Editors' Suggestion

Robustness of Measurement, Discrimination Games, and Accessible Information

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We introduce a resource theory of measurement informativeness. This allows us to define an associated quantifier, which we call the robustness of measurement. It describes how much "noise" must be added to a measurement before it becomes completely uninformative. We show that this geometric quantifier has operational significance in terms of the advantage the measurement provides over guessing at random in a suitably chosen state discrimination game and that it is the single-shot generalization of the accessible information of a certain quantum-to-classical channel. Using this insight, we further show that the recently introduced robustness of asymmetry or coherence is the single-shot generalization of the accessible information of an ensemble. Finally, we discuss more generally the connection between robustness-based measures, discrimination problems, and information-theoretic quantities.

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Introduction.—Although quantum states provide a complete description of a physical system at a given time, it is through the process of measurement that classical information about the state of the system is obtained. How much information is obtained depends upon the nature of the measurement made. Intuitively, some measurements are more informative than others, depending on how much correlation can be generated between the measurement outcomes and the state of the quantum system. Measurements which are not able to generate strong correlations, i.e., those which lead to almost uniform measurement outcomes for all quantum states, are naturally less informative than measurements which can lead to deterministic results.

The study of how informative a quantum measurement is, is not new. There has been a series of papers studying this question from an information-theoretic perspective [1–14]. The novel approach we adopt here comes from introducing a resource theory of measurement informativeness.

In recent years, there has been much interest coming from quantum information in studying quantum properties and phenomena taking a resource-theoretic perspective, whereby one treats the property or phenomenon of interest as a resource and tries to quantify it from an operational perspective. The prototypical example of such a quantum resource theory is the theory of entanglement [15,16], but there have been many other resource theories put forward recently, including asymmetry [17,18], coherence [19,20], purity [21], thermodynamics [22–24], magic states [25], nonlocality [26], steering [27], contextuality [28–30], knowledge [31], and projective simulability [32,33]. For a recent review article, see Ref. [34].

Here we are interested in returning to the question of how informative a measurement is, starting from such a resource-theoretic perspective. Previous works have considered quantifying measurements in terms of resources such as randomness or classical communication [2–4,10,12]. Here, in contrast, we would like to consider a resource theory of measurements, in particular, where we consider as the resource those measurements which are informative. A number of questions arise. Which measurements are most informative? How can we compare the informativeness of one measurement to another from this perspective?

To that end, we introduce here a way of quantifying the informativeness of a measurement by introducing what we call the robustness of a measurement, which, roughly speaking, corresponds to the amount of "noise" that has to be added to a measurement before it ceases to be informative at all. After showing that this quantity has the usual desirable properties that one would expect, such as faithfulness, convexity, and nonincrease under processing, we go on to show that it has a natural operational interpretation from the perspective of state discrimination, where it quantifies the best advantage that the measurement can provide over randomly guessing the state. Moreover, we also show that the robustness of measurement is naturally related to a single-shot generalization of the accessible information of a quantum-classical channel. Thus, although our starting point was different from previous work, we indeed find a close connection to many ideas already explored [1–14], as one might expect.

Using this insight, we return to a similar quantifier that was recently introduced in the context of the resource theory of coherence or asymmetry [35,36]. We show that the robustness of coherence is a single-shot generalization of the accessible information of an ensemble. We believe this signals a more general connection between robustness-based measures of resources, information-theoretic quantities, and discrimination-type problems, as we discuss in the Conclusions.

Robustness of measurement.—Let us think about (destructive) quantum measurements starting from a resource-theoretic perspective. Imagine a scenario with only one specific measuring device. That is, a box which accepts as input an arbitrary quantum state ρ (of fixed dimension d) and performs the measurement $\mathbb{M} = \{M_a\}_a$ with o outcomes on the system, where each M_a is a positive-semidefinite operator $M_a \geq 0$, [a positive-operator-valued measure (POVM) element], which collectively sum to the identity $\sum_a M_a = \mathbb{1}$. The box returns the measurement outcome a with probability $p(a) = \mathrm{tr}[M_a\rho]$.

Even if we only have access to the single measurement $\mathbb{M}$, we still naturally have access to another type of box, which performs a "trivial" measurement. That is, we consider a box $\mathbb{T} = \{T_a\}_a$ which accepts quantum states but produces random outcomes, i.e., which returns a supposed measurement outcome a with probability $q(a)$ independent of the quantum state measured. Such a measurement has POVM elements $T_a = q(a)\mathbb{1}$.

Using resource-theoretic language, we think of the set of all trivial measurements as being the *free* measurements and any measurement which is not of this form as being a *resourceful* measurement, i.e., one which genuinely performs a quantum measurement. We shall refer to this as a resource theory of measurement informativeness.

It is natural to look at quantitative properties of measurements from this perspective. In particular, given a particular measurement $\mathbb{M}$, one can quantify to what extent it is a resourceful measurement. Intuitively, ideal von Neumann measurements, where each POVM element is a rank-1 projector $M_a = \Pi_a$, should be among the most resourceful measurements.

Here we focus on a single measure, which we term the robustness of measurement (ROM), which is the analogue of robustness measures which have been widely studied in the many quantum information-theory contexts; for example, Refs. [35–39]. This particular measure has many nice properties and a compelling operational interpretation.

The ROM is defined by the minimal amount of noise that needs to be added to the measurement such that it becomes a trivial measurement. In particular, if instead of always performing the measurement $\mathbb{M}$, sometimes a different measurement $\mathbb{N} = \{N_a\}_a$ is performed, then of interest is the minimal probability of this other measurement which would make the overall measurement trivial. Formally,

$$\begin{aligned} R(\mathbb{M}) = \min_{r,\mathbb{N},\mathbf{q}} \ & r \\ \text{s.t.} \ & \frac{M_a + rN_a}{1+r} = q(a)\mathbb{1} \quad \forall \ a, \\ & N_a \geq 0 \quad \forall \ a, \qquad \sum_a N_a = \mathbb{1}. \end{aligned} \tag{1}$$

In the above equation, the minimization is over all noise measurements $\mathbb{N} = \{N_a\}_a$ and all probability distributions $\mathbf{q} = \{q(a)\}_a$. In order to have a number of convenient mathematical properties, we use the convention whereby the probability of noise is given by $r/(1+r)$.

Properties.—As is often the case for robustness-based measures, the robustness of measurements has a number of desirable properties: (i) It is *faithful*, meaning that it vanishes if and only if the measurement is trivial, i.e.,

$$R(\mathbb{M}) = 0 \Leftrightarrow M_a = q(a)\mathbb{1} \quad \forall \ a. \tag{2}$$

(ii) It is *convex*, meaning that one cannot have a larger robustness by classically choosing between two measurements, i.e., for $0 \leq p \leq 1$,

$$R[p\mathbb{M}_1 + (1-p)\mathbb{M}_2] \leq pR(\mathbb{M}_1) + (1-p)R(\mathbb{M}_2). \tag{3}$$

(iii) It is nonincreasing under any allowed measurement simulation [33]. That is, given access only to a single measurement $\mathbb{M}$, we can simulate any other measurement $\mathbb{M}' = \{M'_b\}_b$ (with an arbitrary number of outcomes b) such that

$$M'_b = \sum_a p(b|a)M_a, \tag{4}$$

where $p(b|a)$ form a set of conditional probability distributions [such that the matrix $[D]_{ab} = p(b|a)$ is a stochastic matrix]; i.e., the measurement $\mathbb{M}$ is performed and then the outcome is postprocessed. For any such simulated $\mathbb{M}'$, we have

$$R(\mathbb{M}') \leq R(\mathbb{M}). \tag{5}$$

These three properties can be easily shown and follow the same logic as in other robustness measures. Proofs are included in the Supplemental Material [40].

It turns out that $R(\mathbb{M})$ can be evaluated explicitly. By defining $\tilde{\mathbf{q}} = \{\tilde{q}(a)\}_a$ with $\tilde{q}(a) \coloneqq (1+r)q(a)$, and using the first equality in Eq. (1) to solve for N_a, $R(\mathbb{M})$ can be equivalently written as

$$\begin{aligned} R(\mathbb{M}) = \min_{\tilde{\mathbf{q}}} \ & \sum_a \tilde{q}(a) - 1 \\ \text{s.t.} \ & \tilde{q}(a)\mathbb{1} \geq M_a \quad \forall \ a, \end{aligned} \tag{6}$$

which is explicitly in the form of a semidefinite program [41]. However, by inspection, the optimal solution of this optimization problem can be identified: $\tilde{q}(a)$ will be minimized when equal to the operator norm $\|M_a\|_\infty$ (since M_a is positive semidefinite), and hence, we arrive at the exact expression

$$R(\mathbb{M}) = \sum_a \|M_a\|_\infty - 1. \tag{7}$$

In order to be a valid POVM element, it is necessary to satisfy the operator inequality $\mathbb{1} \geq M_a$ from which it follows that $1 \geq \|M_a\|_\infty$ and hence, $R(\mathbb{M}) \leq o - 1$. Consider also the pair

$$N_a = \frac{\mathrm{tr}[M_a]\mathbb{1} - M_a}{d-1}, \qquad q(a) = \frac{1}{d}\mathrm{tr}[M_a], \qquad (8)$$

which for any measurement $\mathbb{M}$ forms a valid measurement $\mathbb{N} = \{N_a\}_a$ and probability distribution $\mathbf{q} = \{q(a)\}_a$. This pair directly implies that $R(\mathbb{M}) \leq d - 1$. Putting both bounds together, we have

$$R(\mathbb{M}) \leq \min(o, d) - 1. \qquad (9)$$

This shows that in dimension d the largest robustness that can be achieved is $R(\mathbb{M}) = d - 1$, which can occur only for measurements with at least d outcomes.

It is interesting to identify which measurements achieve this maximum and are maximally robust. First, for ideal projective von Neumann measurements $M_a = \Pi_a$, we have $\|\Pi_a\|_\infty = 1$ for all a, and hence, $R(\mathbb{M}) = d - 1$. Consider furthermore any rank-1 measurement (with an arbitrary number of outcomes $o > d$), where $M_a = \alpha_a \Pi_a$. To be a valid measurement, $\alpha_a \geq 0$ and $\sum_a \alpha_a = d$. Such measurements are also seen to be maximally robust $R(\mathbb{M}) = d - 1$. We will return to the meaning of this later.

Finally, we saw previously that the ROM can be formulated as a semidefinite program (SDP) in Eq. (6). This provides a second representation of the ROM in terms of the dual formulation of the SDP [41], which will prove insightful for the operational significance of the ROM. As demonstrated explicitly in the Supplemental Material [40], strong duality holds, and the dual formulation of Eq. (6) is

$$\begin{aligned} R(\mathbb{M}) = \max_{\{\rho_a\}_a} & \sum_a \mathrm{tr}[\rho_a M_a] - 1 \\ \text{s.t. } & \rho_a \geq 0, \quad \mathrm{tr}[\rho_a] = 1 \quad \forall\ a, \end{aligned} \qquad (10)$$

where the maximization is now over the dual variables $\{\rho_a\}_a$, which, due to the nature of the constraints, are seen to correspond to quantum states.

As with the primal formulation, the dual can be solved explicitly by inspection. In particular, ρ_a should be chosen as a projector onto any state in the eigenspace of the maximal eigenvalue of M_a. With such a choice, then $\mathrm{tr}[M_a \rho_a] = \|M_a\|_\infty$ and $R(\mathbb{M}) = \sum_a \|M_a\|_\infty - 1$ as required.

Operational significance.—We now turn our attention to the operational significance of the ROM. Originally, it was introduced as a distance-based quantifier. Here, we will see that the ROM is also the advantage that can be achieved in a state discrimination problem over guessing at random if only the measurement $\mathbb{M}$ is available.

Consider a situation where one of a set of known states $\{\sigma_x\}_x$ is prepared with probability $\mathbf{p} = \{p(x)\}_x$. Such a situation is described by an ensemble $\mathcal{E} = \{\sigma_x, p_x\}_x$. The goal, as in any state discrimination problem, is to guess which of the states has been prepared in a given round. In each round, a guess g will be made of which state was prepared. Our figure of merit will be the average probability of guessing correctly, i.e., $p_{\mathrm{guess}}(\mathcal{E}) = \sum_x p(x)p(g = x|x) = \sum_{x,g} p(x)p(g|x)\delta_{x,g}$, where $p(g|x)$ is the conditional probability of guessing the state σ_g, given that the state σ_x was actually prepared. We will consider two situations: (i) the trivial situation where it is not possible to measure the quantum states prepared and (ii) where only a fixed measurement $\mathbb{M}$ can be performed in order to produce a guess.

In case (i), the optimal strategy is to always guess that the most probable state was prepared, i.e., the state σ_x such that $p(x) = \max_y p(y)$ (which may not be unique). If we denote by $p_{\max} = \max_x p(x)$, then in this case the probability to guess correctly is precisely $p^C_{\mathrm{guess}}(\mathcal{E}) = p_{\max}$.

In case (ii), after measuring the state prepared by using the measurement $\mathbb{M}$, the most general strategy is to produce a guess based upon the outcome according to some distribution $P(g|a)$, which will lead to a guessing probability of

$$P^Q_{\mathrm{guess}}(\mathcal{E}, \mathbb{M}) = \max_{\{P(g|a)\}} \sum_{x,a,g} p(x)\mathrm{tr}[\sigma_x M_a]P(g|a)\delta_{g,x}. \qquad (11)$$

We are then interested in the state discrimination problem which maximizes the ratio between these two guessing probabilities, i.e., the discrimination problem for which having access to the measurement $\mathbb{M}$ provides the biggest advantage over having to guess at random. Formally, we are interested in the advantage

$$\max_{\mathcal{E}} \frac{P^Q_{\mathrm{guess}}(\mathcal{E}, \mathbb{M})}{P^C_{\mathrm{guess}}(\mathcal{E})}. \qquad (12)$$

In the Supplemental Material [40], we show that the advantage is specified completely by the ROM, in particular, that

$$\max_{\mathcal{E}} \frac{P^Q_{\mathrm{guess}}(\mathcal{E}, \mathbb{M})}{P^C_{\mathrm{guess}}(\mathcal{E})} = 1 + R(\mathbb{M}) \qquad (13)$$

and that the optimal discrimination problem is to choose uniformly at random from a set of o states $\{\rho^*_a\}_a$ which are optimal variables for the dual SDP (10).

Considering specific examples, for an ideal von Neumann measurement, we can use it to perfectly guess which out of d states were prepared, whereas without the ability to perform a measurement, we would have to guess (uniformly at random), and hence, the advantage is $p^Q_{\mathrm{guess}}/p^C_{\mathrm{guess}} = d$. As a second example, consider a rank-1 measurement $\mathbb{M} = \{\alpha_a \Pi_a\}_a$. For the discrimination problem with the o states associated with this measurement, the guessing probability is $p^Q_{\mathrm{guess}} = d/o$, while the classical probability is $p^C_{\mathrm{guess}} = 1/o$, and again the advantage is d, as expected. This shows why such measurements still have maximal robustness, since they still allow for a d times advantage in this context.

Single-shot information theory.—We now demonstrate a second way of interpreting the operational significance of the ROM by making a connection to single-shot information theory by viewing a measurement alternatively as a quantum channel which produces classical outputs.

Given a general quantum channel, i.e., a general completely positive and trace-preserving map $\Lambda(\cdot)$ that maps quantum states to quantum states, a basic quantity of interest is the accessible information, the maximal amount of classical information that can be conveyed by the channel [44]

$$I^{\mathrm{acc}}(\Lambda(\cdot)) = \max_{\mathcal{E},\mathbb{D}} I(X:G), \tag{14}$$

where $\mathcal{E} = \{\sigma_x, p(x)\}_x$, with σ_x the input states to the channel, which are chosen with probability $p(x)$, $\mathbb{D} = \{D_g\}_g$ forms a POVM which is measured on the output of the channel to produce a symbol g with probability $p(g|x) = \mathrm{tr}[D_g \Lambda(\sigma_x)]$, and $I(X:G) = H(X) - H(X|G)$ is the classical mutual information of the distribution $p(x,g) = p(x)p(g|x)$. The accessible information quantifies the maximal amount of classical mutual information that can be generated between the input and output of the channel, optimizing over all encodings (input ensembles) and decodings (measurements).

Since it is based upon the Shannon entropy, I^{acc} is an asymptotic measure of a channel. Here, we will consider an analogous quantity in a single-shot regime, where the channel will only be used a single time. We consider the following single-shot variant of the mutual information [45]

$$I_{\min}(X:G) = H_{\min}(X) - H_{\min}(X|G), \tag{15}$$

where $H_{\min}(X) = -\log \max_x p(x)$ and $H_{\min}(X|G) = -\log \sum_g \max_x p(x,g)$ are the min-entropy and conditional min-entropy, respectively, and are the entropies associated with the guessing probability [46,47]. We then define the accessible min-information as

$$I^{\mathrm{acc}}_{\min}(\Lambda(\cdot)) = \max_{\mathcal{E},\mathbb{D}} I_{\min}(X:G), \tag{16}$$

where $\mathcal{E} = \{\sigma_x, p(x)\}_x$ and $\mathbb{D} = \{D_g\}_g$ are all as before.

A special class of quantum channels are those which correspond to measurements, i.e., quantum channels which take as input a quantum state ρ and produce as output the state $\sum_a \mathrm{tr}[M_a\rho]|a\rangle\langle a|$, where $\{|a\rangle\}$ forms an arbitrary orthonormal basis for the Hilbert space of the output. We denote by $\Lambda_{\mathbb{M}}(\cdot)$ the channel associated with the measurement $\mathbb{M} = \{M_a\}_a$ in this way.

We show in the Supplemental Material [40], that given this viewpoint, we can express the previous result that the ROM is the advantage in a state discrimination problem as

$$I^{\mathrm{acc}}_{\min}(\Lambda_{\mathbb{M}}(\cdot)) = \log[1 + R(\mathbb{M})]; \tag{17}$$

that is, the ROM is also equivalent to the accessible min-information of the channel $\Lambda_{\mathbb{M}}(\cdot)$ associated with the measurement, which is the maximal amount of mutual min-information that can be generated between the input and output of the channel in a single use.

Robustness of asymmetry and coherence.—We now turn our attention to a closely related robustness-based measure, the robustness of asymmetry (ROA) [36], which has as a special case the robustness of coherence [35]. We will show that the above operational significance of the ROM in terms of accessible min-information of a quantum-to-classical channel has a natural analogue for the ROA, where it will also be shown to be equal to the accessible min-information of an ensemble (for a suitably chosen ensemble), which can be thought of as a classical-to-quantum channel.

The ROA is the minimal amount of noise that needs to be added to a state before it becomes symmetric

$$\begin{aligned} \mathcal{A}_R(\rho) = \min_{s,\tau,\sigma}\ & s \\ \text{s.t.}\ & \frac{\rho + s\tau}{1+s} = \sigma \\ & \tau \geq 0,\ \mathrm{tr}[\tau] = 1,\ \sigma = \frac{1}{|H|}\sum_h U_h \sigma U_h^\dagger, \end{aligned} \tag{18}$$

where U_h forms a unitary representation of a group H, and σ is therefore a symmetric state (see Supplemental Material [40] for more details).

Here we will show that the ROA has an operational interpretation in terms of the accessible min-information of the ensemble $\mathcal{E}_\rho = \{U_h \rho U_h^\dagger, 1/|H|\}_h$. In particular, for an ensemble $\mathcal{E} = \{\sigma_h, q(h)\}_h$, the accessible min-information can be defined (in analogy to the accessible information [44]) as

$$I^{\mathrm{acc}}_{\min}(\mathcal{E}) = \max_{\mathbb{M}} I_{\min}(H:Y), \tag{19}$$

where $\mathbb{M} = \{M_y\}_y$ is an arbitrary POVM, and $p(h,y) = p(h)tr[\sigma_h M_y]$. Then, for ensembles of the form $\mathcal{E}_\rho = \{U_h \rho U_h^\dagger, 1/|H|\}_h$, it can be shown that

$$I^{\mathrm{acc}}_{\min}(\mathcal{E}_\rho) = \log[1 + \mathcal{A}_R(\rho)]. \tag{20}$$

That is, the ROA quantifies the accessible min-information of the ensemble formed by application of U_h to ρ. A proof of this statement can be found in the Supplemental Material [40].

We finish by noting that while a measurement can be viewed as a quantum-to-classical channel, an ensemble can be thought of as a classical-to-quantum channel, taking the classical random variable h to the quantum state σ_h. As such, the ROM and ROA can be seen as capturing properties of two extremal types of channels, transforming quantum information from or to classical information.

Conclusions.—Here we addressed the question of how informative a measurement is. We introduced a quantifier of informativeness, which we termed the robustness of measurement. Our first main finding is to show that this quantifier exactly characterizes the advantage that a measurement provides (over guessing at random) in the task of state discrimination. Second, when viewing a measurement as a quantum-to-classical channel, our quantifier is also equal to a single-shot generalization of the accessible information of the channel.

Our starting point was to introduce a resource theory of measurement informativeness, where we are only able to perform a single measurement $\mathbb{M}$, and the free operations are to postprocess the measurement (this should be contrasted to a recently introduced resource theory of measurement nonprojectiveness, where the resource is the nonprojective nature of a measurement [32,33,48]). The ROM was shown in Eq. (5) to be a *monotone* in this respect, that is, nonincreasing under the allowed operation. A natural question is what other monotones exist for this resource theory of measurements and to find a complete set of monotones which characterize whether or not a measurement $\mathbb{M}'$ is a postprocessing of $\mathbb{M}$ (i.e., to establish the partial order). In the Supplemental Material [40], we show that a complete set of monotones exists and is given by the success probability over the set of all discrimination games [49]. That is, $\mathbb{M}'$ is a postprocessing of $\mathbb{M}$ if and only if

$$P^{Q}_{\text{guess}}(\mathcal{E},\mathbb{M}) \geq P^{Q}_{\text{guess}}(\mathcal{E},\mathbb{M}') \quad \text{for all } \mathcal{E}, \tag{21}$$

where $\mathcal{E} = \{\sigma_x, p_x\}_x$ is any ensemble of states (on the same Hilbert space as the measurement $\mathbb{M}$), which should be guessed as well as possible in the discrimination game.

There are a number of interesting directions which we leave for future work. First, we focused on a particular choice of quantifier here, which we showed had desirable properties and interesting operational significance. One can nevertheless define other quantifiers starting from the resource-theory perspective taken here, e.g., based upon relative entropy or other distance-based measures. It would be interesting to understand how the use of other quantifiers can lead to further insights into the informativeness of a measurement.

Second, our work fits into a strand or research which connects robustness-based measures of resources with discrimination-type problems [36,39,50]. Although it was also known for the case of entanglement that the robustness-based measure was connected to single-shot information theory through the maxrelative entropy of entanglement [51], we believe that this is the first time where this triangle of connections has been made explicit more generally (see Fig. 1). It would be very interesting to understand just how far this triangle of robustness-based quantifier, discrimination problem, and information-theoretic quantity can be applied. For example, we conjecture that for any channel, single-shot accessible information is associated with a robustness of some type and moreover to a discrimination-type game. Indeed, we can ask if this triangle of associations holds very generally: Given any example of a vertex of the triangle, can one find the associated other two vertices (either in the single-shot or asymptotic scenario)?

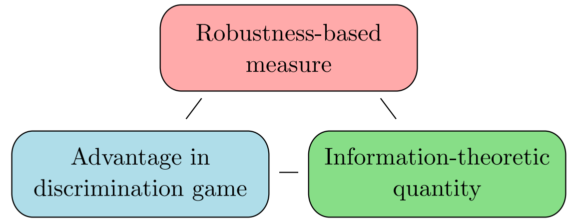

FIG. 1. Triangle of associations found here. A robustness-based measure is found to give the optimal advantage in a suitably chosen discrimination game. This, in turn, is found to be equivalent to a suitably defined information-theoretic quantity—the single-shot accessible information of a suitably defined channel. It is interesting to ask how general this triangle of associations is.

Third, here we have only considered the measurement outcomes and not the postmeasurement state. It would be interesting to extend the results here to the case of quantum instruments, where we also keep the postmeasurement state. In particular, since we know there is an information-disturbance trade-off, it would be interesting to investigate this phenomenon using the robustness of measurement as the quantifier of information gain.

Finally, in single-shot information theory it is usually necessary to introduce the possibilities of small errors—and therefore, approximations—in order to obtain meaningful results through the use of smoothed entropies. Here, however, we have not had to introduce such approximations and smoothing. It would be interesting to consider the role of approximation and smoothing when considering the measurements from this resource-theoretic perspective.

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Note added.—Recently, the independent work of Takagi *et al.* appeared online [52]. In that work, the authors show a general connection between robustness-based measures for states and discrimination games, as we conjectured here in our discussion as one link in the triangle.

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