D. Proment, S. Nazarenko, and M. Onorato, Physica (Amsterdam) **241D**, 304 (2012).
[88] K. Fujimoto and M. Tsubota, Phys. Rev. A **91**, 053620 (2015).
[89] K. Fujimoto and M. Tsubota, Phys. Rev. A **93**, 033620 (2016).
[90] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, Nature (London) **539**, 72 (2016).
[91] N. Navon, C. Eigen, J. Zhang, R. Lopes, A. L. Gaunt, K. Fujimoto, M. Tsubota, R. P. Smith, and Z. Hadzibabic, arXiv:1807.07564.
[92] S. V. Manakov, Sov. Phys. JETP **38**, 248 (1974).
[93] Th. Busch and J. R. Anglin, Phys. Rev. Lett. **87**, 010401 (2001).
[94] T. Tsuchida, Prog. Theor. Phys. **111**, 151 (2004).
[95] D. Yan, J. J. Chang, C. Hamner, M. Hoefer, P. G. Kevrekidis, P. Engels, V. Achilleos, D. J. Frantzeskakis, and J. Cuevas, J. Phys. B **45**, 115301 (2012).
[96] The measure n_{sol} of the soliton number is a lattice number of regions where $\hat{S}_{z,j}^2$ is larger than S_{th}^2. When we calculate the soliton number, n_{sol} should be divided by width of a soliton. The typical width is $3 \sim 8$ lattice points.
[97] L. Isella and J. Ruostekoski, Phys. Rev. A **72**, 011601(R) (2005).
[98] L. Isella and J. Ruostekoski, Phys. Rev. A **74**, 063625 (2006).
[99] A. D. Martin and J. Ruostekoski, New J. Phys. **12**, 055018 (2010).
[100] The correlation function $N_{xy}(z,t)$ is defined by replacing the lattice index k and the summation $\sum_k$ in Eq. (4) with the coordinate z and the integral $\int dz$, respectively.
[101] The experiment of Ref. [74] investigated the density correlation function after a weak quench. This situation is different from our case with many magnetic solitons, and it may show different universal relaxation caused by a wave-turbulent cascade and provide a direct relation between the density and nematic correlation functions as discussed in the Supplemental Material [81].
[102] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. **83**, 5198 (1999).
[103] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature (London) **417**, 150 (2002).
[104] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).

[105] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, New J. Phys. **5**, 73 (2003).
[106] C. Becker, S. Stellmer, P. Soltan-Panahi, S. Dörscher, M. Baumert, E.-M. Richter, J. Kronjäger, K. Bongs, and K. Sengstock, Nat. Phys. **4**, 496 (2008).
[107] D. Yan, J. J. Chang, C. Hamner, P. G. Kevrekidis, P. Engels, V. Achilleos, D. J. Frantzeskakis, R. Carretero-Gonzàlez, and P. Schmelcher, Phys. Rev. A **84**, 053630 (2011).
[108] T. Yefsah, A. T. Sommer, M. J. H. Ku, L. W. Cheuk, W. Ji, W. S. Bakr, and M. W. Zwierlein, Nature (London) **499**, 426 (2013).
[109] S. Donadello, S. Serafini, M. Tylutki, L. P. Pitaevskii, F. Dalfovo, G. Lamporesi, and G. Ferrari, Phys. Rev. Lett. **113**, 065302 (2014).
[110] T. M. Bersano, V. Gokhroo, M. A. Khamehchi, J. DAmbroise, D. J. Frantzeskakis, P. Engels, and P. G. Kevrekidis, Phys. Rev. Lett. **120**, 063202 (2018).
[111] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-Gonzàlez, *Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment* (Springer, Berlin, 2008).
[112] T. Busch and J. R. Anglin, Phys. Rev. Lett. **84**, 2298 (2000); **87**, 010401 (2001).
[113] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. **86**, 2926 (2001).
[114] J. Brand and W. P. Reinhardt, Phys. Rev. A **65**, 043612 (2002).
[115] L. D. Carr and J. Brand, Phys. Rev. Lett. **92**, 040401 (2004).
[116] N. S. Ginsberg, J. Brand, and Lene V. Hau, Phys. Rev. Lett. **94**, 040403 (2005).
[117] S. Stellmer, C. Becker, P. Soltan-Panahi, E.-M. Richter, S. Dörscher, M. Baumert, J. Kronjäger, K. Bongs, and K. Sengstock, Phys. Rev. Lett. **101**, 120406 (2008).
[118] A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis, and P. G. Kevrekidis, Phys. Rev. Lett. **101**, 130401 (2008).
[119] I. Shomroni, E. Lahoud, S. Levy, and J. Steinhauer, Nat. Phys. **5**, 193 (2009).
[120] R. V. Mishmash and L. D. Carr, Phys. Rev. Lett. **103**, 140403 (2009).
[121] C. Hamner, J. J. Chang, P. Engels, and M. A. Hoefer, Phys. Rev. Lett. **106**, 065302 (2011).
[122] M. A. Hoefer, J. J. Chang, C. Hamner, and P. Engels, Phys. Rev. A **84**, 041605 (2011).
[123] C. Hamner, Y. Zhang, J. J. Chang, C. Zhang, and P. Engels, Phys. Rev. Lett. **111**, 264101 (2013).
[124] M. J. H. Ku, W. Ji, B. Mukherjee, E. Guardado-Sanchez, L. W. Cheuk, T. Yefsah, and M. W. Zwierlein, Phys. Rev. Lett. **113**, 065301 (2014).
[125] M. J. H. Ku, B. Mukherjee, T. Yefsah, and M. W. Zwierlein, Phys. Rev. Lett. **116**, 045304 (2016).
[126] J. H. V. Nguyen, D. Luo, and R. G. Hulet, Science **356**, 422 (2017).
[127] M. O. D. Alotaibi and L. D. Carr, Phys. Rev. A **96**, 013601 (2017).