# Optimal Probabilistic Storage and Retrieval of Unitary Channels

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We address the question of quantum memory storage for quantum dynamics. In particular, we design an optimal protocol for $N \to 1$ probabilistic storage and retrieval of unitary channels on $d$-dimensional quantum systems. If we access the unknown unitary gate only $N$ times, the optimal success probability of perfect single-use retrieval is $N/(N-1+d^2)$. The derived size of the memory system exponentially improves the known upper bound on the size of the program register needed for probabilistic programmable quantum processors. Our results are closely related to probabilistic perfect alignment of reference frames and probabilistic port-based teleportation.

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*Introduction.*—Since the discovery of the first quantum algorithms [1,2] and protocols [3,4], information processing with quantum systems has challenged basic paradigms and existing limitations of computer science. In the last few decades we have discovered that quantum information cannot be cloned [5], its "logical value" cannot be inverted [6], quantum processors cannot be universally programmed [7], and universal multimeters do not exist [8,9]. No doubt, any of these programmable devices would represent a very useful piece of quantum technology; thus, their approximate realizations are of foundational interest [8–15]. The no-go restrictions imposed by quantum theory are treated in two ways. Either we ask for an approximate performance or we allow that the perfect performance happens with some probability of failure.

Studies of optimal approximate cloners initiated by Bužek and Hillery [10] demonstrated that such nonideal devices are of practical relevance, and this motivated the study of other universal devices. In particular, it was shown that quantum theory limits the fidelity of $1 \to N$ clones of qubits to $(2N+1)/3N$ [16]. For quantum processors Nielsen and Chuang [7] proved that perfect (error-free) implementation of $k$ distinct unitary transformations requires at least a $k$-dimensional program register. Recently, cloning was also considered for quantum transformations [17,18]. This unveiled an unexpected feature called superreplication [19,20]. In this protocol, starting with $N$ copies of a qubit unitary transformation $U$, one deterministically generates up to $N^2$ copies of $U$ with an exponentially small error rate. While studying the cloning of unitaries, it was realized that there is a closely related task of storage and retrieval (SAR), which differs only in the causal order of available resources. While in the cloning case the cloned device is available after the input states are at their disposal, one can consider also a task where this order is reversed; thus, the device is available only before the input states. In such a case, we need to learn [21] and somehow *store* the action of the device and *retrieve* it once the input states are available.

*Problem formulation.*—The devices transforming states of a $d$-dimensional quantum system associated with a Hilbert space $\mathcal{H}$ are formalized as quantum channels, i.e., completely positive trace-preserving linear maps on the space $\mathcal{L}(\mathcal{H})$ of linear operators on $\mathcal{H}$. Suppose that an unknown channel $\mathcal{U}$ is provided for experiments and that we may access it $N$ times. However, we are asked to apply $\mathcal{U}$ on an unknown state $\xi$ only after we have lost access to this channel. Therefore, our aim is to find an optimal strategy that stores $\mathcal{U}$ in a state of a quantum memory (associated with Hilbert space $\mathcal{H}_M$) and allows us to retrieve its action when needed. In the approximative setting, this task (for unitary channels) was studied in Ref. [22].

Our goal is to investigate the probabilistic version of the SAR problem; in particular, we aim to find the optimal $N \to 1$ probabilistic storage and retrieval procedure (PSAR). Moreover, we require the retrieved channel to be implemented perfectly and with the same probability of success ("covariance" property) for all considered channels. We will design the strategy maximizing the probability for the set of unitary channels, i.e., $\mathcal{U}(\xi) = U\xi U^\dagger$ for some unitary operator $U$. Owing to the no-programming theorem [7], the retrieving part of any PSAR strategy cannot be deterministic. Thus, the successful retrieval is described by a trace-nonincreasing completely positive linear map (quantum operation) $\mathcal{T}_U : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ proportional to the unknown unitary channel, $\mathcal{T}_U = \lambda_U \mathcal{U}$. Consequently, the success probability is $\lambda_U = \mathrm{tr}[\mathcal{T}_U(\xi)]$, and the condition of covariance implies that $\lambda_U = \lambda$ for all $U$.

 

*One-to-one probabilistic storage and retrieval.*—In such a case the unknown unitary $U$ is applied on a suitably chosen state $|\psi\rangle$ (in general bipartite and entangled), which yields state $|\psi_U\rangle \in \mathcal{H}_M$ and concludes the storing phase. Afterwards, once we want to apply unitary $U$ on some state $\xi$, we employ a retrieving quantum instrument $\mathbf{R} = \{\mathcal{R}_s, \mathcal{R}_f\}$, which acts on $\xi \otimes |\psi_U\rangle\langle\psi_U|$ and, in the case of success, outputs an subnormalized state $\lambda U \xi U^\dagger$, i.e., $\mathcal{R}_s : \mathcal{L}(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_M) \to \mathcal{L}(\mathcal{H}_{\mathrm{out}})$ with $\mathcal{H} = \mathcal{H}_{\mathrm{in}} = \mathcal{H}_{\mathrm{out}}$. The retrieving quantum instrument plays the role of a probabilistic programmable processor and the state $|\psi_U\rangle$ programs a unitary transformation $U$ to be performed on a state $\xi$.

Using the Choi isomorphism [23] and requiring perfect retrieval, we have $\mathcal{R}_s(\xi \otimes |\psi_U\rangle\langle\psi_U|) = \mathrm{tr}_{\mathrm{in},M}[(I \otimes \xi^T \otimes |\psi_U\rangle\langle\psi_U|^T) R_s] = \lambda \mathrm{tr}_{\mathrm{in}}[(I \otimes \xi^T)|U\rangle\rangle\langle\langle U|] = \lambda U \xi U^\dagger$, where $R_s \in \mathcal{L}(\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_M)$ and $|U\rangle\rangle = \sqrt{d}(U \otimes I)|\psi_+\rangle$ with $|\psi_+\rangle = d^{-1/2} \sum_j |j\rangle \otimes |j\rangle$ (vectors $\{|j\rangle\}$ form an orthonormal basis of $\mathcal{H} = \mathcal{H}_{\mathrm{in}} = \mathcal{H}_{\mathrm{out}}$). Since the above identity must hold for any $\xi$ and $|\psi_U\rangle\langle\psi_U|^T = |\psi_U^*\rangle\langle\psi_U^*|$ (both the transposition and the conjugation are defined with respect to the same basis of $\mathcal{H}_M$), we obtain the following *perfect retrieval condition*

$$\langle\psi_U^*|R_s|\psi_U^*\rangle = \lambda |U\rangle\rangle\langle\langle U| \quad \forall \ U \in SU(d). \tag{1}$$

Already this simple case shows that the maximization of probability of success $\lambda$ involves the simultaneous optimization of the storing phase (choice of $|\psi\rangle$) and the retrieving phase (choice of quantum instrument $\mathbf{R}$). It turns out that the optimal performance is achieved by the (incomplete) quantum teleportation protocol [4] that is a known example of a universal probabilistic quantum processor [24]. Let us note that this is similar to quantum gate teleportation invented by Gottesman and Chuang [25], yet it is different because PSAR must work perfectly for any unitary transformation. In particular, the optimal state for storage is $|\psi\rangle = |\psi_+\rangle$ (see Fig. 1). Then the optimal retrieval is achieved by a quantum teleportation of state $\xi$ using the stored state $|\psi_U\rangle = d^{-1/2}|U\rangle\rangle$. The generalized Bell measurement performed on $\xi$ and one part of $|\psi_U\rangle$ results in an outcome $k$ with probability $1/d^2$. In such a case we are left with the second part of $|\psi_U\rangle$ in the state $U\sigma_k \xi \sigma_k U^\dagger$, where $\sigma_k$ are generalized Pauli operators. In case of $\sigma_k = I$ (associated with the Bell measurement projection onto $|\psi_+\rangle$) the stored unitary channel is successfully retrieved. For all of the other outcomes, the unwanted $\sigma_k$ rotation cannot be undone because the unitary $U$ is unknown. In conclusion, the teleportation-based PSAR succeeds with probability $1/d^2$. Its optimality follows from our subsequent discussion of the optimal $N \to 1$ PSAR.

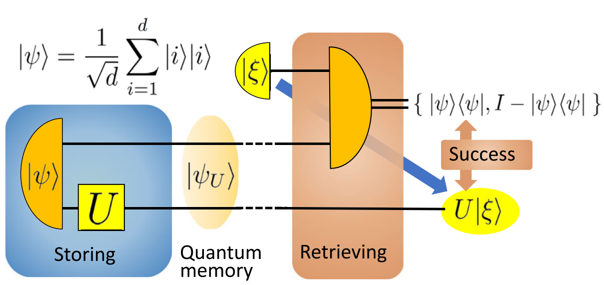


FIG. 1. Optimal $1 \to 1$ PSAR of unitary channels.

*N-to-one probabilistic storage and retrieval.*—The general PSAR strategy with $N$ uses of a channel in the storing phase involves all combinations of their parallel, successive, and adaptive processing and corresponds to a quantum circuit with open slots, where the $N$ uses of a channel can be inserted. Such a framework is described within the theory of *quantum networks* [26–29] and any quantum circuit with open slots is represented by a positive operator (see the Supplemental Material [30] for a short introduction). The storing network $\mathcal{S}$ accepts $N$ channels as its input, and it outputs a memory state $|\psi_U\rangle \in \mathcal{H}_M$ [see Fig. 2(a)]. As in the $1 \to 1$ case, the retrieving phase is described by a two-valued instrument $\mathbf{R} = \{\mathcal{R}_s, \mathcal{R}_f\}$. The overall action of PSAR is a composition of $\mathcal{S}$ and $\mathbf{R}$ determining a generalized quantum instrument $\mathbf{L} = \{\mathcal{L}_s, \mathcal{L}_f\}$. In the Choi picture the input of PSAR corresponds to $|U\rangle\rangle\langle\langle U|^{\otimes N} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $L_s \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}})$, where $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}^{\otimes N}$. The perfect retrieval condition [as in Eq. (1)] is

$$\langle\langle U^*|^{\otimes N} L_s |U^*\rangle\rangle^{\otimes N} = \lambda |U\rangle\rangle\langle\langle U| \quad \forall \ U \in SU(d), \tag{2}$$

where $\lambda$ gives the success probability. Let us stress that the probability of success, i.e., the value of $\lambda$, is required to be the same for all $U \in SU(d)$. Thanks to this assumption, we can without loss of generality apply the methods of Ref. [22] to

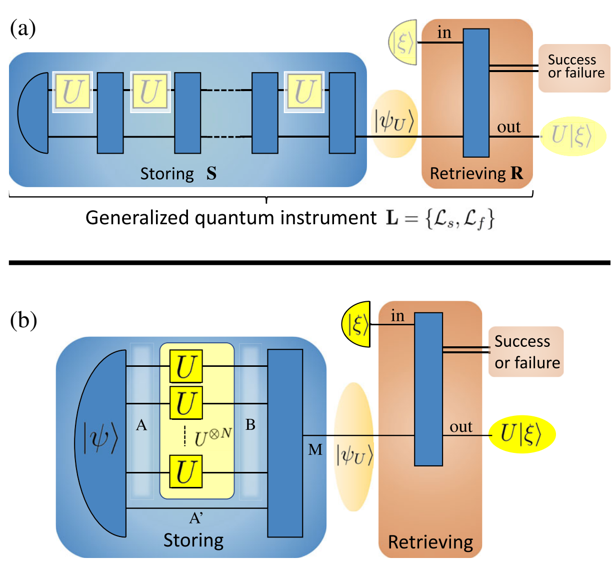


FIG. 2. Illustration of $N \to 1$ PSAR. (a) PSAR with the most general strategy. (b) PSAR with parallel use of unitary channels.

conclude that the optimal storing phase is *parallel*, as illustrated in Fig. 2(b). Consider the decomposition $U^{\otimes N} = \bigoplus_{j\in \mathrm{irrep}(U^{\otimes N})} U_j \otimes I_{m_j}$ into irreducible representations (irreps), where $U_j$ is a unitary operator on $\mathcal{H}_j$, and $I_{m_j}$ denotes the identity operator on the multiplicity space. This corresponds to the following decomposition of the Hilbert space $\mathcal{H}_A \coloneqq \bigoplus_{j\in \mathrm{irrep}(U^{\otimes N})} \mathcal{H}_j \otimes \mathcal{H}_{m_j}$, and we set $d_j = \dim(\mathcal{H}_j)$. The result of Ref. [22] implies that the memory state $|\psi\rangle$ can be taken to be of the following form:

$$|\psi\rangle \coloneqq \bigoplus_j \sqrt{\frac{p_j}{d_j}} |I_j\rangle\rangle \in \mathcal{H}_M, \qquad p_j \geq 0, \sum_j p_j = 1, \qquad (3)$$

where $I_j$ denotes the identity operator on $\mathcal{H}_j$, and $\mathcal{H}_M \coloneqq \bigoplus_{j\in \mathrm{irrep}(U^{\otimes N})} \mathcal{H}_j \otimes \mathcal{H}_j \subseteq \mathcal{H}_A \otimes \mathcal{H}_{A'}$. The state $|\psi\rangle$ undergoes the action of the unitary channels and becomes $|\psi_U\rangle \coloneqq \bigoplus_j \sqrt{p_j/d_j}|U_j\rangle\rangle$. Clearly, $|\psi_U\rangle \in \mathcal{H}_M$ for any $U$.

Let us now focus on the retrieving quantum instrument $\mathbf{R}$ from $\mathcal{L}(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_M)$ to $\mathcal{L}(\mathcal{H}_{\mathrm{out}})$, where subscripts in and out refer to the system on which the retrieved channel is applied. The perfect retrieval condition is again given by Eq. (1) with $|\psi_U^*\rangle = \bigoplus_j \sqrt{p_j/d_j}|U_j^*\rangle\rangle$. As a consequence of Eq. (2), the optimal Choi operator $R_s$ can be chosen to satisfy the commutation relation

$$[R_s, U'^* V' \otimes U_{\mathrm{in}} \otimes V_{\mathrm{out}}^*] = 0, \qquad (4)$$

where $U' \coloneqq \bigoplus_j U_j \otimes I_j$, $V' \coloneqq \bigoplus_j I_j \otimes V_j$. Thanks to Eq. (4), $U'|\psi\rangle = |\psi_U\rangle$ and $|\psi_I^*\rangle = |\psi\rangle$, so the perfect retrieval condition becomes

$$\langle\psi|R_s|\psi\rangle = \lambda |I\rangle\rangle\langle\langle I|, \qquad (5)$$

where $\lambda = (1/d^2)\langle\langle I|\langle\psi|R_s|\psi\rangle|I\rangle\rangle$ is the success probability. Let us now consider the decomposition

$$U_j^* \otimes U = \bigoplus_{J\in \mathrm{irrep}(U_j^*\otimes U)} U_J \otimes I_{m_J^{(j)}}, \qquad (6)$$

which induces the Hilbert space decomposition $\mathcal{H}_j \otimes \mathcal{H} = \bigoplus_{J\in \mathrm{irrep}(U_j^*\otimes U)} \mathcal{H}_J \otimes \mathcal{H}_{m_J^{(j)}}$. Let us denote by $\mathrm{j}_{JK}$ the set of values of $j$ such that $U_J \otimes V_K$ is in the decomposition of $U_j^* \otimes V_j \otimes U \otimes V^*$. Using Eqs. (4) and (6), we can assume [30] that $R_s = \bigoplus_J I_J \otimes I_J \otimes s^{(J)}$, where $s^{(J)} \coloneqq \sum_{j,j'\in \mathrm{j}_{JJ}} s_{jj'}^{(J)} |I_{m_J^{(j)}}\rangle\rangle\langle\langle I_{m_J^{(j')}}|$. Given this, the left-hand side of Eq. (5) reads

$$\langle\psi|R_s|\psi\rangle = \sum_J \lambda_J |I\rangle\rangle\langle\langle I| + \nu_J\left(I - \frac{1}{d}|I\rangle\rangle\langle\langle I|\right), \qquad (7)$$

where $\nu_J$ are specified in the Supplemental Material [30], $\lambda_J = (d_J/d^2)\langle\phi_J|s^{(J)}|\phi_J\rangle$, and $|\phi_J\rangle = \bigoplus_{j\in \mathrm{j}_{JJ}} \sqrt{p_j/d_j}|I_{m_J^{(j)}}\rangle\rangle$.

Since $R_s \geq 0$, the perfect learning condition of Eq. (5) holds only if $\nu_J = 0$ for all $J$, in which case the success probability is $\lambda = \sum_J \lambda_J$. The following result translates the optimization of $\lambda$ from an operator optimization problem into a linear program.

*Theorem 1.*—For optimal PSAR the success probability $\lambda$ is given by the following linear programming problem:

$$\begin{aligned} &\underset{\mu_J, p_j}{\text{maximize}} \quad \lambda = \sum_{J\in C} d_J^3 \mu_J, \\ &\text{subject to} \quad 0 \leq d_J \mu_J \leq \frac{p_j}{d_j^2} \quad \forall\ j \in \mathrm{j}_{JJ} \quad \forall\ J \in C \\ &\qquad\qquad p_j \geq 0 \sum_{j\in \mathrm{irrep}(U^{\otimes N})} p_j = 1, \end{aligned} \qquad (8)$$

where $C = \{J \in \mathrm{irrep}(U^{\otimes N} \otimes U^*) | d d_J = \sum_{j\in \mathrm{j}_{JJ}} d_j\}$.

*Proof.*—We will sketch only the key steps. The complete proof is in the Supplemental Material [30]. First, one shows that $J \notin C$ implies that $s^{(J)} = 0$. Then (for any $J \in C$), $\nu_J = 0$ and $s^{(J)} \geq 0$ imply that $\sqrt{p_j p_{j'}} s_{jj'}^{(J)} = \mu_J \sqrt{d_j^3 d_{j'}^3}$ for some $\mu_J \geq 0$. Thus, $\lambda = \sum_{J\in C} \sum_{j,j'\in \mathrm{j}_{JJ}} (d_J \mu_J / d^2) d_j d_{j'} = \sum_{J\in C} d_J^3 \mu_J$. The constraint that $\mathcal{R}_s$ is a quantum operation gives $\mathrm{tr}_{\mathrm{out}}[R_s] \leq I$. Equation (4) implies that $[\mathrm{tr}_{\mathrm{out}}[R_s], U'V' \otimes U_{\mathrm{in}}^*] = 0$ and $\mathrm{tr}_{\mathrm{out}}[R_s] = \bigoplus_J \bigoplus_{j\in \mathrm{j}_{JJ}} I_J \otimes I_j (d_J/d_j) s_{jj}^{(J)}$. Thus, $d_J \mu_J (d_j^2/p_j) \leq 1$ must hold for all $J$ and $j \in \mathrm{j}_{JJ}$. Conditions on $p_j$ are from Eq. (3). □

*Case study: $N \to 1$ PSAR for qubit channels.*—In the qubit ($d = 2$) case the decomposition of $U^{\otimes N}$ into irreducible representations (irreps) of $SU(2)$ reads $U^{\otimes N} = \bigoplus_{j=(N \bmod 2)/2}^{N/2} U_j \otimes I_{m_j}$, where $m_j = [(2j+1)/(N/2+j+1)]\binom{N}{N/2+j}$ [34] and $U_j$ are the irreps of spin $j$ with dimension $d_j = 2j+1$. For convenience we work with even $N$ (for odd $N$ see the Supplemental Material [30]), so $j = 0, 1, \ldots, N/2$. For $SU(2)$ the complex conjugate representation $U_j^*$ is equivalent to irrep $U_j$. Thus, in Eq. (6) we get either $J = j + 1/2$ or $J = j - 1/2$. Altogether, $J$ can have values $J \in C = \{1/2, \ldots, (N-1)/2\}$ or $J = (N+1)/2 \notin C$ because $\sum_{j\in \mathrm{j}_{JJ}} d_j = d_{J-1/2} + d_{J+1/2} = d d_J$ and $d_{N/2} \neq 2 d_{(N+1)/2}$. The constraints in Eq. (8) imply for any $j$ but $j = 0, N/2$ the following two inequalities:

$$\mu_{j+1/2} d_j^2 d_{j+1/2} \leq p_j, \qquad (9)$$

$$\mu_{j-1/2} d_j^2 d_{j-1/2} \leq p_j. \qquad (10)$$

For $j=0,N/2$ only one of them exists. Let us define $f_j \in [0,1]$ for $j=0,\ldots,N/2$ as $f_j = (1/2)[2j/(2j+1)][(2j+2)/N+1]$. Since $f_0 = 0$ and $f_{N/2} = 1$, we can multiply Eq. (9) by $1 - f_j$ and Eq. (10) by $f_j$ and take the sum for all $j$. A straightforward calculation gives the upper bound:

$$\frac{N+3}{N}\sum_{J=\frac{1}{2}}^{(N-1)/2} d_J^3\mu_J \le 1 \Leftrightarrow \lambda \le \frac{N}{N+3}. \qquad (11)$$

Finally, by choosing $p_j = (2j+1)^2/L$, $\mu_{j+1/2} = 1/(L(2j+2))$ (where $L = (N+1)(N+2)(N+3)/6$), one proves that conditions in Eq. (8) are satisfied and the upper bound (11) is achieved. The knowledge of $\mu_J$ and $p_j$ completely specifies the state $|\psi\rangle$ and the retrieving operation $\mathcal{R}_s$, which can be explicitly expressed [see Fig. 2(b)]. Let $|j, j_z\rangle \in \mathcal{H}_j$ with $j_z \in \{-j, \ldots, j\}$ be an orthonormal basis of the spin $j$ irrep. By definition $|I_j\rangle\rangle = \sum_{j_z=-j}^{j} |j, j_z\rangle \otimes |j, j_z\rangle$. Consequently, from Eq. (3), the dimension of the quantum memory is $\dim \mathcal{H}_M = \sum_{j=0}^{N/2} d_j^2 = L$, and the optimal input state for storage is $|\psi\rangle = \bigoplus_{j=0}^{N/2} \sqrt{(2j+1)/L}|I_j\rangle\rangle$.

*Optimal PSAR for qudit unitary transformations.*—The optimization of $N \to 1$ PSAR of qudit channels follows similar steps as the qubit case, and it exploits a combinatorial identity (Proposition 3 in Refs. [35,36]) which was discovered and proved as a by-product of this analysis.

*Theorem 2.*—The optimal probability of success of $N \to 1$ probabilistic storage and retrieval of a unitary channel $\mathcal{U}(.) = U.U^\dagger$, $U \in SU(d)$ equals $\lambda = N/(N-1+d^2)$. The optimal state for storage is $|\psi\rangle \coloneqq \bigoplus_j \sqrt{(d_j/L)}|I_j\rangle\rangle$ [see Eq. (3)], where $L \coloneqq \sum_j d_j^2$ and $j \in \mathrm{irrep}(U^{\otimes N})$.

The proof is given in the Supplemental Material [30]. Clearly, as $N$ goes to infinity, $\lambda \sim 1 - (d^2-1)/N$, and $\lambda \approx \frac{1}{2}$ implies that $N \approx d^2$. Recalling that a $d$-dimensional unitary transformation has $d^2$ parameters, we see that roughly one use per unknown parameter is needed for reliable storage and retrieval of the transformation. Let us note that the storage state in Theorem 2 is optimal also for the estimation of a group transformation in the maximum likelihood approach [37]. Further, it is worth stressing that the optimal PSAR protocol is achieved by a coherent retrieval; hence, the quantum memory is essential. By contrast, optimal approximate SAR [22] is equivalent to quantum estimation in the maximum fidelity approach, and classical memory is sufficient as an output of the storing phase. Use of the optimal storage state in the design of an approximate SAR leads to fidelity that scales as $1 - O(N^{-1})$; however, for the optimal approximate SAR the fidelity scales as $1 - O(N^{-2})$ [22]. This $O(N)$ difference is the price to pay for the perfect retrieval in the case of PSAR.

*Alignment of reference frames (ARF) [38].*—Let us note that the correction of alignment errors can be modeled as a PSAR protocol in which $N$ uses of an unknown $\mathcal{U}$ are stored and the aim is to retrieve the inverse transformation $\mathcal{U}^\dagger$. For $SU(2)$ we can show that, given $N$ uses of $\mathcal{U}$, the inverse transformation $\mathcal{U}^{-1}$ can be perfectly retrieved with the same optimal probability of success $\lambda$ (see Fig. 3 and the Supplemental Material [30]). It follows that the success

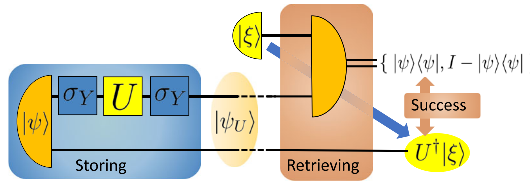


FIG. 3. A modified optimal $1 \to 1$ PSAR in which $\mathcal{U}$ is stored and the inverse transformation $\mathcal{U}^\dagger$ is retrieved [$SU(2)$ case]. The generalization to the $N \to 1$ case is straightforward.

probability of the probabilistic ARF protocol [38] achieves the optimal scaling $O(N^{-1})$ (see the Supplemental Material [30]).

*Probabilistic port-based teleportation (PPBT).*—As the first step of PPBT [39], Alice and Bob share $N$ suitably entangled pairs of quantum systems. Their goal is to teleport an unknown state $\xi$ to Bob in a way that this state appears in one of his systems (called ports [40,41]). In order to achieve this goal (see also Fig. 4), Alice performs a specific measurement resulting in $n \in \{0, 1, \ldots, N\}$ (0 labels the failure of the protocol) and communicates this information to Bob who selects the system from the $n$th port to accomplish the teleportation. If Bob applies a channel $\mathcal{U}$ on each of his ports (storing phase) and Alice starts the teleportation (retrieving phase) of $\xi$ afterwards, the $n$th port will output $\mathcal{U}(\xi)$. Strictly speaking, we swap the $n$th port into a fixed quantum system, and effectively we achieve $N \to 1$ PSAR. Let us stress that while any PPBT protocol can be turned into a PSAR protocol, the converse does not hold. In a sense, the PPBT scheme provides a structurally simple realization of an optimal PSAR protocol. Our results show that the optimal probability of PPBT [42] coincides with the optimal success probability of PSAR. However, the memory dimension $\dim \mathcal{H}_M$ of the optimal PSAR is exponentially smaller (see the following paragraph) in comparison with $2N$ qudits used in PPBT construction.

*Implications for covariant probabilistic programmable processors.*—Up to now the best bound on the size of the program register for universal covariant probabilistic processors was provided by a family of PPBT processors for which $\dim \mathcal{H}_M \approx (d^{2(d^2-1)})^{1/f}$, where $f = 1 - \lambda$ is the failure probability. By contrast, the retrieving phase of optimal $N \to 1$ PSAR defines a class of processors for

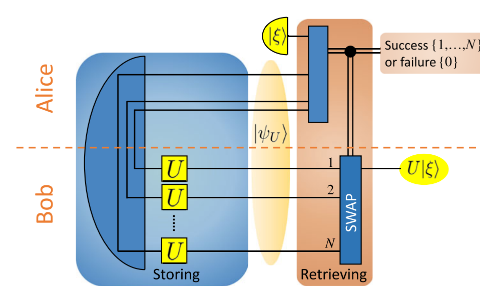


FIG. 4. Use of port-based teleportation scheme for PSAR.

which the program register size reads $\dim\mathcal{H}_M = \sum_{j\in\mathrm{irrep}(U^{\otimes N})} d_j^2 = \binom{N+d^2-1}{N}$, where we used Schur's result [43]. In terms of the failure probability, it reads $\dim\mathcal{H}_M \propto (1/f)^{(d^2-1)}$, which is exponentially smaller (for fixed $d$ and $f\to 0$) than provided by PPBT-based processors. This result can be viewed as a quantification of achievable trade-offs imposed by the no-programming theorem [7] on universal covariant probabilistic processors. Although PSAR provides only an upper bound on the size of the program register, we conjecture that the lower bound will have the same scaling. However, this question remains open.

*Summary.*—We showed that optimal probabilistic storage and retrieval of unknown unitary channels on $d$-dimensional quantum systems can be designed with success probability $\lambda = N/(N-1+d^2)$, where $N$ is the number of uses of the channel in the storing phase. This probability coincides with the success probability for probabilistic port-based teleportation [42], and, for the $SU(2)$ case, with the probability of success for probabilistic alignment of reference frames. Optimal PPBT can be rephrased as an optimal protocol for PSAR, but for the PSAR protocol designed here the storing memory system is exponentially smaller and optimal in this parameter. However, $N\to 1$ PPBT-based PSAR implements all quantum channels (not only unitary ones), and therefore its performance is universal. The question of a potential reduction of the memory system while keeping universality for all channels remains open. A natural extension of this Letter would be to consider storage and retrieval in the presence of noise in the action of the stored unitary channels. We performed a preliminary analysis of the noise robustness of the optimal $2\to 1$ PSAR protocol under the influence of unbiased depolarizing noise [44] and uncovered a surprising phenomenon. As expected, the success probability decreases as the noise level increases; however, for any noise level and any dimension $d$, we observed a suppression of the noise in the successfully retrieved channel. How the noise suppression behaves for arbitrary $N$ is an open question left for further investigation.

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