Ubiquity of Superconducting Domes in the Bardeen-Cooper-Schrieffer Theory with Finite-Range Potentials

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Based on recent progress in mathematical physics, we present a reliable method to analytically solve the linearized Bardeen-Cooper-Schrieffer (BCS) gap equation for a large class of finite-range interaction potentials leading to s-wave superconductivity. With this analysis, we demonstrate that the monotonic growth of the superconducting critical temperature T_c with the carrier density n predicted by standard BCS theory, is an artifact of the simplifying assumption that the interaction is quasilocal. In contrast, we show that *any* well-defined nonlocal potential leads to a "superconducting dome," i.e., a nonmonotonic $T_c(n)$ exhibiting a maximum value at finite doping and going to zero for large n. This proves that, contrary to conventional wisdom, the presence of a superconducting dome is not necessarily an indication of competing orders, nor of exotic superconductivity.

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Introduction.—It is well known that the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [1] predicts a critical temperature that increases monotonically with the density of quasiparticles. However, since the discovery of high-temperature superconductors, a growing number of superconducting systems have been revealed to possess critical temperatures, T_c, that have a nonmonotonic dependence on either the carrier density or pressure, including $SrTiO_3$ [2,3], the cuprates [4–7], the pnictides [8], and heavy fermion superconductors [9]. This nonmonotonic critical temperature presents itself as a dome of superconductivity in the phase diagram of these systems, and in many cases these domes appear to occur in the neighborhood of a quantum critical point (QCP) [10,11]. This concomitance is so prevalent that it has resulted in the often quoted rule that beneath every dome there is a QCP of some critical order. In some systems, this is likely the case because the presence of a QCP can induce soft bosonic excitations which can act as a "glue" leading to the formation of a superconducting state. However, superconducting domes have also been observed in doped band insulators [12] and magic-angle graphene superlattices [13] with no sign of competing orders. In these cases, a more conventional explanation may be necessary.

One reason for the incredible success of so many predictions of BCS theory can be attributed to *universality*, that is, certain predictions of the theory are independent of model details and thus accurately predicted by simplified models [14]. Famous examples of such universal features of BCS theory include [15] the ratio of the superconducting gap at zero temperature to the critical temperature: $\Delta(0)/T_c \approx 1.76$; and the temperature dependence of the gap for temperatures close to T_c: $\Delta(T)/T_c \approx 3.07\sqrt{1-T/T_c}$ (we set $k_B = \hbar = 1$ throughout this Letter). This being said, T_c is nonuniversal, and accurate predictions of T_c are notoriously difficult; see e.g., Ref. [16] for a classic reference and Ref. [17] for a recent discussion. Therefore, it is not clear, *a priori*, what universal statements can be made about the dependence of T_c on doping or other control parameters. However, recent developments in mathematical physics [18,19] have significantly improved the mathematical toolkit we can use to extract reliable analytic results for critical temperatures from BCS-like theories.

In this Letter, we take advantage of these recent mathematical insights [19] to address the general question: when do superconducting domes arise in isotropic BCS models? Surprisingly, we find that superconducting domes arise ubiquitously whenever the electron-electron interaction responsible for the superconductivity has nontrivial spatial dependence and satisfies certain convergence criteria. In this way, we show that the monotonic T_c predicted

by BCS theory is actually an artifact of the trivial spatial dependence of the interaction. Furthermore, we present analytic solutions of the linearized gap equation, applicable to a broad class of long-range BCS models, and show that the explicit T_c equation thus obtained is numerically accurate.

The ubiquity of these domes can be understood to arise from an interplay between the length scale determining the range of interaction, ℓ, and the average interparticle separation, $\sim k_F^{-1}$. At low densities, when $k_F^{-1} \gg \ell$, the interaction becomes effectively local, and the pairing is well described by standard BCS theory; in this regime, T_c grows as the density increases. Whereas, at high densities, when $\ell \gg k_F^{-1}$, the pairing between electrons at the Fermi surface becomes weaker with increasing k_F due to the decay of the interaction in Fourier space, which suppresses T_c toward zero. Therefore, in the crossover regime, where $\ell \sim k_F^{-1}$, a superconducting dome arises. This simple explanation of the physics of superconducting domes does not rely on quantum criticality or any other exotic physics. The only necessary ingredient is a mathematically well-behaved electron-electron interaction with nonzero spatial range.

From a mathematical point of view, our contribution is to extend recent results for T_c [19] from 0th order to arbitrary order in a small parameter expansion. This extension is of great importance for the assessment of the numerical accuracy of results obtained using these methods. Additionally, because we use simpler mathematical arguments, we hope that the present Letter can act as a bridge between the mathematical physics community working on BCS theory and the broader community of physicists working on superconductivity.

Generalized BCS model.—To study the superconducting critical temperature, we employ the standard quantum many-body Hamiltonian

$$H = \int \sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_{\sigma,\mathbf{r}} \left(-\frac{\nabla^2}{2m^*} - \mu \right) \psi_{\sigma,\mathbf{r}} d^3r + \frac{1}{2} \iint \sum_{\sigma,\sigma'=\uparrow,\downarrow} \psi^\dagger_{\sigma,\mathbf{r}} \psi^\dagger_{\sigma',\mathbf{r}'} V(|\mathbf{r}-\mathbf{r}'|) \psi_{\sigma',\mathbf{r}'} \psi_{\sigma,\mathbf{r}} d^3r d^3r', \tag{1}$$

where $\psi^\dagger_{\sigma,\mathbf{r}}$ ($\psi_{\sigma,\mathbf{r}}$) creates (annihilates) a fermion with spin σ at position $\mathbf{r}$, μ and m^* are the chemical potential and effective mass, respectively, and $V(r)$ is an attractive nonlocal interaction potential depending on the interparticle distance $r = |\mathbf{r}-\mathbf{r}'|$.

The standard textbook BCS model corresponds to the special case where the interaction is quasilocal in position space $V(\mathbf{r}) = -g\delta^3(\mathbf{r})$ with $g > 0$ the coupling strength (we write "quasilocal" because the strict local interaction leads to diverging integrals which need to be regularized, as discussed below). We generalize this approach by allowing for finite-range potentials $V(|\mathbf{r}|)$ of the form

$$V(r) = -g\ell^{-3} W(r/\ell), \tag{2}$$

where $W(x)$ is a function of the dimensionless variable $x = r/\ell \geq 0$ normalized so that $4\pi \int_0^\infty W(x) x^2 dx = 1$, and $\ell > 0$ is the length scale associated with the decay of the interaction in position space. Thus, in the limit $\ell \to 0$, one obtains the textbook BCS model, independent of the function $W(x)$.

In addition to simple normalization, in this Letter we assume that the functions $W(x)$ also satisfy the two technical conditions [19] that are as follows:

$$(i) \quad \hat{W}(q) \geq 0, \qquad (ii) \quad \int_0^\infty |W(x)|^p x^2 dx < \infty \quad \text{for } 1 \leq p \leq \frac{3}{2}, \tag{3}$$

where $\hat{W}(q) = (4\pi/q) \int_0^\infty W(x) \sin(xq) x dx$ is the Fourier transform of $W(x)$. To understand the significance of these conditions, we note that, although the model defined in Eq. (1) for a local potential always leads to s-wave superconductivity, this is not the case for nonlocal potentials; see Ref. [20] for counter examples. However, it is known that, if the Fourier transform of the pairing potential, $\hat{V}_{\mathbf{k},\mathbf{k}'}$, is nonpositive, one always obtains s-wave superconductivity [21,22]. This is equivalent to Eq. (3) (i). Eq. (3) (ii) guarantees that the BCS gap equation in Eq. (4) is well defined [21]; i.e., it rules out potentials that are too singular or that do not decay fast enough at large distances. Some familiar examples of functions that satisfy both of these criteria are the Gaussian distribution, the Lorentzian distribution, and the Yukawa potential (see Table I).

TABLE I. Examples of functions $W(x)$ determining finite-range potentials as in Eq. (2), together with their Fourier transforms $\hat{W}(q)$ and associated functions $f_W(\varepsilon)$.

	$W(x)$	$\hat{W}(q)$	$f_W(\varepsilon)$
Gaussian	$(2\pi)^{-3/2} e^{-x^2/2}$	$e^{-q^2/2}$	$2(1-e^{-\varepsilon/2})$
Lorentzian	$\pi^{-2}(1+x^2)^{-2}$	$e^{-\lvert q\rvert}$	$2[1-e^{-\sqrt{\varepsilon}}(1+\sqrt{\varepsilon})]$
Yukawa	$(4\pi x)^{-1} e^{-x}$	$(1+q^2)^{-1}$	$\ln(1+\varepsilon)$
k-box [23]	$(2\pi^2 x^3)^{-1}[\sin(x) - x\cos(x)]$	$\theta(1-\lvert q\rvert)$	$\varepsilon\theta(1-\varepsilon) + \theta(\varepsilon-1)$

T_c for finite-range potentials.—To find the superconducting critical temperature, T_c, associated with the model in Eq. (1), we will solve the linearized BCS gap equation [14,21]

$$\Delta(\epsilon,T) = -\int \hat{V}(\epsilon,\epsilon')N(\epsilon')\frac{\tanh\frac{\epsilon'}{2T}}{2\epsilon'}\Delta(\epsilon',T)d\epsilon', \quad (4)$$

where $\Delta(\epsilon,T)$ is the gap function, depending on energy $\epsilon = \epsilon_{\mathbf{k}} = (k^2/2m^*) - \mu$ and temperature T, $N(\epsilon) = [2m^*/(2\pi)^2]\theta(1+\epsilon/\mu)k(\epsilon)$ is the electronic density of states, and $\hat{V}(\epsilon,\epsilon')$ is the average of $\hat{V}_{\mathbf{k},\mathbf{k}'}$ over the energy surfaces $\epsilon = \epsilon_{\mathbf{k}}$ and $\epsilon' = \epsilon_{\mathbf{k}'}$. With these definitions, it is straightforward to show that $\hat{V}(\epsilon,\epsilon')$ is given by

$$\hat{V}(\epsilon,\epsilon') = \theta(1+\epsilon/\mu)\theta(1+\epsilon'/\mu) \times \frac{f_W\{\ell^2[k(\epsilon)+k(\epsilon')]^2\} - f_W\{\ell^2[k(\epsilon)-k(\epsilon')]^2\}}{4\ell^2 k(\epsilon)k(\epsilon')}, \quad (5)$$

where $\theta(x)$ is the Heaviside function, $k(\epsilon) = k_F\sqrt{1+\epsilon/\mu}$, $k_F = \sqrt{2m^*\mu}$ is the Fermi momentum, and $f_W(\varepsilon) = \int_0^{\varepsilon}\hat{W}(\sqrt{\varepsilon'})d\varepsilon'$ is a special function determined by the interaction potential. In many cases of interest, one can find simple explicit formulas for the function $f_W(\varepsilon)$; see Table I for examples.

To put this problem in perspective, recall that, for separable potentials of the form $\hat{V}(\epsilon,\epsilon') = -g\eta(\epsilon)\eta(\epsilon')$, the energy dependence of the gap is trivially determined by the potential $\Delta(\epsilon,T) = \Delta(T)\eta(\epsilon)$. Thus, after insertion into Eq. (4) and canceling $\Delta(T)$, one obtains an equation involving an integral of known functions and parameters that can be solved for T. This was the strategy employed by BCS in their seminal work [1], using $\eta(\epsilon) = \theta(\omega_D - |\epsilon|)$ where $\omega_D > 0$ is the Debye energy. As mentioned above, this can be interpreted as Eq. (4) with a local potential and an energy cutoff, ω_D, introduced to regularize a diverging integral. For nonlocal potentials satisfying Eq. (3), no such *ad hoc* regularization is needed: the integral in Eq. (4) is mathematically well defined [21]. However, the price we must pay is computational difficulty: to solve the gap equation in Eq. (4) for nonseparable potentials, one must keep track of the energy dependence of the gap.

Our main result is an explicit formula for T_c in terms of $f_W(\varepsilon)$, obtained by solving Eq. (4) analytically. As we will show in the next section, T_c is given by

$$T_c = \frac{2e^{\gamma}}{\pi}\mu\exp\left(-\frac{1}{\lambda} + a_0 + a_1\lambda + a_2\lambda^2 + \cdots\right), \quad (6)$$

where γ is the Euler-Mascheroni constant, $2e^{\gamma}/\pi \approx 1.13$, the coefficients a_n are given by Eqs. (16) and (17), and λ is a parameter defined as

$$\lambda = -N(0)\hat{V}(0,0) = \frac{2m^*}{(2\pi)^2}k_F g\frac{f_W([2k_F\ell]^2)}{[2k_F\ell]^2}. \quad (7)$$

As explained below, such an explicit formula for T_c can be obtained mainly because the energy scale for superconductivity is exponentially smaller than the chemical potential μ. This is true even in the low-density limit $\mu \to 0$ [24]. Indeed, it follows from our result that T_c/μ goes like $e^{-1/\lambda}$ with $\lambda \propto k_F^3 g/\mu$ vanishing like $\sqrt{\mu}$ as $\mu \to 0$, and if λ is sufficiently small, such corrections are negligible. We stress that λ can be small even in cases where the coupling strength, g, is large. It is an emergent small parameter in the problem whose maximum value occurs at a finite doping such that $k_{F,\max} = q_0/2\ell$, where q_0 is a numerical value determined only by the form of the interaction potential (see the Supplemental Material [25]).

Derivation of T_c equation.—We present our method for solving Eq. (4). This section can be skipped without loss of continuity if one is only interested in the results.

To solve Eq. (4), we start from the ansatz

$$\Delta(\epsilon) = -\hat{V}(\epsilon,0)N(0)\Delta(0)\log(\Omega_T(\epsilon)/T), \quad (8)$$

which serves as a definition of the function $\Omega_T(\epsilon)$:

$$\Omega_T(\epsilon) = T\exp\left(\int\frac{\tanh\frac{\epsilon'}{2T}}{2\epsilon'}G(\epsilon,\epsilon')d\epsilon'\right) \quad (9)$$

where $G(\epsilon,\epsilon') = \hat{V}(\epsilon,\epsilon')N(\epsilon')\Delta(\epsilon')/\hat{V}(\epsilon,0)N(0)\Delta(0)$. In a sense, all we have done is rewrite Eq. (4); however, using the definition of λ in Eq. (7), it is clear that T_c is given exactly by $T_c = \Omega_{T_c}(0)e^{-1/\lambda}$. Therefore, the main objective is now to obtain a general method for solving Eq. (9).

We now claim that $\Omega_T(\epsilon)$ has a well-defined limit $T \to 0$, and $\Omega_T(\epsilon)$ can be replaced by $\Omega_0(\epsilon)$ up to negligible corrections. To see this, consider the auxiliary quantity $\Omega_T^{(0)} = T\exp\{\int[\tanh(\epsilon'/2T)/2\epsilon']\theta(\mu-|\epsilon'|)d\epsilon'\}$. It is well known that $\Omega_T^{(0)} \to (2e^{\gamma}/\pi)\mu$ as $T \to 0$ [1,22], and it is easy to take the limit $T \to 0$ in the ratio $\Omega_T(\epsilon)/\Omega_T^{(0)}$. Thus

$$\Omega_0(\epsilon) = \frac{2e^{\gamma}}{\pi}\mu\exp\left(\int\frac{G(\epsilon,\epsilon') - \theta(\mu-|\epsilon'|)}{2|\epsilon'|}d\epsilon'\right), \quad (10)$$

which is well defined because the integrand remains finite as $\epsilon' \to 0$. It can be proven that $\log[\Omega_T(\epsilon)/\Omega_0(\epsilon)]$ vanishes like $(T/\mu)^2$ for small T/μ (see the Supplemental Material [25]). Because $T/\mu < T_c/\mu$ and T_c/μ is proportional to $e^{-1/\lambda}$, which is negligible for sufficiently small λ, we can replace $\Omega_T(\epsilon)$ by $\Omega_0(\epsilon)$ in the following.

Inserting Eq. (10) to Eq. (8), for $\epsilon = 0$, we can solve for T_c. Ignoring corrections $\propto e^{-1/\lambda}$, we find

$$T_c = \frac{2e^{\gamma}}{\pi}\mu \exp\left[-\frac{1}{\lambda} + \int \frac{G(0,\epsilon') - \theta(\mu - |\epsilon'|)}{2|\epsilon'|} d\epsilon'\right]. \quad (11)$$

To compute $G(0,\epsilon')$, we use

$$\frac{\Delta(\epsilon)}{\Delta(0)} = \frac{\hat{V}(\epsilon,0)}{\hat{V}(0,0)}\left(1 - \lambda \log\left[\frac{\Omega_{T_c}(\epsilon)}{\Omega_{T_c}(0)}\right]\right), \quad (12)$$

implied by Eq. (8) and $T_c = \Omega_{T_c} e^{-1/\lambda}$. Replacing $\log[\Omega_{T_c}(\epsilon)/\Omega_{T_c}(0)]$ by $\log[\Omega_0(\epsilon)/\Omega_0(0)]$, ignoring corrections $\propto e^{-1/\lambda}$, we can write this as

$$\frac{\Delta(\epsilon)}{\Delta(0)} = \frac{\hat{V}(\epsilon,0)}{\hat{V}(0,0)} + \lambda \int K(\epsilon,\epsilon')\frac{\Delta(\epsilon')}{\Delta(0)} d\epsilon' \quad (13)$$

with the integral kernel

$$K(\epsilon,\epsilon') = \frac{\hat{V}(\epsilon,0)}{\hat{V}(0,0)}\frac{1}{2|\epsilon'|}\left[\frac{\hat{V}(\epsilon,\epsilon')}{\hat{V}(\epsilon,0)} - \frac{\hat{V}(0,\epsilon')}{\hat{V}(0,0)}\right]\frac{N(\epsilon')}{N(0)}. \quad (14)$$

This integral equation can be solved by the following iteration: $\Delta(\epsilon)/\Delta(0) = F_0(\epsilon) + F_1(\epsilon)\lambda + F_2(\epsilon)\lambda^2 + \cdots$ with

$$F_0(\epsilon) = \frac{\hat{V}(\epsilon,0)}{\hat{V}(0,0)}, \qquad F_n(\epsilon) = \int K(\epsilon,\epsilon')F_{n-1}(\epsilon')d\epsilon', \quad (15)$$

for $n = 1, 2, \ldots$. Recalling the definition of $G(0,\epsilon')$, we use Eq. (11) to obtain Eq. (6) with

$$a_0 = \int \frac{1}{2|\epsilon'|}\left(\frac{\hat{V}(0,\epsilon')N(\epsilon')}{\hat{V}(0,0)N(0)}F_0(\epsilon') - \theta(\mu - |\epsilon'|)\right)d\epsilon' \quad (16)$$

and

$$a_n = \int \frac{\hat{V}(0,\epsilon')N(\epsilon')}{\hat{V}(0,0)N(0)}\frac{F_n(\epsilon')}{2|\epsilon'|}d\epsilon' \quad (17)$$

for $n = 1, 2, \ldots$ (note that all $a_{n\geq 0}$ are well defined because the integrands in Eqs. (16) and (17) remain finite as $\epsilon' \to 0$).

Superconducting domes.—To gain some insight into the universal properties of the formula for T_c, Eq. (6), in Fig. 1 we plot T_c as a function of the chemical potential, μ, [26] for the four examples appearing in Table I, using the same coupling constant g for each case. Although the spatial dependence of the interaction, $W(x)$, obviously has a large effect on the finer structure of each phase diagram, clearly, all four examples exhibit superconducting domes. These domes appear, despite the fact that the examples possess wildly different spatial dependence, for example: the

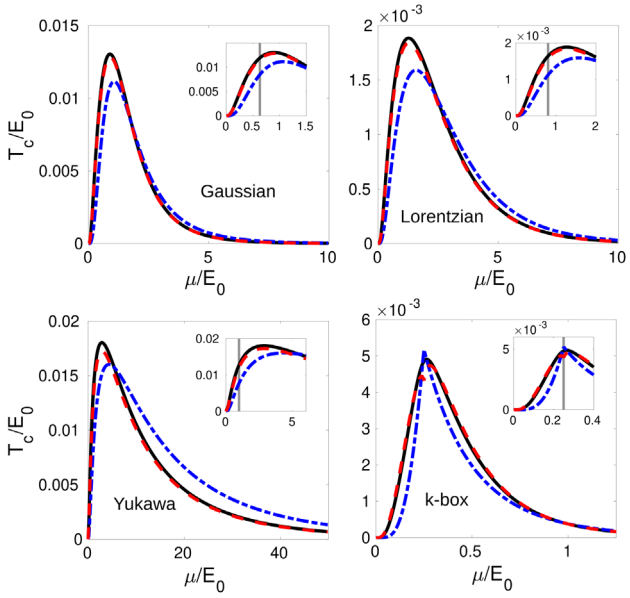

FIG. 1. Plots of the critical temperature T_c as a function of the chemical potential μ for the finite-range potential examples given in Table I. In each case we present (blue/dashed-dotted) $T_c^{(-1)}$, computed by truncating Eq. (6) at the $1/\lambda$ order; (red/dashed) $T_c^{(0)}$, computed by truncating Eq. (6) at the a_0 order; and (black/solid) in which the first order correction $a_1\lambda$ is included. In each case, the latter two curves agree remarkably well, and the first is a reasonable approximation capturing the qualitative behavior. All energies are reported in units of $E_0 = 1/2m^*\ell^2$ where ℓ is the interaction range, and the coupling constant is set so that $m^*g/(2\pi)^2\ell = 0.5$. The insets show the small μ (low concentration) behavior of the respective T_c with the vertical line indicating the μ value at which λ has its maximum.

Lorentzian distribution decays much more slowly in space, and the "k-box" potential is actually oscillatory in space.

The emergence of these superconducting domes is a direct consequence of the doping dependence of λ which is the product of two factors: the density of states at the Fermi level, $N(0)$, and the interaction strength between quasiparticles at the Fermi level, $\hat{V}(0,0)$. Each of these factors has different doping dependences, while $N(0)$ increases monotonically with doping, $\hat{V}(0,0)$ gets weaker at large doping, due to decay of the Fourier coefficients of the interaction potential at large momenta. This can be understood more rigorously by considering the second equality in Eq. (7), which implies $\lambda \propto m^*g\ell^{-1}f_W(\varepsilon)/\sqrt{\varepsilon}$ with $\varepsilon = (2k_F\ell)^2 = 4\mu/E_0$. One can show that $f_W(\varepsilon)/\sqrt{\varepsilon} \to 0$ in both limits $\varepsilon \to 0$ and $\varepsilon \to \infty$ provided the condition in Eq. (3) (ii) holds true. Because the behavior of T_c is dominated by the factor $e^{-1/\lambda}$ for small λ, this implies that T_c vanishes both in the low- and high-density limits, as described in the introduction. Therefore, we conclude that superconducting domes are ubiquitous in BCS theory with finite-range potentials such that the BCS gap equation is well defined.

It is interesting to note that the vanishing of T_c in the large-density limit is related to the short-distance behavior of the potential $V(r)$: in order for the ratio $f_W(\varepsilon)/\sqrt{\varepsilon}$ to not approach zero as $\varepsilon \to \infty$, $V(r)$ must have a $1/r^\alpha$-singularity at $r \to 0$ with $\alpha \geq 2$. However, such singular potentials violate Eq. (3) (ii), and, thus, the BCS equation in (4) is not mathematically well defined. This means that, for well-defined potentials with finite spatial range, the pairing between electrons at the Fermi surface becomes weaker at high doping due to the decay of the interaction in Fourier space.

Although superconducting domes are ubiquitous, the precise spatial dependence of the potential can be responsible for some significant features in the finer structure of the phase diagram. For example, in Fig. 1, we see that the different potentials give rise to critical temperatures of very different energy scales, differing by orders of magnitude for the four examples. This demonstrates another key point of our results: it is not just the coupling strength that determines the magnitude of T_c, the precise form of the interaction potential can make a huge difference. Additionally, we note that, in the case of the k-box potential, if we approximate the T_c formula by truncating the series in Eq. (6) at order -1 or 0, the phase diagram in Fig. 1 exhibits a cusp, resulting from the oscillatory spatial dependence of this potential. However, this cusp disappears at order 1, illustrating the point that, for certain potentials, the higher order corrections can be significant even for weak coupling.

Conclusions.—BCS theory and its generalization due to Eliashberg have provided a remarkably successful theoretical framework to describe many superconducting materials. Still, some properties of superconductors have remained difficult to explain from first principles, or even from a knowledge of the normal state. One such property is the superconducting critical temperature T_c. In this Letter, we have called attention to one microscopic detail that has limited the accuracy of T_c predictions: the spatial dependence of the pairing interaction. We presented a reliable method to take this into account in BCS theory, and we demonstrated its importance, both quantitatively and qualitatively.

Importantly, we showed that a nontrivial spatial dependence of the pairing interaction in BCS theory leads to superconducting domes. Although the exact scale of T_c depends sensitively on interaction details, we found a wide class of "reasonable" potentials which induce domes with $T_c(n)$ dependences that look remarkably similar to one another upon scaling.

The superconducting domes we find are controlled by the ratio of interparticle distance to the effective range of the potential, and they do not rely on any competing order or quantum critical fluctuations of the competing phases in the vicinity of superconducting state. These findings give greater confidence to the applicability of standard BCS results and establish the interesting possibility that superconducting domes can occur in simple BCS superconductors with no competing orders.

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[23] This example decays only like $1/r^2$ as $r \to \infty$ and thus violates Eq. (3) (ii). However, we have checked that our result is still well defined and numerically accurate. This is not the case for potentials which are more singular for $r \to 0$ than allowed by Eq. (3) (ii).
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[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.157001 for some mathematical details used in our derivation of the T_c equation, together with numerical results exploring the accuracy of various approximations to the full T_c equation.
[26] We find it convenient to plot T_c as a function of the chemical potential μ. It is easy to convert μ into n if T_c is much smaller than all other energy scales: One then can identify μ with $k_F^2/2m^*$ and n with $4\pi k_F^3/3$ (volume of Fermi sphere), and thus $n/n_0 = (\mu/E_0)^{3/2}$. For larger T_c values there is a more complicated relation between μ and n.