Comment on "Relativistic Quantum Dynamics of Twisted Electron Beams in Arbitrary Electric and Magnetic Fields"

In a recent Letter [1], the authors studied the motion of relativistic electron beams described by the Dirac equation employing the Foldy-Wouthuysen (FW) representation. This representation has been used in many other papers referenced in this work. In our Comment, we show explicitly that in the relativistic case studied in [1] the FW transformation distorts the probability density.

The FW transformation is unitary. It does not modify the energy spectrum, but it changes the wave function. As a result, the FW wave function $\Psi_{\rm FW}$ does not describe correctly the charge distribution of the electron (except in the unphysical case of a single monochromatic plane wave). The interpretation of $\rho_D = \Psi_D^\dagger \Psi_D$ as the probability density has been explicitly stated over and over again by the founders of relativistic quantum mechanics [2]. The density $\rho_{\rm FW} = \Psi_{\rm FW}^\dagger \Psi_{\rm FW}$ defined in terms of the FW wave function cannot describe correctly the probability density because it does not obey the continuity equation. It is the probability density and the current density evaluated in the *Dirac* representation that satisfy $\partial_\mu j^\mu = 0$ and couple *locally* to the electromagnetic field through the expression $j^\mu(\boldsymbol{r},t)A_\mu(\boldsymbol{r},t)$. Moreover, at the level of quantum field theory the operators $\rho_{\rm FW}$ become nonlocal (they do not commute at spacelike separations) and this would play havoc with relativistic QED.

This shortcoming of the FW representation has been stressed already by Foldy and Wouthuysen who wrote [3] in connection with their formula (20): "In the old representation the Dirac particle interacted with the electromagnetic field only at its position. However, a particle which in the old representation was located at a point is in the new representation spread out over a region of dimensions of the order of a Compton wavelength."

We shall use the case of the relativistic Bessel beam [4] to analyze these problems. The general solution $\Psi_D(\boldsymbol{r},t)$ of the Dirac equation in the Dirac representation $(i\gamma_D^\mu \partial_\mu - m)\Psi_D(\boldsymbol{r},t) = 0$, describing the Bessel beam with the projection of the total angular momentum on the z axis equal to $\hbar(l+1/2)$ is

$$\Psi_D(\boldsymbol{r},t) = \exp[-i(Et - q_z z - l\phi)] \times \begin{bmatrix} [a(1+E-q_z) + ibq_\perp]J_l(q_\perp\rho) \\ [-iaq_\perp + b(1+E+q_z)]e^{i\phi}J_{l+1}(q_\perp\rho) \\ [a(1-E+q_z) - ibq_\perp]J_l(q_\perp\rho) \\ [iaq_\perp + b(1-E-q_z)]e^{i\phi}J_{l+1}(q_\perp\rho) \end{bmatrix}, \quad (1)$$

where E is the beam energy, q_z is the momentum along the direction of propagation, $q_\perp = \sqrt{q_x^2 + q_y^2}$ is the transverse momentum, ϕ is the polar angle, and (a,b) are arbitrary complex amplitudes. We used the following dimensionless variables: the space-time variables $\{ct, x, y, z\}$ are measured in the electron Compton wavelength, the momenta are measured in mc, and the energy in these units is $E = \sqrt{1 + q_z^2 + q_\perp^2}$. The transverse position variable is $\rho = \sqrt{x^2 + y^2}$. The form of the solution $\Psi_D(\boldsymbol{r},t)$ is different than the one appearing in [5] because there we used the Weyl representation of γ matrices and here we use the Dirac representation.

The connection between the bispinor in the Dirac and in the FW representation [3] can be written in the form

$$\Psi_{\rm FW}(\boldsymbol{r},t) = \left(\hat{c}_+ + \frac{\beta\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}}{\hat{p}}\hat{c}_-\right)\Psi_D(\boldsymbol{r},t), \quad (2)$$

where $\hat{p}$ is the length of momentum operator and

$$\hat{c}_\pm = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{1}{\sqrt{1+\hat{p}^2}}}. \quad (3)$$

For monochromatic solutions of the Dirac equation, and Bessel beams belong to this category, we can evaluate the transformation (2) directly; we just replace the momentum operators in (2) by their eigenvalues

$$\hat{p} \to \sqrt{E^2 - 1}, \qquad \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} \to E - \beta, \quad (4)$$

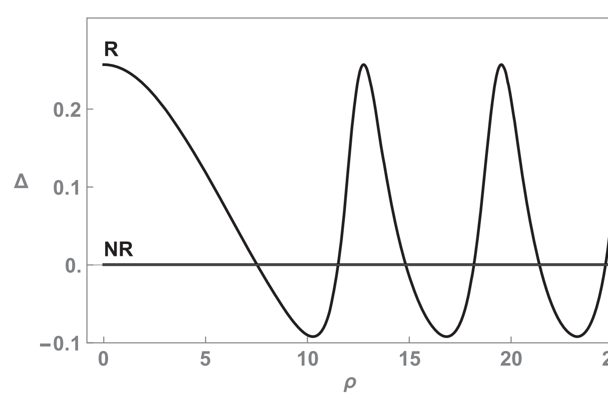

FIG. 1. Relative difference $\Delta = (\varrho_D - \varrho_{\rm FW})/\varrho_D$ between the probability densities calculated from the wave functions in the Dirac and in the Foldy-Wouthuysen representations for $l = 2, a = 1, b = 2i$. The difference is negligible for nonrelativistic electrons ($q_\perp = 0.02, q_z = 0.03$, lower curve) but it clearly becomes significant for relativistic electron beams ($q_\perp = 0.5, q_z = 1.0$, upper curve). The corresponding relativistic kinetic energy is $E - mc^2 = 250$ keV.

and we obtain the following final result:

$$\Psi_{\mathrm{FW}}(\boldsymbol{r},t)=\sqrt{\frac{2}{1+1/E}}\exp\left[-i(Et-q_z z-l\phi)\right]\begin{bmatrix}[a(1+E-q_z)+ibq_\perp]J_l(q_\perp\rho)\\ [-iaq_\perp+b(1+E+q_z)]e^{i\phi}J_{l+1}(q_\perp\rho)\\ 0\\ 0\end{bmatrix}. \tag{5}$$

The probability densities in the Dirac representation $\varrho_D(\rho)$ and in the FW representation $\varrho_{\mathrm{FW}}(\rho)$ have the same structure,

$$\varrho(\rho)=cJ_l(q_\perp\rho)^2+dJ_{l+1}(q_\perp\rho)^2, \tag{6}$$

but the coefficients are quite different,

$$c_D=2|a|^2[1+(E-q_z)^2]+2|b|^2q_\perp^2+4\mathrm{Im}(ab^*)q_\perp(E-q_z), \tag{7a}$$

$$d_D=2|b|^2[1+(E+q_z)^2]+2|a|^2q_\perp^2+4\mathrm{Im}(ab^*)q_\perp(E+q_z), \tag{7b}$$

$$c_{\mathrm{FW}}=[2|a|^2(E-q_z+1)^2+2|b|^2q_\perp^2+4\mathrm{Im}(ab^*)q_\perp(E-q_z+1)]/(1+1/E), \tag{7c}$$

$$d_{\mathrm{FW}}=[2|b|^2(E+q_z+1)^2+2|a|^2q_\perp^2+4\mathrm{Im}(ab^*)q_\perp(E+q_z+1)]/(1+1/E). \tag{7d}$$

In summary, the use of the Foldy-Wouthuysen representation in [1] for relativistic beams is problematic because, as illustrated in Fig. 1, the beam shape (charge distribution) is not correctly reproduced.

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