Editors' Suggestion

# Free-Surface Variational Principle for an Incompressible Fluid with Odd Viscosity

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We present variational and Hamiltonian formulations of incompressible fluid dynamics with a free surface and nonvanishing odd viscosity. We show that within the variational principle the odd viscosity contribution corresponds to geometric boundary terms. These boundary terms modify Zakharov's Poisson brackets and lead to a new type of boundary dynamics. The modified boundary conditions have a natural geometric interpretation describing an additional pressure at the free surface proportional to the angular velocity of the surface itself. These boundary conditions are believed to be universal since the proposed hydrodynamic action is fully determined by the symmetries of the system.

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*Introduction.*—The variational principle in hydrodynamics has a long history. We refer to Refs. [1,2] and references therein for an introduction to the topic. In particular, Luke's variational principle (LVP) is a variational principle of an inviscid and incompressible fluid with a free surface [3,4]. LVP provides both bulk hydrodynamic equations for an irrotational flow as well as kinematic and dynamic boundary conditions at the free surface boundary [4]. Such a principle was later extended to include surface tension and bulk vorticity (for a recent summary see Ref. [5]). In this Letter, we present a further extension of LVP which accounts for the presence of odd viscosity in isotropic two-dimensional fluids with broken parity.

In three dimensions, parity odd terms in the viscosity tensor have been known for a long time in the context of plasma in a magnetic field [6] and in hydrodynamic theories of superfluid He-3*A* [7], where the fluid anisotropy plays a major role. In two dimensions, however, the odd viscosity is compatible with isotropy of the fluid [8]. The odd viscosity is the parity violating nondissipative part of the stress-strain rate response of a two-dimensional fluid. The recent interest in odd viscosity is motivated by the seminal paper by Avron, Seiler, and Zograf [9] where it was shown that, in general, quantum Hall states have nonvanishing odd viscosity. The role of odd viscosity (a.k.a. Hall viscosity) in the context of quantum Hall effect has been an active area of research [10–33], but is out of the scope of this work.

In Ref. [8], Avron has initiated the search for odd viscosity effects in classical 2D hydrodynamics. These effects are subtle in the case when the classical two-dimensional fluid is incompressible. Recent works have outlined some of observable consequences of the odd viscosity for incompressible flows [34–39]. In particular, in Ref. [39] the equations governing the Hamiltonian dynamics of surface waves were derived in the case where bulk vorticity is absent.

Let us start by summarizing the main equations of an incompressible fluid dynamics with odd viscosity. In the following we assume that the fluid density is constant and take it as unity. We also neglect all thermal effects. Then, the hydrodynamic equations are the incompressibility condition and the Euler equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0, \tag{1}$$

$$\partial_t\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=\boldsymbol{\nabla}\otimes\boldsymbol{T}. \tag{2}$$

Here, $\boldsymbol{v}(\boldsymbol{x},t)$ is a two-component velocity vector field and $\boldsymbol{T}$ is the stress tensor of the fluid. In components the rhs of the Euler equation, Eq. (2), reads $(\boldsymbol{\nabla}\otimes\boldsymbol{T})_i=\nabla_j T_{ij}$. In flat space and in Cartesian coordinates, the stress tensor assumes the following form:

$$T_{ij}=-\delta_{ij}p+\nu_o(\partial_i v_j^*+\partial_i^* v_j). \tag{3}$$

The first term of Eq. (3) is standard and describes the contribution to the stress from isotropic fluid pressure $p$. The second term, however, is quite different from the conventional dissipative shear viscosity $\nu_e(\partial_i v_j+\partial_j v_i)$ (here $\nu_e$ is the shear or "even" viscosity coefficient). The last term of Eq. (3), instead, is the contribution of the odd viscosity, with $\nu_o$ being the kinematic odd viscosity coefficient. Differently from $\nu_e$, we can assign either sign to the odd viscosity $\nu_o$, since it multiplies a dissipationless term. In Eq. (3) and in the following we use the "star operation" so that the vector $\boldsymbol{a}^*$ is the vector $\boldsymbol{a}$ rotated 90° clockwise or in components $a_i^* \equiv \epsilon_{ij} a_j$. This operation explicitly breaks parity and a nonvanishing $\nu_o$ is only allowed in parity breaking fluids.

Euler Eq. (2) with the stress tensor Eq. (3) takes the form of the Navier-Stokes equation with odd viscosity term replacing the conventional viscosity term

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p + \nu_o \Delta \boldsymbol{v}^*. \tag{4}$$

Bulk hydrodynamic Eqs. (1) and (4) must be supplemented by boundary conditions. For a free surface we should use one kinematic and two dynamic boundary conditions:

$$(\partial_t \Gamma)_n = v_n|_\Gamma, \tag{5}$$

$$T_{ij} n_j|_\Gamma = 0, \tag{6}$$

where $\boldsymbol{n}$ is the unit vector normal to the free 1D surface $\Gamma = \partial \mathcal{M}$ of the 2D fluid domain $\mathcal{M}$. The kinematic boundary condition (KBC), Eq. (5), states that the velocity of the free surface in its normal direction is equal to the normal component of the velocity flow taken at the surface. The set of two dynamical boundary conditions (DBCs) given by Eq. (6) imposes that both components of stress force acting on the segment of the surface vanish. These conditions are appropriate for interfaces with vacuum or air, assuming that the latter cannot maintain nonvanishing forces on the surface of the fluid.

For a rather general class of fluid flows it is not possible to satisfy both DBCs [Eq. (6)] with the stress tensor [Eq. (3)] by smooth velocity configurations. A singular boundary layer is formed. One can see it, for example, in a linear approximation [39] and the phenomenon is very similar to a formation of a boundary layer for fluid with infinitesimal shear viscosity [40]. A nonvanishing shear viscosity $\nu_e$ or finite compressibility, characterized by a finite sound velocity $c_s$, results in a finite thickness of the boundary layer proportional to $\sqrt{\nu_e}$ [39] or to $1/c_s$ [41], respectively. If one assumes that at least for finite times the boundary layer is stable and very thin, the motion of the fluid surface should be defined by effective boundary conditions imposed on the interior part of the fluid. Colloquially speaking, the latter boundary conditions can be obtained by "integrating out" the boundary layer. As a result, instead of two independent DBCs [Eq. (6)], one should consider a single effective normal dynamic boundary condition

$$\tilde{p}|_\Gamma \equiv p - \nu_o \omega|_\Gamma = 2\nu_o \partial_s v_n, \tag{7}$$

where $\partial_s v_n = -n_i^* \partial_i v_n$ is the derivative of normal velocity along the boundary and we introduced a notation $\tilde{p}$—pressure modified by vorticity $\omega = \partial_i v_i^*$.

While the precise way in which the tangent stress part of DBCs [Eq. (6)] is satisfied depends on the exact structure of the boundary layer, here we show that the effective normal stress boundary condition is universal and is given by Eq. (7). We obtain this universal statement by taking the variational principle for an ideal incompressible fluid and modifying it by adding a boundary term, which is lowest order in gradient expansion, breaks parity but preserves other symmetries of the system. We show that this boundary term produces Eq. (7) justifying the expectation of universality.

Let us start by rewriting Eq. (4) as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} \tilde{p} \tag{8}$$

using the incompressibility of the fluid, Eq. (1). Equation (8) is indistinguishable from the conventional Euler equation [42]. Therefore, we can start from Luke's variational principle to produce the bulk hydro equations together with perfect fluid boundary conditions and look for boundary corrections to LVP to obtain the modified DBC on the fluid which are in agreement with Eq. (7).

In contrast with Ref. [39], here we do not use any expansions in $\nu_e$ and our results do not rely on small surface angle approximations or on any assumption about the structure of the boundary layer.

*Luke's variational principle.*—Let us start from the simplest case of the incompressible potential fluid flow, that is, $\boldsymbol{v} = \boldsymbol{\nabla}\theta$. Luke's variational principle is written in terms of the velocity potential $\theta$ as follows:

$$S_\mathcal{M} = -\int dt \int_\mathcal{M} d^2x \left(\partial_t \theta + \frac{1}{2}(\partial_i \theta)^2\right), \tag{9}$$

where $\mathcal{M}$ is the 2D fluid domain with the boundary [43]. Variation over $\theta$ in the bulk gives $\Delta\theta = 0$—the incompressibility condition. It is also straightforward to obtain Eq. (8) as an identity if the modified pressure is identified as

$$\tilde{p} = -\partial_t \theta - \frac{1}{2}(\partial_i \theta)^2. \tag{10}$$

Thus, the action Eq. (9) produces bulk equations [Eqs. (1) and (8)] for a potential flow. The bulk vorticity of such flow vanishes identically $\omega = 0$, implying $\tilde{p} = p$. Let us now keep track of boundary terms and assume that the bulk equation of motion $\Delta\theta = 0$ is satisfied. Hence, varying Eq. (9) over the velocity potential $\theta$ and over the shape of the fluid domain $\mathcal{M}$, we obtain that all the nontrivial dynamics resides on the fluid boundary and the action variation becomes [for details see Supplemental Material (SM) [44] ]

$$\delta S_{\mathcal{M}} = \int dt \int_\Gamma ds[\delta\theta((\partial_t\Gamma)_n - \partial_n\theta) + (\delta\Gamma)_n\tilde{p}]. \quad (11)$$

Here $\Gamma = \partial\mathcal{M}$ is the spatial boundary of the fluid domain and $s$ is the natural parameter along the boundary so that $dx^2 + dy^2 = ds^2$. The variation over the boundary values of the potential $\theta$, i.e., the first term in the integrand, gives the KBC Eq. (5). The variation of the boundary, i.e., the second term in the integrand in Eq. (11), gives the vanishing pressure boundary condition $\tilde{p}|_\Gamma = 0$ well known for ideal fluids. The latter is markedly different from the effective DBC [Eq. (7)] derived in Ref. [39]. Therefore, while the variational principle Eq. (9) produces all equations and boundary conditions for ideal fluid it does not account for the contributions from odd viscosity.

*Boundary term.*—The main result of this work is that in order to obtain the effective DBC [Eq. (7)], the following boundary term should be added to LVP:

$$S_\Gamma = \nu_o \int dt \int_\Gamma ds(\partial_t\Gamma)_n\alpha, \quad (12)$$

where $s$ is the natural parameter along the boundary and $\alpha$ is the angle between the surface and some fixed direction, check Fig. 1. Two remarks are in order. (i) The term Eq. (12) is constructed purely from the local boundary geometry data and does not contain, e.g., velocity potential $\theta$. This means that the KBC Eq. (5) is not modified by this term. (ii) Naively, Eq. (12) contains the preferred direction —the reference axis for $\alpha$. However, shifting $\alpha$ by constant does not change Eq. (12) because $\int_\Gamma ds(\partial_t\Gamma)_n = 0$ due to the KBC and incompressibility of the fluid. Later on we will present a covariant way of writing Eq. (12).

Let us first consider the example of a half-plane geometry when the fluid domain $\mathcal{M}$ is given by $y \le h(x,t)$. We choose the reference direction to be the $x$ direction and write the angle $\alpha$ explicitly as $\alpha = \tan^{-1} h_x$. In this geometry $ds = \sqrt{1+h_x^2}dx$ and for normal velocity of the boundary we have

$$(\partial_t\Gamma)_n = \frac{h_t}{\sqrt{1+h_x^2}} = v_n, \quad (13)$$

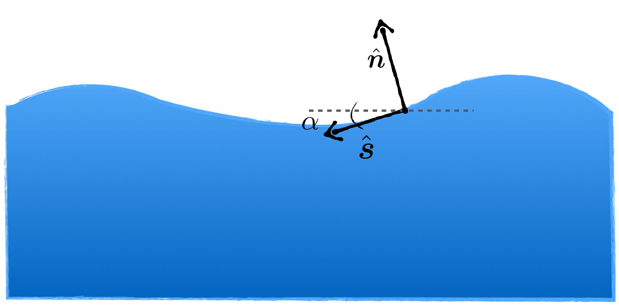


FIG. 1. The choice of normal $\hat{n}$ and tangent $\hat{s}$ unit vectors, and the angle $\alpha$ for the case of half-plane geometry.

so that Eq. (12) can be written as

$$S_\Gamma = \nu_o \int dt \int_{\mathbb{R}} dx\, h_t\alpha = -\nu_o \int dt \int_{\mathbb{R}} dx\, h\alpha_t, \quad (14)$$

with $\alpha_t = (h_{xt}/1 + h_x^2)$. Computing the variation of Eq. (14) we obtain (for details see SM [44])

$$\delta S_\Gamma = -2\nu_o \int dt \int_\Gamma ds(\delta\Gamma)_n\partial_s(\partial_t\Gamma)_n, \quad (15)$$

where $(\partial_t\Gamma)_n$ is given by Eq. (13) and $(\delta\Gamma)_n = \delta h/\sqrt{1+h_x^2}$.

It is easy to see that the variation $\delta(S_{\mathcal{M}} + S_\Gamma)$ over $(\delta\Gamma)_n$ given by Eqs. (11) and (15) gives the modified boundary condition Eq. (7). Notice that the rhs of Eq. (15) is purely geometric, since $\partial_s(\partial_t\Gamma)_n$ is the angular velocity of the boundary.

The same analysis can be repeated for the geometry of a disk, i.e., a simply connected droplet producing again Eq. (15) [45]. Therefore, the variational principle with the action

$$S = S_{\mathcal{M}} + S_\Gamma \quad (16)$$

defined in Eqs. (9) and (12) produces an incompressibility condition and the Euler equation, Eq. (8), with KBC Eq. (5) and effective DBC, Eq. (7). Explicitly, the full set of equations can be written as

$$\Delta\theta = 0, \qquad x \in \mathcal{M}, \quad (17)$$

$$\partial_n\theta = (\partial_t\Gamma)_n, \qquad x \in \Gamma, \quad (18)$$

$$\partial_t\theta + \frac{1}{2}(\partial_i\theta)^2 = -2\nu_o\partial_s(\partial_t\Gamma)_n, \qquad x \in \Gamma. \quad (19)$$

The obtained hydrodynamics describes incompressible potential flows of the fluid with odd viscosity. This is the main result of this work. We will remove the requirement of potentiality of the flow later in this letter.

*Effective contour dynamics.*—In the case of an irrotational bulk flow, the full dynamics is completely determined by the boundary motion. One can express Eqs. (18) and (19) purely in terms of boundary fields using Eq. (17). To do that we introduce the boundary field $\tilde{\theta} = \theta|_\Gamma$ or explicitly $\tilde{\theta}(s,t) = \theta(x(s,t), y(s,t), t)$ with boundary $\Gamma$ given parametrically by functions of the natural parameter $s$ along the boundary. We introduce the material derivative at the boundary $D_t = \partial_t - (\partial_t\Gamma)_s\partial_s$ and use the identity

$$D_t\tilde{\theta} = \partial_t\theta|_\Gamma + (\partial_n\theta)(\partial_t\Gamma)_n \quad (20)$$

in Eq. (19) together with Eq. (18) and obtain

$$D_t\tilde{\theta} + \frac{1}{2}(\partial_s\tilde{\theta})^2 - \frac{1}{2}(\partial_t\Gamma)_n{}^2 = -2\nu_o\partial_s(\partial_t\Gamma)_n. \quad (21)$$

Equation (18) can also be expressed in terms of boundary fields using Eq. (17). It has a form

$$(\partial_t\Gamma)_n = \widehat{DN}\,\tilde{\theta}, \quad (22)$$

where $\widehat{DN}$ is a Dirichlet to Neumann operator which depends on the shape of the domain and can be expressed in terms of the Dirichlet Green function of the Laplace operator [46] (see SM [44]) as

$$\widehat{DN}\,\tilde{\theta}(s) = \int_\Gamma ds'[\partial_n\partial_{n'}G(x,x')]\tilde{\theta}(s'). \quad (23)$$

Equations (21) and (22) fully determine boundary dynamics of the fluid domain. They can be considered as equations for fields $\tilde{\theta}(s,t)$, $x(s,t)$, and $y(s,t)$ specifying both the position of the boundary and the boundary value of the potential. The reparametrization invariance of Eqs. (21) and (22) can be used to remove one of the degrees of freedom. For example when the domain is given by $y \le h(x,t)$ one can rewrite Eqs. (21) and (22) in terms of two fields $\tilde{\theta}(x,t)$ and $h(x,t)$. For this case one also can find $\widehat{DN}$ as an expansion in $h$ and obtain [39]

$$\widehat{DN}\theta = -\tilde{\theta}_x^H - [h\tilde{\theta}_x + (h\tilde{\theta}_x^H)^H]_x + \ldots, \quad (24)$$

where the Hilbert transform is defined as $f^H(x) = ⨍(dx'/\pi)[f(x')/(x'-x)]$.

Equations (21) and (22) are exact expressions given by the action, Eq. (16). The approximate versions of these equations using Eq. (24) can be found in Ref. [39].

It is even easier to derive the effective one-dimensional action corresponding to Eqs. (21) and (22). We integrate Eq. (16) by parts and use the bulk incompressibility of the fluid $\Delta\theta = 0$ to obtain (for details see SM [44])

$$S_{1\mathrm{D}} = \int dt\left(\int_\Gamma ds(\partial_t\Gamma)_n(\tilde{\theta} + \nu_o\alpha) - H\right), \quad (25)$$

$$H = \frac{1}{2}\int_\Gamma ds(\theta\partial_n\theta)_\Gamma = \frac{1}{2}\int_\Gamma ds\tilde{\theta}\,\widehat{DN}\,\tilde{\theta}. \quad (26)$$

The Hamiltonian, Eq. (26), is nothing but the total kinetic energy of the fluid given by the second term of Eq. (9). The variations of Eq. (25) with respect to $\tilde{\theta}$ and displacements of the boundary produce equations of motion, Eqs. (21) and (22). These variations can be computed using the Hadamard's variational formula, defined in Refs. [47,48]; however the most straightforward way to calculate such variations is to rewrite Eq. (26) in its local form as a two-dimensional integral.

*Hamiltonian structure of contour dynamics.*—Instead of studying the boundary dynamics for a general fluid domain $\mathcal{M}$, let us focus here on the particular case when $\mathcal{M}$ is given by $y \le h(x,t)$. Then, the action Eq. (25) can be rewritten as

$$S_{1\mathrm{D}} = \int dt\left[\int_{\mathbb{R}} dx h_t(\tilde{\theta} + \nu_o\alpha) - H\right], \quad (27)$$

where the Hamiltonian is given by Eq. (26), with $ds = \sqrt{1+h_x^2}dx$. Let us turn our attention to the first term of Eq. (27). We immediately see that $h$ and $\tilde{\theta} - \nu_o\alpha$ are canonically conjugated variables so that Poisson brackets become [49]

$$\{h, h'\} = 0, \qquad \{\tilde{\theta}, h'\} = \delta(x-x'), \quad (28)$$

$$\{\tilde{\theta}, \tilde{\theta}'\} = \nu_o\left(\frac{1}{1+h_x^2} + \frac{1}{1+h_x'^2}\right)\partial_x\delta(x-x'). \quad (29)$$

Note that the Poisson structure reduces to the well-known Zakharov's Poisson structure [50] when $\nu_o = 0$. In the limit of small slopes $h_x \ll 1$ the bracket Eq. (29) was obtained in Ref. [39]. However, we emphasize here that the Poisson structure (28), (29) is an exact consequence of the variational principle (16), (9), (14) without any additional approximations.

*Boundary term and geometry.*—The boundary term Eq. (12) involves some arbitrariness in choosing a reference direction. In this paragraph, we aim to give a more covariant way of this form and to provide a geometrical picture associated with this boundary action. For that, it is convenient to express $S_\Gamma$ in terms of differential forms. Since $\hat{\boldsymbol{n}} = (\sin\alpha, -\cos\alpha)$, we can associate the derivatives of the angle $\alpha$ to the boundary extrinsic curvature one-form $K = K_\mu dx^\mu$ (for details, *vide* Ref. [29])

$$K_\mu = n_i\partial_\mu s_i = n_i\partial_\mu n_i^* = \partial_\mu\alpha. \quad (30)$$

Integrating by parts, we can rewrite Eq. (12) as

$$S_\Gamma = -\nu_o\int_{\mathbb{R}\times\Gamma} A \wedge K, \quad (31)$$

where $A$ is a one-form whose exterior derivative is the plane volume-form, that is, $dA = dx \wedge dy$. There is an ambiguity in the definition of $A$, since $A' = A + d\Lambda$ gives us $dA' = dA$. However, this gauge freedom does not affect the boundary action Eq. (31) [51].

As an example let us consider $A = -ydx$ for $\mathcal{M}$ given by $y \le h(x,t)$. Then, Eq. (31) reproduces Eq. (14).

For the droplet case, $\mathcal{M}$ is defined in polar coordinates by $r \le R(\varphi,t)$. If we take $A = \frac{1}{2}r^2 d\varphi$, we then obtain:

$$S_\Gamma = -\frac{\nu_o}{2}\int_{\mathbb{R}\times\Gamma} R^2\alpha_t dt \wedge d\varphi. \quad (32)$$

*Conclusions.*—We presented a variational principle which accounts for odd viscosity effects in incompressible fluid dynamics. The boundary part of the proposed action is purely geometrical and fully determined by the symmetries of the system. Therefore, we expect the boundary condition Eq. (7) to be universal and independent of the exact structure of the boundary layer, given this boundary layer to be sufficiently thin. In particular, Eq. (7) reproduces the approximate equations obtained in Ref. [39], which were derived in the limit of very small, but nonvanishing shear viscosity. We also expect the same boundary conditions assuming the boundary layer structure to be determined by a finite compressibility of the fluid. If the fluid is compressible, the odd viscosity affects the flow of the fluid in the bulk as well. While it is relatively straightforward to construct a variational principle for the compressible fluid, its connection to the incompressible limit is subtle and will be discussed elsewhere.

The variational principle [Eqs. (16) and (12)] gives hydrodynamic equations for an incompressible fluid with odd viscosity under the assumption that the tangent stress-free surface boundary conditions can be satisfied by a thin boundary layer. This is not the case for all fluid flows. For example, in the geometry of an expanding air bubble exact solutions show strong dependence of the bulk flow on shear viscosity [37]. Also, even if the assumption of a thin boundary layer is satisfied initially it might break at finite time [39]. The applicability of the thin boundary layer assumption is beyond of the scope of this Letter.

In the irrotational case, the degrees of freedom reside on the boundary and the effective dynamics is one dimensional and Hamiltonian, albeit nonlocal. The derived Hamiltonian structure modifies the well-known Hamiltonian structure of incompressible ideal fluids [50]. While, for simplicity, we presented here only the irrotational case, the generalization to more general flows, with nonzero vorticity, is straightforward and requires the addition of more Clebsch variables [*vide* SM for more details [44] ].

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[42] Remember that in incompressible fluids pressure $p$ is not a thermodynamic variable, but, instead, is fully determined by the flow $\boldsymbol{v}$. From this point of view $p \to \tilde{p}$ is just a change of notations.

[43] Note that the first term in Eq. (9) is not trivial since the domain $\mathcal{M}$ is time dependent. This term should be integrated by parts using the Leibniz integral rule.

[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.154501 for computation details and some generalizations of our results.

[45] Assuming that the boundary action [Eq. (12)] is well defined (see the following sections) there is, actually, no need to repeat the calculations. Any surface can be locally parameterized as $y = h(x,t)$ and all variational calculations we perform are local. Nevertheless, we give explicit formulas for the droplet geometry in the SM for future references.

[46] Alternatively one can invert the Eq. (22) and write it as $\theta = \hat{S}(\partial_t \Gamma)_n$ using the Neumann operator (see SM).

[47] S. E. Warschawski, On Hadamard's variation formula for Green's function, J. Math. Mech. **9**, 497 (1960).

[48] J. Peetre, On Hadamard's variational formula, J. Differ. Equations **36**, 335 (1980).

[49] We use abbreviated notations $h = h(x)$, $h' = h(x')$ etc. For details of the derivation see SM.

[50] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys. **9**, 190 (1972).

[51] It is assumed here that $\Lambda$ is single valued. To allow for large gauge transformations one should include additional bulk terms involving spin connection [29].