Exactly Solvable Record Model for Rainfall

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Daily precipitation time series are composed of null entries corresponding to dry days and nonzero entries that describe the rainfall amounts on wet days. Assuming that wet days follow a Bernoulli process with success probability p, we show that the presence of dry days induces negative correlations between record-breaking precipitation events. The resulting nonmonotonic behavior of the Fano factor of the record counting process is recovered in empirical data. We derive the full probability distribution $P(R,n)$ of the number of records R_n up to time n, and show that for large n, it converges to a Poisson distribution with parameter $\ln(pn)$. We also study in detail the joint limit $p \to 0$, $n \to \infty$, which yields a random record model in continuous time $t = pn$.

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It is widely expected that global climate change leads to an increase in the frequency of extreme weather conditions such as heat waves, droughts, and heavy precipitation [1–5]. The public perception of weather extremes is particularly sensitive to record-breaking events, which often receive extensive media coverage. Records are extremes that are defined not relative to some threshold value, but relative to all preceding events. Their analysis provides a useful tool for the distribution-free inference of trends, because the temporal record statistics of sequences of independent random variables drawn from a continuous probability distribution is manifestly universal [6–15]. This observation has motivated a number of recent studies aimed at detecting and quantifying the effects of a warming climate on the frequency of temperature records [16–23].

In comparison, the effects of climatic trends on precipitation records are more complex and have generally received less attention [24–26]. To detect such trends, the null model describing a stationary climate has to account for the specific structure of a precipitation time series. In contrast to temperature, which is well described as a Gaussian random variable [18,23], the amount of daily rainfall at a specific location has a positive probability of being exactly zero. Stochastic precipitation models incorporate this feature by combining an *occurrence process* that determines whether a given day is dry (zero precipitation) or wet (nonzero precipitation) with an *amount process* that specifies the amount of rainfall on a wet day [27].

In this Letter we show that the presence of dry days has a profound effect on the occurrence statistics of precipitation records in a stationary climate. Assuming that the wet days follow a Bernoulli process with success probability p, we find that record events become negatively correlated when $p < 1$. This is in marked contrast to the well-known property of record events from sequences of independent, identically and continuously distributed (IICD) random variables to be stochastically independent [8,13–15]. As a consequence, we show below that the ratio of the variance and the mean of the record counting process, known as the Fano factor, displays a minimum at intermediate times when $q = 1 - p$ is sufficiently large. This minimum is an unequivocal signature of correlations between record events, and we demonstrate that it can be clearly identified in empirical data. For this comparison we use time series comprising rainfall amounts on a given calendar day over several decades, which justifies the assumption of uncorrelated occurrence and amount processes. We expect that the mechanism giving rise to correlations in the Bernoulli model is of broader relevance also beyond the specific context of precipitation records, and provide a detailed analysis of the model including the full distribution of the number of records.

*Bernoulli model.—*Within the Bernoulli model a dry day with zero precipitation occurs with probability q, and a wet day with probability $p = 1 - q$. For a wet day, the amount of precipitation x is a random variable drawn from a continuous probability density $p_W(x)$ with support on the positive real axis. The full probability density of precipitation x on any randomly selected day thus reads

$$p(x) = q\delta(x) + (1 - q)p_W(x). \quad (1)$$

The δ function at $x = 0$ implies that the corresponding cumulative distribution function

$$P(x) = \int_0^x dx' p(x') = q\theta(x) + (1 - q)P_W(x) \quad (2)$$

is discontinuous at the origin, as indicated by the Heaviside theta function. We are interested in the statistics of the number of record events R_n that have occurred up to time n. It is convenient to introduce a binary indicator variable σ_m for the mth day such that $\sigma_m = 1$ if a record occurs on the mth day, and $\sigma_m = 0$ otherwise. Clearly

$$R_n = \sum_{m=1}^{n} \sigma_m. \tag{3}$$

We note one important point: If a record occurs on the mth day, then the mth day is necessarily wet.

The expected number of records is given by

$$\langle R_n \rangle = \sum_{m=1}^{n} \langle \sigma_m \rangle = \sum_{m=1}^{n} r_m, \tag{4}$$

where the record rate r_m denotes the probability that a record occurs on the mth day. The latter, assuming that the entries of the time series are independent, is given by

$$r_m = (1-q)\int_0^\infty dx\, p_W(x) P(x)^{m-1}, \tag{5}$$

with the following interpretation: the probability density that the mth day is wet with precipitation $x > 0$ is $(1-q)p_W(x)$, and in order for this to be a record all the previous $m-1$ days must have precipitation less than x. To perform the integration we make the substitution $x \to u = P(x)$, noting that $u \in [q, 1]$ and $du = (1-q)p_W(x)dx$ for $x > 0$. The resulting expression

$$r_m = \int_q^1 du\, u^{m-1} = \frac{1-q^m}{m} \tag{6}$$

is independent of the distribution $p_W(x)$ and reduces to the classic result $r_m = 1/m$ for IICD random variables when $q \to 0$. Correspondingly, the expected number of records up to day n is given by

$$\langle R_n \rangle = \sum_{m=1}^{n} \frac{1-q^m}{m}. \tag{7}$$

For large n and fixed $q = 1-p$, it is easy to show that $\langle R_n \rangle \approx \ln(pn) + \gamma_E$, where $\gamma_E = 0.57721\ldots$ is the Euler constant (see [28] for details). Thus, at late times the record sequence looks like a "diluted" IICD record process where the effective number of random variables that have been presented up to time n is reduced by a factor p. We will see below that this observation applies also to the variance as well as to the full distribution of R_n.

To compute the second moment of R_n, we square and average Eq. (3), using that $\sigma_m^2 = \sigma_m$. This gives

$$\langle R_n^2 \rangle = \langle R_n \rangle + 2\sum_{l_1=1}^{n-1}\sum_{l_2=1}^{n-l_1} \langle \sigma_{l_1}\sigma_{l_1+l_2} \rangle, \tag{8}$$

where $\langle \sigma_{l_1}\sigma_{l_1+l_2} \rangle$ is the joint probability of two records occurring on day l_1 and $l_1 + l_2$. To compute this, let the record at day l_1 have value x_1 and the one at $l_1 + l_2$ have value x_2 with $x_2 > x_1$. Evidently, both days have to be necessarily wet. All the days before l_1 must have precipitation values less than x_1, and all the days between l_1 and $l_1 + l_2$ must have precipitation values less than x_2. Writing down the corresponding probability in analogy to Eq. (5) and performing the substitution $x \to P(x)$ (see [28]) leads to the simple form

$$\langle \sigma_{l_1}\sigma_{l_1+l_2} \rangle = \int_q^1 du_2 \int_q^{u_2} du_1 u_1^{l_1-1} u_2^{l_2-1}, \tag{9}$$

with $l_2 \geq 1$. For $l_2 = 0$, $\langle \sigma_{l_1}^2 \rangle = r_{l_1}$. Combining Eq. (9) with the result (6) for the record rate yields the connected correlation function of record events,

$$\begin{aligned} g_{l_1,l_1+l_2} &\equiv \langle \sigma_{l_1}\sigma_{l_1+l_2} \rangle - r_{l_1} r_{l_1+l_2} \\ &= -\frac{q^{l_1}}{l_1}\int_q^1 du\, u^{l_2-1}(1-u^{l_1}) \\ &= -\frac{q^{l_1}}{l_1}\left(\frac{1-q^{l_2}}{l_2} - \frac{1-q^{l_1+l_2}}{l_1+l_2}\right), \end{aligned} \tag{10}$$

which is universal [independent of $p_W(x)$] for all $l_1 \geq 1$ and $l_2 \geq 1$. The second equality in Eq. (10) manifestly shows that the correlation is negative for all l_1, l_2, and $0 < q < 1$. Thus, the record events become anticorrelated when $q > 0$. The origin of these correlations ultimately lies in the discontinuity of the distribution function (2), which reduces the domain of integration in Eqs. (6) and (9) compared to the IICD case. A simple explanation can be provided for $n = 2$ [28]. If the first day is wet it is a record by definition, and the same holds for the second day if the first day is dry. However, if both days are wet, the second day is a record only if the corresponding precipitation amount exceeds that of the first day, which is true with probability $\frac{1}{2}$. Thus, the presence of a record on the first day reduces the record probability on the second day.

For fixed l_1 and large l_2, the connected correlation function decays as a power law, $g_{l_1,l_1+l_2} \sim -q^{l_1}/[l_2(l_1+l_2)] \sim l_2^{-2}$. This indicates that the record breaking events are rather strongly correlated. Inserting Eq. (9) into Eq. (8) and performing the double sum yields, after a substantial amount of algebra, the expression [28]

$$\begin{aligned} V_n(q) &= \langle R_n^2 \rangle - \langle R_n \rangle^2 = \langle R_n \rangle \\ &\quad + 2\int_q^1 \frac{du\, u^n}{1-u}\left(\int_q^u dv \frac{1-v^n}{1-v} - \int_{q/u}^1 dv \frac{1-v^n}{1-v}\right) \end{aligned} \tag{11}$$

for the variance of the number of records up to day n. Asymptotically for large n with fixed q, we find that [28] $V_n(q) \to \langle R_n \rangle - \pi^2/6 \approx \ln(pn) + \gamma_E - \pi^2/6$.

Random record model.—In order to arrive at a more tractable expression for V_n, we now analyze the problem in the scaling limit $p \to 0$, $n \to \infty$ at fixed $t = pn$. In this limit the Bernoulli sequence of wet days becomes a Poisson process of unit intensity in continuous time t. In the mathematical literature this setting is known as the random record model [8,29,30], see also [31,32]. For the expected number of records (7) the limit $q \to 1$, $n \to \infty$ yields $\langle R_n \rangle \to \mu(pn)$ with

$$\mu(t) = \int_0^t dy \frac{1 - e^{-y}}{y} = \ln t + \gamma_E + \int_t^\infty \frac{e^{-z}}{z} dz, \qquad (12)$$

where the last identity can be found in [33]. The asymptotics are $\mu(t) \to t - t^2/4$ as $t \to 0$ and $\mu(t) \to \ln t + \gamma_E$ as $t \to \infty$. Thus, the scaling function describes a crossover in the expected number of records from an early time linear growth $\langle R_n \rangle \approx pn$ where the number of records is limited by the number of events, to a late time logarithmic growth $\langle R_n \rangle \approx \ln(pn) + \gamma_E$. Taking the scaling limit of the expression (11) is not straightforward, but eventually leads to the relatively simple form

$$V_n(q) \to \mu(t) + 2 \int_0^t \frac{dz}{z} e^{-z} [\mu(t) - \mu(z) - \mu(t-z)], \qquad (13)$$

where $\mu(t)$ is given in Eq. (12) (see [28]).

Fano factor.—To quantify the correlations between record events, we introduce the Fano factor [34] or index of dispersion [35] defined as the ratio of the variance to the mean of the record counting process, $F_n = V_n / \langle R_n \rangle$. This ratio measures the deviation of the counting process from a Poisson process, for which $F_n = 1$. We first prove that F_n, for an arbitrary time series, must be an increasing function of n if record events are uncorrelated. Let $\langle \sigma_m \rangle = r_m$ denote the record rate at step m. In the absence of correlations between record events, $\langle \sigma_l \sigma_m \rangle = r_m \delta_{l,m} + r_l r_m (1 - \delta_{l,m})$, which implies using (8) that $V_n = \sum_{m=1}^n r_m (1 - r_m)$. As a consequence $F_{n+1} - F_n = S_n / \langle R_n \rangle - S_{n+1} / \langle R_{n+1} \rangle$, where $S_n = \sum_{m=1}^n r_m^2$. Based on this, it is easy to show that $F_{n+1} - F_n > 0$ provided $r_{n+1} < r_m$ for all $m \le n$, which only requires the record rate to be monotonically decreasing. Thus a nonmonotonic behavior of F_n is an unambiguous signature of correlations.

Using the results from Eqs. (12) and (13), we find that in the scaling limit the Fano factor converges to a scaling form, $F_n(q) \to F(t = pn)$ with

$$F(t) = 1 + \frac{2}{\mu(t)} \int_0^t \frac{dz}{z} e^{-z} [\mu(t) - \mu(z) - \mu(t-z)]. \qquad (14)$$

The scaling function $F(t)$ is clearly nonmonotonic, showing that the strong correlations between record events persist in the scaling limit (Fig. 1). It starts at $F(0) = 1$, decreases with increasing t, reaches a minimum around $t^* \approx 4.4$, and converges slowly back to $F = 1$ as $t \to \infty$. Its asymptotic behaviors can be easily computed from the exact expression in Eq. (14), and we obtain $F(t) \to 1 - t/2 + O(t^2)$ as $t \to 0$ and $F(t) \to 1 - \pi^2/(6 \ln t)$ as $t \to \infty$. The figure also shows estimates for F_n at finite $p > 0$ obtained from simulations. The minimum is even more pronounced at positive p, and the simulation results are indistinguishable from the asymptotic prediction (14) for $p = 0.02$.

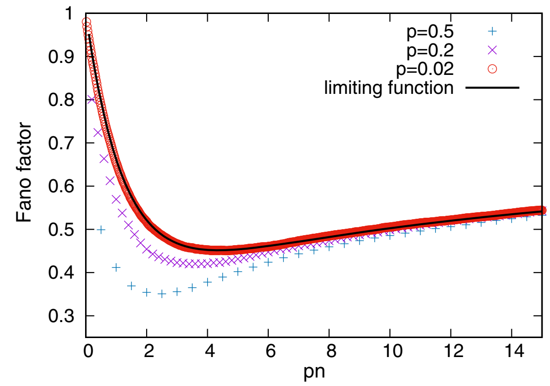

FIG. 1. The Fano factor of the record process obtained from simulations (symbols) is compared to the analytic limit function $F(t)$ in Eq. (14) (full line). Note that the numerical estimates start at $F_1 = 1 - p$.

Comparison to precipitation data.—In order to test the predictions of the Bernoulli model we analyzed a large set of daily precipitation data compiled by the German weather service (DWD). The full data set comprises rainfall amounts from 5400 weather stations positioned throughout Germany. Out of these, 417 stations were selected which provided complete daily precipitation time series for the period 1974 to 2013 [36]. The average rainfall probability for this data set is close to $p = 0.5$ with some variability between stations. In order to minimize the effects of the variability in p, we further restricted the analysis to those stations where the time-averaged precipitation probability lies in the interval $p \in [0.48, 0.52]$. This leaves 144 stations covering the 40 year period. For each station we extracted 365 time series corresponding to precipitation amounts on a given calendar day.

Figure 2 shows the Fano factor of the number of precipitation records obtained from the empirical data, compared to simulations of the Bernoulli model with $p = 0.5$. The simulation data were averaged over 5×10^4 runs, which is close to the total number of empirical time series ($144 \times 365 = 52560$). We have checked that allowing p to vary over the interval [0.48, 0.52] in the simulations does not significantly affect the results. The empirically determined Fano factor displays a pronounced minimum and the overall shape is in good agreement with the model. The remaining discrepancy at longer times is probably not of a statistical nature and could be related to features that are ignored in the model, such as spatial

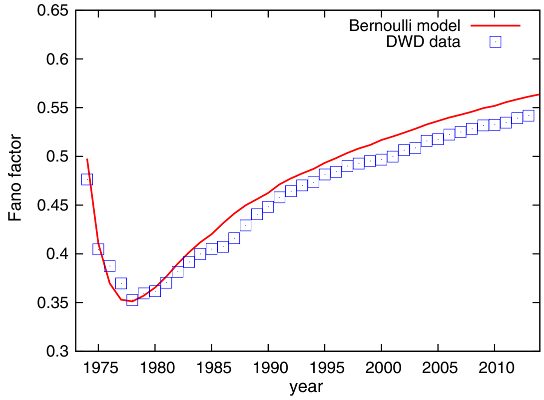

FIG. 2. Blue squares show the Fano factor of precipitation records estimated from daily rainfall amounts at 144 German weather stations. For comparison, the full line shows simulation results obtained from the Bernoulli model with the average rainfall probability $p = 0.5$. A similar analysis covering the time period 1961 to 2013 can be found in [28].

correlations between weather stations or trends in the model parameters [36,37] (see also [18,23,38] for a discussion of trends in record processes).

Distribution of the number of records.—Having derived the mean and the variance of the record number R_n, one may naturally investigate its full distribution $P(R,n) = \mathbb{P}[R_n = R]$. Exploiting the renewal structure of the record process in the Bernoulli model, we were able to derive a compact exact expression for the double generating function (see [28])

$$\sum_{n=0}^{\infty}\sum_{R=0}^{\infty} P(R,n)\lambda^R z^n = \frac{(1-qz)^{\lambda-1}}{(1-z)^{\lambda}}. \tag{15}$$

For $q = 0$, the right-hand side reduces to $(1-z)^{-\lambda}$, a known result for the IICD case [39–43]. From Eq. (15), one can in principle compute all the moments. Moreover, by analyzing Eq. (15) for large n (with fixed $p = 1 - q$ and $R \geq 1$), we can show (see [28]) that $P(R,n)$ converges to the Poisson distribution

$$P(R,n) \approx \frac{1}{pn}\frac{[\ln(pn)]^{R-1}}{(R-1)!}. \tag{16}$$

We conclude that the record occurrence events become a Poisson process in "time" $\ln(pn)$ for large n, as was observed previously for the IICD case $q = 0$ [42,44].

Interestingly, in the limit $R \to \infty$, $n \to \infty$, but keeping the ratio $x = R/\ln(pn)$ fixed, the Poisson distribution in Eq. (16) admits a large deviation form

$$P(R,n) \sim e^{-\ln(pn)\Phi\{R/[\ln(pn)]\}} \tag{17}$$

with an explicit rate function

$$\Phi(x) = 1 - x + x\ln x; \quad x \geq 0. \tag{18}$$

Typically in statistical physics problems one finds large deviation principles of the form $\sim\exp[-L\Phi(R/L)]$, where L represents the "size" of the system. In the present problem, the effective size L is not n, but rather the average number of records $\langle R_n\rangle \sim \ln(pn)$. Similar "anomalous" large deviation forms appeared before in the context of the distribution of the number of zero crossings of smooth Gaussian fields or equivalently in the distribution of the number of real roots of a class of random polynomials [45–47], and more recently in the distribution of entanglement in random quantum spin chains [48]. The rate function (18) is independent of q. Typical fluctuations of R_n are described by the quadratic approximation of the rate function $\Phi(x)$ around its unique minimum at $x^* = 1$. Substituting this quadratic form in Eq. (17), we find that the typical fluctuations are described by a Gaussian with mean and variance $\ln(pn)$. Thus, despite the power law correlations between the indicator variables σ_m, their sum $R_n = \sum_{m=1}^{n}\sigma_m$ satisfies a central limit theorem.

Conclusions.—Motivated by the statistics of rainfall, we have investigated a simple extension of the classic IICD record problem where the non-negative random variables forming the time series take the value zero with a positive probability $q > 0$. Our key finding is that this induces long-ranged correlations between record events, which lead to a pronounced minimum in the Fano factor of the record counting process. Importantly, our conclusions would not change if the dry days were replaced by days with nonzero precipitation amounts that are too small to be detected [28]. The effect that we describe therefore applies quite generally to situations where a fraction of events is not recorded because of measurement uncertainty or limited experimental resolution [49]. The emergence of correlations between record events has been observed previously, e.g., for records drawn from distributions that broaden [11] or shift [12,38,50] in time, or as a consequence of rounding effects [51]. Taken together, these results highlight the fact that the stochastic independence between record events in the standard IICD setting is a highly nongeneric and fragile feature.

The comparison with the empirical data in Fig. 2 shows that the Bernoulli model qualifies as a null model for precipitation time series comprising daily rainfall amounts on a given calendar day over a sequence of years. It could also be applied to data aggregated over weeks or months, which would however reduce the probability of dry events and correspondingly the strength of the anticorrelation. On the other hand, the model clearly fails to describe the time series of rainfall amounts on consecutive days, which are characterized by strongly correlated spells of dry and wet days arising from large-scale weather patterns. This kind of data can be modeled by an alternating renewal process, where dry and wet spell lengths are drawn independently from two different probability distributions [27]. The record occurrence statistics is then again universal with respect to the amount distribution $p_W(x)$ but depends explicitly on the spell length distributions. Results for this model will be reported elsewhere [52].

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[1] D. R. Easterling, G. A. Meehl, C. Parmesan, S. A. Changnon, T. R. Karl, and L. O. Mearns, Science **289**, 2068 (2000).
[2] J. Hansen, M. Sato, and R. Ruedy, Proc. Natl. Acad. Sci. U.S.A. **109**, E2415 (2012).
[3] D. Coumou and S. Rahmstorf, Nat. Clim. Change **2**, 491 (2012).
[4] E. M. Fischer and R. Knutti, Nat. Clim. Change **5**, 560 (2015).
[5] B. I. Cook, T. R. Ault, and J. E. Smerdon, Sci. Adv. **1**, e1400082 (2015).
[6] F. G. Foster and A. Stuart, J. R. Stat. Soc. Ser. B **16**, 1 (1954).
[7] N. Glick, Am. Math. Mon. **85**, 2 (1978).
[8] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *Records* (Wiley, New York 1998).
[9] B. Schmittmann and R. K. P. Zia, Am. J. Phys. **67**, 1269 (1999).
[10] R. E. Benestad, Glob. Planet. Change **44**, 11 (2004).
[11] J. Krug, J. Stat. Mech. (2007) P07001.
[12] J. Franke, G. Wergen, and J. Krug, Phys. Rev. Lett. **108**, 064101 (2012).
[13] G. Wergen, J. Phys. A **46**, 223001 (2013).
[14] G. Schehr and S. N. Majumdar, Exact record and order statistics of random walks via first-passage ideas, in *First-Passage Phenomena and Their Applications*, edited by R. Metzler, G. Oshanin, and S. Redner (World Scientific, Singapore, 2013), p 226–251.
[15] C. Godrèche, S. N. Majumdar, and G. Schehr, J. Phys. A **50**, 333001 (2017).
[16] S. Redner and M. R. Petersen, Phys. Rev. E **74**, 061114 (2006).
[17] G. A. Meehl, C. Tebaldi, G. Walton, D. Easterling, and L. McDaniel, Geophys. Res. Lett. **36**, L23701 (2009).
[18] G. Wergen and J. Krug, Europhys. Lett. **92**, 30008 (2010).
[19] S. Rahmstorf and D. Coumou, Proc. Natl. Acad. Sci. U.S.A. **108**, 17905 (2011).
[20] C. M. Rowe and L. E. Derry, Geophys. Res. Lett. **39**, L16703 (2012).
[21] D. Coumou, A. Robinson, and S. Rahmstorf, Clim. Change **118**, 771 (2013).
[22] B. Christiansen, J. Clim. **26**, 7863 (2013).
[23] G. Wergen, A. Hense, and J. Krug, Clim. Dyn. **42**, 1275 (2014).
[24] R. E. Benestad, Climate research **25**, 3 (2003).
[25] R. E. Benestad, J. Clim. **19**, 630 (2006).
[26] J. Lehmann, D. Coumou, and K. Frieler, Clim. Change **132**, 501 (2015).
[27] D. S. Wilks and R. L. Wilby, Prog. Phys. Geography **23**, 329 (1999).
[28] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.158702 for the details of the calculations described in the main text.
[29] M. Westcott, Proc. R. Soc. A **356**, 529 (1977).
[30] J. A. Bunge and H. N. Nagaraja, J. Appl. Probab. **29**, 587 (1992).
[31] J. Davidsen, P. Grassberger, and M. Paczuski, Phys. Rev. E **77**, 066104 (2008).
[32] S.-C. Park and J. Krug, J. Stat. Mech. (2008) P04014.
[33] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1979), see identity 8.212 in p. 925.
[34] U. Fano, Phys. Rev. **72**, 26 (1947).
[35] D. R. Cox and P. A. W. Lewis, *The Statistical Analysis of Series of Events* (Wiley, New York, 1966).
[36] P. von Bomhard, Rekordstatistik von Niederschlägen, Bachelor thesis, University of Cologne, 2014.
[37] P. von Bomhard, Development of a statistical daily precipitation model and its application to precipitation records, Master thesis, University of Cologne, 2017, available online at https://kups.ub.uni-koeln.de/9185/.
[38] J. Franke, G. Wergen, and J. Krug, J. Stat. Mech. (2010) P10013.
[39] A. Rényi, Théorie des éléments saillants d'une suite d'observations, in *Colloquium on Combinatorial Methods in Probability Theory* (Mathematical Institute, Aarhus University, Aarhus, Denmark, 1962), pp. 104–117.
[40] C. M. Goldie, Math. Proc. Cambridge Philos. Soc. **106**, 169 (1989).
[41] P. Flajolet and R. Sedgewick, *Analytic Combinatorics* (Cambridge University Press, Cambridge, England, 2009).
[42] C. Godrèche and J. M. Luck, J. Stat. Mech. (2008) P11006.
[43] S. N. Majumdar, K. Mallick, and S. Sabhapandit, Phys. Rev. E **79**, 021109 (2009).
[44] P. Sibani, M. Brandt, and P. Alstrøm, Int. J. Mod. Phys. B **12**, 361 (1998).
[45] G. Schehr and S. N. Majumdar, Phys. Rev. Lett. **99**, 060603 (2007).
[46] G. Schehr and S. N. Majumdar, J. Stat. Phys. **132**, 235 (2008).
[47] M. Poplavskyi and G. Schehr, Phys. Rev. Lett. **121**, 150601 (2018).
[48] T. Devakul, S. N. Majumdar, and D. A. Huse, Phys. Rev. B **95**, 104204 (2017).
[49] Y. Edery, A. B. Kostinski, S. N. Majumdar, and B. Berkowitz, Phys. Rev. Lett. **110**, 180602 (2013).
[50] G. Wergen, J. Franke, and J. Krug, J. Stat. Phys. **144**, 1206 (2011).
[51] G. Wergen, D. Volovik, S. Redner, and J. Krug, Phys. Rev. Lett. **109**, 164102 (2012).
[52] S. N. Majumdar, P. von Bomhard, and J. Krug (to be published).