Contextuality and the Single-Qubit Stabilizer Subtheory

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Contextuality is a fundamental nonclassical property of quantum theory, which has recently been proven to be a key resource for achieving quantum speed-ups in some leading models of quantum computation. However, which of the forms of contextuality, and how much thereof, are required to obtain a speed-up in an arbitrary model of quantum computation remains unclear. In this Letter, we show that the relation between contextuality and a computational advantage is more complicated than previously thought. We achieve this by proving that generalized contextuality is present even within the simplest subset of quantum operations, the so-called single-qubit stabilizer theory, which offers no computational advantage and was previously believed to be completely noncontextual. However, the contextuality of the single-qubit stabilizer theory can be confined to transformations. Therefore, our result also demonstrates that the commonly considered prepare-and-measure scenarios (which ignore transformations) do not fully capture the contextuality of quantum theory.

Introduction.—Contextuality [1–8], which includes the better-known concept of Bell nonlocality as a special case, is often regarded as the fundamental nonclassical property of quantum theory. Furthermore, contextuality has emerged as an intriguing explanation for the power of quantum computation: Contextuality is required [9–12] to achieve an exponential quantum speed-up by injecting magic states into Clifford circuits [13], and it also quantifies the computational advantage that can be obtained [11,14–17] in both the magic-state and measurement-based models of quantum computation [18].

In addition, contextuality has proved key in understanding quantum correlations, formalized in the Cabello-Severini-Winters graph formalism [8]. This formalism gives experimentally testable contextuality inequalities [19–21], paralleling nonlocality inequalities such as Clauser-Horne-Shimony-Holt inequality [22,23]. Contextuality has also found applications in quantifying the memory cost of simulating quantum processes [24–26].

These considerations motivate us to understand the scope of phenomena that exhibit contextuality, with the aim of identifying which features of contextual phenomena enable quantum computational speed-up. However, one of the primary obstacles to understanding how contextuality powers a quantum computer is that the multiqubit stabilizer subtheory [27] exhibits contextuality and yet can be efficiently simulated on a classical computer [28,29].

There are two leading definitions of contextuality: traditional contextuality [1–5,8], often referred to as Bell-Kochen-Specker contextuality, and generalized contextuality [6,7]. In this Letter, we show that generalized contextuality [6] is present even in the single-qubit stabilizer subtheory of quantum theory, a fact missed by previous work [30–32]. We further demonstrate that the contextuality present in the single-qubit stabilizer subtheory can be confined to only appear in the transformations. This contradicts the common—and often implicit—assumption that an operational theory can be classified as contextual or noncontextual by only considering the preparations and measurements [11,33–37].

Operational theories and ontological models thereof.—An operational theory is noncontextual under a given definition if there exists an ontological model of the operational theory satisfying a specific property that we describe below.

An operational theory is a mathematical framework for predicting the outcomes of an experimental procedure, that is, a sequence of preparations, transformations, and measurements. These elements fully determine the experimental statistics, that is, the conditional probabilities $\Pr(k|P, T, M)$ of observing the outcome k when the preparation P, transformation T, and measurement M are performed sequentially. In quantum mechanics, the conditional probabilities for an experiment consisting of preparing a density matrix ρ, applying a completely positive and trace-preserving (CPTP) map Φ, and measuring a positive-operator-valued measure (POVM) $\{E_k\}$ are $\Pr(k|\rho, \Phi, E_k) = \mathrm{Tr}[E_k \Phi(\rho)]$ by the Born rule.

We can describe the underlying physical processes that generate the experimental statistics using the ontological model formalism. Here, we follow the treatment in Ref. [38]. An ontological model is defined by a measurable space Λ of possible physical states, with an associated σ-algebra Σ, and sets of measures or measurable functions over Λ are used to represent preparations, transformations, and measurements in the ontological model. For simplicity, we assume that there exists a measure that dominates all other measures in the model [39] (see the Appendix for a proof of the main theorem without this assumption). This allows us to express an ontological model in terms of probability densities, stochastic matrices, and response functions.

When a system is prepared via some procedure P, the physical properties of the system are probabilistically assigned values, which are completely encoded by the physical states $\lambda \in \Lambda$. Mathematically, we associate each preparation procedure P with a probability density over Λ, $\mu_P : \Lambda \to [0, 1]$, where $\int_\Lambda \mu_P(\lambda) d\lambda = 1$ as a system is always in some physical state. In other words, the probability that a physical state, $\lambda \in \Lambda$, was prepared via P is $\mu_P(\lambda)$.

Similarly, when a transformation is applied to a system, the physical properties of the system dynamically evolve according to some stochastic map. Formally, we associate each transformation procedure T with a stochastic map $\Gamma_T : \Lambda \times \Lambda \to [0, 1]$, where the conditional probability that some λ is sent to another λ' by T is $\Gamma_T(\lambda', \lambda)$. As every physical state is mapped to some physical state by a transformation, $\int_\Lambda \Gamma_T(\lambda', \lambda) d\lambda' = 1$ for all $\lambda \in \Lambda$.

Finally, when a system is measured via some procedure M, the probability that outcome k occurs is specified by the physical state. In other words, a measurement M is equivalent to a set of conditional probability functions $\{\xi_k^M : \Lambda \to [0, 1]\}_k$. As some measurement outcome always occurs, $\sum_k \xi_k^M(\lambda) = 1$ for all $\lambda \in \Lambda$. To correctly reproduce the experimental statistics of the operational theory, the distributions must satisfy

$$\Pr(k|P, T, M) = \int_\Lambda \xi_k^M(\lambda') \Gamma_T(\lambda', \lambda) \mu_P(\lambda) d\lambda d\lambda'. \quad (1)$$

Ontological models are assumed, often implicitly [40], to be convex linear; that is, a probabilistic implementation of a set of operations is represented by the probabilistic mixture of the corresponding probability densities.

Generalized contextuality.—We now review generalized contextuality. The (experimental) setting of an operation is the set of other operations that are performed with the operation during an experiment. Two operations are operationally equivalent, denoted $\cong$, if they produce the same outcome statistics in all settings. (i) Two preparations P and P' are equivalent, $(P \cong P')$, if $\Pr(k|P, T, M) = \Pr(k|P', T, M) \ \forall \ T, M$. (ii) Two transformations T and

T' are equivalent, $(T \cong T')$, if $\Pr(k|P, T, M) = \Pr(k|P, T', M) \ \forall \ P, M$. (iii) Two measurement outcomes $k \in M$ and $k \in M'$ are equivalent, $[(k, M) \cong (k, M')]$, if $\Pr(k|P, T, M) = \Pr(k|P, T, M') \ \forall \ P, T$. Note that the definition of operational equivalence differs slightly from that of Ref. [6] in that we consider operational equivalence of individual measurement outcomes. However, this definition can be obtained from that of Ref. [6] by coarse graining all measurements into two-outcome POVMs [35,41,42].

An ontological model is preparation noncontextual (PNC) if operationally equivalent preparation procedures are represented by the same probability densities, that is,

$$\mu_P = \mu_{P'} \Leftrightarrow P \cong P'. \quad (2)$$

Similarly, an ontological model is transformation noncontextual (TNC) if

$$\Gamma_T = \Gamma_{T'} \Leftrightarrow T \cong T' \quad (3)$$

and measurement noncontextual (MNC) if

$$\xi_{k,M} = \xi_{k,M'} \Leftrightarrow (k, M) \cong (k, M'). \quad (4)$$

An ontological model is universally noncontextual, in the generalized sense, if it satisfies Eqs. (2)–(4); otherwise it is contextual [6].

Even a single qubit manifests generalized contextuality [6]. However, previous proofs of generalized contextuality for a single qubit have required subtheories strictly larger than the single-qubit stabilizer subtheory.

Previous proofs of generalized contextuality have mostly been focused on the prepare-measure setting, as defined in [38], wherein transformations are considered part of a preparation or measurement procedure, and systems are discarded after measurement. However, this seemingly innocuous operational assumption is insufficient to identify an operational theory as noncontextual because the contextuality of the operational theory *can* be confined to transformations, as we shall see is the case with the single-qubit stabilizer subtheory.

Contextuality in the 8-state model.—We now show that the 8-state model of the single-qubit stabilizer subtheory exhibits transformation contextuality, a feature missed in previous studies of this model [30–32]. The single-qubit stabilizer subtheory consists of preparations and measurements in the eigenbases of the single-qubit Pauli matrices $\{X, Y, Z\}$, the group of unitary transformations that permute the signed single-qubit Pauli matrices (i.e., the single-qubit Clifford group) and convex combinations of these operations. The single-qubit stabilizer subtheory has the property that preparing an eigenstate of one Pauli matrix P with eigenvalue η then measuring another Pauli Q results in the outcome $\eta' = \pm\eta$ if $P = \pm Q$ and otherwise the outcome is either eigenvalue with equal probability.
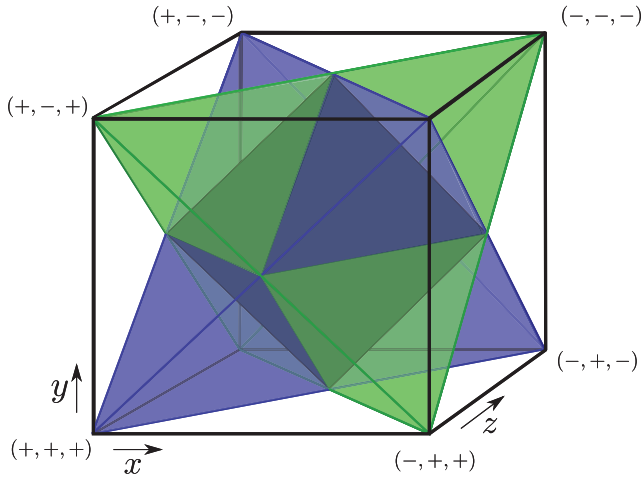
FIG. 1. A graphical representation of the ontic space Λ of the 8-state model, where the tuples (x, y, z) index ontic states. The green and blue tetrahedra are the simplices of the odd- and even-parity ontic states, respectively. The stabilizer polytope is the octahedron defined by the intersection of the two tetrahedra. The transformation $\Gamma_{\mathcal{T}_1}$ from Eq. (6) maps any ontic state in one of the tetrahedra to another ontic state in the same tetrahedra, while the operationally equivalent transformation $\Gamma_{\mathcal{T}_2}$ from Eq. (7) maps any ontic state to an ontic state in the opposite tetrahedra.

The 8-state model, originally developed in Ref. [30], is a natural ontological model for the single-qubit stabilizer subtheory (see Fig. 1). It is defined by setting $\Lambda = \{\pm 1\}^{\times 3}$ and writing $\lambda = (x, y, z)$, where x, y, and z are the eigenvalues of X, Y, and Z, respectively. These ontic states form the extremal points of the classical probability simplex for three random binary variables. Preparing the η eigenstate of X corresponds to setting $x = \eta$ and choosing y and z uniformly at random, etc. Similarly, measuring X returns the value of x, etc. This model is both preparation and measurement noncontextual [30].

In the 8-state model, a transformation corresponds to a permutation that acts on the hidden variable (x, y, z) in the same way that it acts on the Pauli operators (X, Y, Z). For example, conjugation by X maps $(X, Y, Z) \rightarrow (X, -Y, -Z)$, so it is represented by the permutation $\Gamma_X : (x, y, z) \rightarrow (x, -y, -z)$, with the transformations for Y and Z defined in a similar manner. Conjugation by the Hadamard matrix,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

maps $(X, Y, Z) \rightarrow (Z, -Y, X)$, so it is represented by the permutation $\Gamma_H : (x, y, z) \rightarrow (z, -y, x)$. Note that a Pauli operation preserves the parity xyz and the Hadamard swaps it (as does the phase gate P).

The 8-state model can be viewed as a conjunction of two distinct generalized noncontextual 4-state Wigner functions [11,30]. By extending the 4-state Wigner function to the 8-state model, all Clifford operations can be ontologically

represented. As we now show, this increase in the size of the ontology causes the 8-state model to be transformation contextual.

To show that the 8-state model is transformation contextual, let

$$\mathcal{T}_1(\rho) = [\rho + X\rho X + Y\rho Y + Z\rho Z]/4,$$
$$\mathcal{T}_2(\rho) = H\mathcal{T}_1(\rho)H. \tag{5}$$

These two transformations are operationally equivalent, as $\mathcal{T}_1(\rho) = \mathcal{T}_2(\rho) = I/2$ for any input state ρ. However, by convexity we have

$$\Gamma_{\mathcal{T}_1}[(a, b, c), (x, y, z)] = \begin{cases} \frac{1}{4} & \text{if } xyz = abc \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

while, since the Hadamard swaps the sign of xyz,

$$\Gamma_{\mathcal{T}_2}[(a, b, c), (x, y, z)] = \begin{cases} 0 & \text{if } xyz = abc \\ \frac{1}{4} & \text{otherwise.} \end{cases} \tag{7}$$

In other words, $\Gamma_{\mathcal{T}_1} \neq \Gamma_{\mathcal{T}_2}$, as illustrated in Fig. 1.

The single-qubit stabilizer subtheory is contextual.— Above, we demonstrated that the 8-state model for the single-qubit stabilizer subtheory is transformation contextual. We now prove that there is no generalized noncontextual model for the single-qubit stabilizer subtheory, and hence that the single-qubit stabilizer subtheory is contextual. The proof follows by reducing the ontic space of a general preparation noncontextual model of the single-qubit stabilizer subtheory to that of the 8-state model.

Theorem 1: Every ontological model of the single-qubit stabilizer subtheory is either preparation or transformation contextual.

Proof.—Fix an arbitrary preparation noncontextual ontological model of the single-qubit stabilizer subtheory. Let Δ_ρ be the support of the quantum state ρ in the ontological model; that is, the set of physical states ρ has some possibility of preparing

$$\Delta_\rho = \{\lambda | \mu_\rho(\lambda) > 0, \lambda \in \Lambda\}. \tag{8}$$

Deleting any ontic state $\lambda \in \Lambda$ such that $\mu_{I/2}(\lambda) = 0$, we can partition Λ into eight disjoint spanning sets from the assumption of PNC [see Eqs. (11) and (83)–(87) in [6]],

$$\Lambda_{x,y,z} = \Delta_{(I+xX)/2} \cap \Delta_{(I+yY)/2} \cap \Delta_{(I+zZ)/2}. \tag{9}$$

As the model is preparation noncontextual, every quantum state has a unique support. Hence, this partitioning is unique.

Noting that preparing σ and then applying a transformation T, which implements a CPTP map Φ, is a valid preparation procedure for the state $\Phi(\sigma)$. It must be the case

that T maps the support of ρ to the support of $\Phi(\rho)$ in a preparation noncontextual ontological model:

$$\Gamma_T : \Delta_\rho \to \Delta_{\Phi(\rho)}. \qquad (10)$$

Therefore, a Pauli X unitary must be represented by the permutation $\tau_X : \Lambda_{x,y,z} \to \Lambda_{x,-y,-z}$ on the partition $\{\Lambda_{x,y,z}\}$. Similarly, Pauli I, Y, and Z transformations must be represented by the respective permutations $\tau_I : \Lambda_{x,y,z} \to \Lambda_{x,y,z}$, $\tau_Y : \Lambda_{x,y,z} \to \Lambda_{-x,y,-z}$, and $\tau_Z : \Lambda_{x,y,z} \to \Lambda_{-x,-y,z}$. Therefore, by convex linearity there exists an implementation of $\mathcal{T}_1$ that has the same stochastic map as Eq. (6), when defined over the coarse-grained sets $\Lambda_{x,y,z}$.

Similarly, for the Hadamard gate we have the map $\tau_H : \Lambda_{x,y,z} \to \Lambda_{x,-y,z}$. Therefore, there exists an implementation of $\mathcal{T}_2$ that has the same stochastic map as Eq. (7), when defined over the coarse-grained sets $\Lambda_{x,y,z}$. In other words, $\mathcal{T}_1 \cong \mathcal{T}_2$, and yet they cannot be represented by the same stochastic map in any preparation noncontextual model. $\square$

We now show that any model of the single-qubit stabilizer subtheory must be either traditionally contextual [2] or transformation contextual. We do this by proving a stronger result.

Theorem 2: Every ontological model of the single-qubit stabilizer subtheory is either outcome indeterministic or transformation contextual.

Proof.—The proof proceeds in the same manner as Theorem 1, where now we assume outcome determinism rather than preparation noncontextuality. By outcome determinism, we can partition Λ into eight disjoint sets according to measurement outcomes:

$$\tilde{\Lambda}_{xyz}^{X,Y,Z} = \{\lambda | \xi_x^X(\lambda) = 1, \xi_y^Y(\lambda) = 1, \xi_z^Z(\lambda) = 1\},$$

where our choice of measurement contexts X, Y, and Z is arbitrary and is merely used to partition Λ such that the maps representing Clifford transformations are well defined. Using the equivalent to Eq. (10) for measurements, the maps $\Gamma_{\mathcal{T}_1}$ and $\Gamma_{\mathcal{T}_2}$ must be represented as stated in Eqs. (6) and (7). $\square$

Therefore, Theorem 2 implies that any traditionally noncontextual ontological model of the single-qubit stabilizer formalism—for example, Kochen-Specker's original model of a single qubit [2]—must be transformation contextual, as traditional contextuality is implied by the conjunction of outcome determinism and generalized measurement noncontextuality [6]. Finally, we note that Theorem 2 can be used to prove Theorem 1, as preparation noncontextuality implies traditional contextuality for sharp measurements [43].

Discussion.—In this Letter we have shown that the single-qubit stabilizer subtheory, a very simple subtheory of the smallest quantum system, exhibits generalized contextuality. This demonstrates that generalized contextuality is so prevalent that even an essentially trivial quantum subtheory is classified as contextual and is therefore nonclassical. The result shows that, unlike traditional contextuality which may be an important resource for quantum speed-up, generalized contextuality is an extremely weak notion of contextuality that exists in trivial and classical-like models, such as the 8-state model. This may reflect the fact that generalized contextuality is defined in terms of a context dependence at the epistemic level, rather than a context dependence at the ontological level.

The contextuality in the single-qubit stabilizer subtheory is only apparent if all operations are accounted for, that is, all stabilizer states, all stabilizer measurements, and the full Clifford group. Therefore, a universally noncontextual model can only be constructed for strict subtheories of the singe-qubit stabilizer subtheory. For example, the Hadamard and Phase gates are not elements of the toy theory [44] or the standard Wigner function [45]; conversely, the Hadamard gate is an element of the rebit subtheory [10], but Y eigenstates and Y measurements are not.

Our result also demonstrates that the operational reduction of only considering preparations and measurements is less robust than previously recognized. This reduction can conceal key features of the model, such as the presence of some forms of contextuality. It is an interesting open problem to understand how and when this kind of reduction can obscure such important conceptual features of an operational theory. This insight regarding the key role of transformations may have unexplored connections to research the memory cost of quantum simulations [24–26] and the formalism of computational mechanics [46,47].

Another possible route to investigate the role of transformations in physical theories is the Choi-Jamiokowski isomorphism. The isomorphism, in quantum theory, relates transformations to states in a larger Hilbert space. Hence, by using a similar isomorphism for ontological models, we may be able to find a connection between the impossibility of a universal noncontextual ontological model of the single-qubit stabilizer subtheory and the impossibility of a preparation noncontextual ontological model of the two-qubit stabilizer subtheory [5,43].

Appendix: Measure-theoretic treatment of theorem 1.— We now prove Theorem 1 in the more general measure-theoretic framework for ontological models (see [38] definition 8.2).

Theorem 3: Every ontological model of the single-qubit stabilizer subtheory is either preparation or transformation contextual.

Proof.—To adapt the proof to the measure-theoretic setting, we need to change the definition of support of a quantum state as follows. Let Δ_ρ be the support of the quantum state ρ in the ontological model, that is, a (not necessarily unique) set such that for all $S \in \Sigma$,

$$\mu_\rho(S) \begin{cases} = 1 & \text{if } \Delta_\rho \subseteq S \\ < 1 & \text{otherwise.} \end{cases} \tag{A1}$$

As before we delete any measurable set $S \in \Sigma$ such that $\mu_{I/2}(S) = 0$, then partition Λ into eight spanning sets that intersect on sets of measure zero, from the assumption of PNC [see Eqs. (11) and (83)–(87) in [6]]. Having reduced the model to a model over a finite set of states, the rest of the proof follows as described in the main text. $\square$

[1] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[2] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[3] N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
[4] A. Peres, J. Phys. A **24**, L175 (1991).
[5] N. D. Mermin, Rev. Mod. Phys. **65**, 803 (1993).
[6] R. W. Spekkens, Phys. Rev. A **71**, 052108 (2005).
[7] A. W. Simmons, J. J. Wallman, H. Pashayan, S. D. Bartlett, and T. Rudolph, New J. Phys. **19**, 033030 (2017).
[8] A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. **112**, 040401 (2014).
[9] M. Howard, J. J. Wallman, V. Veitch, and J. Emerson, Nature (London) **510**, 351 (2014).
[10] N. Delfosse, P. Allard Guerin, J. Bian, and R. Raussendorf, Phys. Rev. X **5**, 021003 (2015).
[11] J. Bermejo-Vega, N. Delfosse, D. E. Browne, C. Okay, and R. Raussendorf, Phys. Rev. Lett. **119**, 120505 (2017).
[12] R. Raussendorf, D. E. Browne, N. Delfosse, C. Okay, and J. Bermejo-Vega, Phys. Rev. A **95**, 052334 (2017).
[13] S. Bravyi and A. Kitaev, Phys. Rev. A **71**, 022316 (2005).
[14] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, New J. Phys. **14**, 113011 (2012).
[15] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, New J. Phys. **16**, 013009 (2014).
[16] C. Okay, S. Roberts, S. D. Bartlett, and R. Raussendorf, Quantum Inf. Comput. **17**, 1135 (2017).
[17] L. Catani and D. E. Browne, Phys. Rev. A **98**, 052108 (2018).
[18] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[19] A. Cabello, Phys. Rev. Lett. **101**, 210401 (2008).
[20] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. **101**, 020403 (2008).
[21] S. Yu and C. H. Oh, Phys. Rev. Lett. **108**, 030402 (2012).
[22] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[23] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[24] A. Karanjai, J. J. Wallman, and S. D. Bartlett, arXiv:1802.07744.
[25] M. Kleinmann, O. Ghne, J. R. Portillo, J.-k. Larsson, and A. Cabello, New J. Phys. **13**, 113011 (2011).
[26] A. Cabello, M. Gu, O. Gühne, and Z.-P. Xu, Phys. Rev. Lett. **120**, 130401 (2018).
[27] A subtheory of quantum theory is a closed subset of the operations available within quantum theory for a given Hilbert space dimension.
[28] D. Gottesman, Proc. Int. Colloq. Group Theor. Methods Phys. **1**, 32 (1998).
[29] S. Aaronson and D. Gottesman, Phys. Rev. A **70**, 052328 (2004).
[30] J. J. Wallman and S. D. Bartlett, Phys. Rev. A **85**, 062121 (2012).
[31] P. Blasiak, Phys. Lett. A **377**, 847 (2013).
[32] L. Kocia and P. Love, Phys. Rev. A **96**, 062134 (2017).
[33] N. Harrigan and T. Rudolph, arXiv:0709.4266v1.
[34] R. W. Spekkens, Phys. Rev. Lett. **101**, 020401 (2008).
[35] R. Kunjwal and R. W. Spekkens, Phys. Rev. Lett. **115**, 110403 (2015).
[36] A. Krishna, R. W. Spekkens, and E. Wolfe, New J. Phys. **19**, 123031 (2017).
[37] D. Schmid, R. W. Spekkens, and E. Wolfe, Phys. Rev. A **97**, 062103 (2018).
[38] M. S. Leifer, Quanta **3**, 67 (2014).
[39] In other words, $\exists \sigma$ such that $\sigma(\Delta) = 0 \Rightarrow \mu(\Delta) = 0$, $\Delta \in \Sigma$ for all μ in the model.
[40] A. Blass and Y. Gurevich, arXiv:1503.08084.
[41] M. S. Leifer and R. W. Spekkens, Int. J. Theor. Phys. **44**, 1977 (2005).
[42] R. W. Spekkens, Found. Phys. **44**, 1125 (2014).
[43] M. S. Leifer and O. J. E. Maroney, Phys. Rev. Lett. **110**, 120401 (2013).
[44] R. W. Spekkens, Phys. Rev. A **75**, 032110 (2007).
[45] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, Phys. Rev. A **70**, 062101 (2004).
[46] N. Barnett and J. P. Crutchfield, J. Stat. Phys. **161**, 404 (2015).
[47] J. Thompson, A. J. P. Garner, V. Vedral, and M. Gu, npj Quantum Inf. **3**, 6 (2017).