Silenko, Zhang, and Zou Reply: We are grateful to I. and Z. Bialynicki-Birula for the interest in our Letter. The preceding Comment is based on two assertions [1]. First, the radius vector operator $\boldsymbol{r}$ defining particle coordinates in the Dirac representation [but not in the Foldy-Wouthuysen (FW) representation] corresponds to the classical radius vector $\boldsymbol{q}$ describing a particle position. Second, the probability density should be determined in the Dirac representation, $\varrho = \varrho_D = \Psi_D^\dagger \Psi_D$, while the FW representation distorts this quantity. Therefore, the authors of Ref. [1] insist that the Dirac representation corrupting the connection between energy, momentum, and velocity provides the right distribution of the probability density and the FW representation, and restoring the Schrödinger picture of relativistic quantum mechanics (QM) distorts this density.

However, QM leads to the opposite conclusion. The problem of the position operator was definitely solved in the 1960s. The famous work by Newton and Wigner [2] established that the position operators are "related to the structure of the unitary irreducible representations of the Lorentz group" [3] and can be unambiguously determined for any representation. The next developments [3–6] (including those fulfilled by other methods [7–10]) have confirmed the validity of the Newton-Wigner (NW) approach. This approach uses equivalent commutation relations for operators and classical variables (commutators and Poisson brackets, respectively) explained in Refs. [3–5]. We mention an important initial contribution by Pryce [11].

Foldy and Wouthuysen have determined [12] that the NW position operator and the radius vector in the FW representation ("mean-position operator" [12]) are identical. This fundamental conclusion has been confirmed in many papers [3–10,13–16]. For a free particle in the Dirac representation, the NW mean-position operator reads [12,17]

$$\boldsymbol{q} = \boldsymbol{r} - \frac{\boldsymbol{\Sigma}\times\boldsymbol{p}}{2E(E+m)} + \frac{i\boldsymbol{\gamma}}{2E} - \frac{i(\boldsymbol{\gamma}\cdot\boldsymbol{p})\boldsymbol{p}}{2E^2(E+m)}, \qquad (1)$$

where E is the particle energy. It has also been proven [6,14–16,18–20] that the classical spin is equivalent to the NW spin operator and the FW mean-spin operator. In the FW representation, wave packets described by the $(1+1)$-dimensional Dirac equation also behave much more like a classical particle than in the Dirac representation [21,22].

Since the particle position is correctly defined by the radius vector in the FW representation, the probability density should also be determined in this representation (see our Letter [23]), $\varrho = \varrho_{\mathrm{FW}} = \Psi_{\mathrm{FW}}^\dagger \Psi_{\mathrm{FW}}$. Thus, the Dirac representation distorts the probability density and the FW wave function correctly defines it.

Contemporary relativistic QM in the FW representation (see Refs. [24–26] and references therein) presents important additional arguments in favor of this conclusion. Relativistic FW Hamiltonians for a spin-1/2 particle in electromagnetic fields are very similar to the corresponding classical Hamiltonians. When the fields are uniform, the gauge $\Phi = -\boldsymbol{E}\cdot\boldsymbol{r}, \boldsymbol{A} = (\boldsymbol{B}\times\boldsymbol{r})/2$ can be used and the relativistic FW Hamiltonian [24,26–28] reads ($\hbar = 1$, $c = 1$)

$$\mathcal{H}_{\mathrm{FW}} = \beta\sqrt{m^2 + \left(\boldsymbol{p} - \frac{e}{2}\boldsymbol{B}\times\boldsymbol{r}\right)^2} - e\boldsymbol{E}\cdot\boldsymbol{r} + \boldsymbol{\Omega}\cdot\boldsymbol{s}, \qquad (2)$$

where $\boldsymbol{s} = \boldsymbol{\Sigma}/2$ is the spin operator and the operator $\boldsymbol{\Omega}$ defines the angular velocity of spin precession.

The classical limit of this Hamiltonian and the corresponding classical Hamiltonian *coincide* [24,26,28,29]. This coincidence covering spin-dependent terms confirms that just the FW radius vector is a counterpart of the classical particle position. The validity of this conclusion for spin-1 particles can be easily shown using Refs. [30,31].

The basic role of the FW representation in nonstationary QM has been proven in Ref. [32]. The classical time-dependent energy corresponds to the time-dependent expectation value of the energy operator. The latter is the Hamiltonian in the Schrödinger QM and the FW representation (but not in the Dirac representation) [32]. The energy expectation values are defined by $E(t) = \langle \mathcal{H}_{\mathrm{FW}}(t)\rangle \neq \langle \mathcal{H}_D(t)\rangle$ [32].

The arguments presented by the authors do not substantiate their point of view. The mentioned spread of a particle location [1] manifesting in the Darwin interaction is a real physical effect but not a shortcoming of the FW representation. In the Dirac representation, it appears in *Zitterbewegung* [33].

The probability density *obeys* the continuity equation in both the FW representation and Schrödinger QM because this representation extends Schrödinger QM to the relativistic region.

Equations (1) and (5)–(7) in the Comment are right. The probability density depends on a representation [34]. The result [1] is a particular case of a general connection between the Dirac and FW wave functions at the exact FW transformation (upper spinors in the two representations differ only by constant factors and lower FW spinors vanish) [35].

This work was supported by the Belarusian Republican Foundation for Fundamental Research (Grant No. Φ18D-002), by the National Natural Science Foundation of China (Grants No. 11575254 and No. 11805242), by the National Key Research and Development Program of China (No. 2016YFE0130800), and by the Heisenberg-Landau program of the German Federal Ministry of Education and Research (Bundesministerium für Bildung und Forschung). A. J. S. also acknowledges hospitality and support by the Institute of Modern Physics of the Chinese Academy of Sciences.

0031-9007/19/122(15)/159302(3)

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Received 26 December 2018; published 18 April 2019
DOI: 10.1103/PhysRevLett.122.159302

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