Decoupled Cascades of Kinetic and Magnetic Energy in Magnetohydrodynamic Turbulence

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Magnetic energy (ME) and kinetic energy (KE) in ideal incompressible magnetohydrodynamics are not global invariants and, therefore, it has been justified to discuss only the cascade of their sum total energy. We provide a physical argument based on scale locality, along with compelling evidence that ME and KE budgets statistically decouple beyond a transitional "conversion" range. This arises because magnetic field-line stretching is a large-scale process which vanishes on average at intermediate and small scales within the inertial-inductive range, thereby allowing each of the mean ME and KE to cascade conservatively and at an equal rate, yielding a turbulent magnetic Prandtl number of unity over these scales.

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Magnetohydrodynamic (MHD) turbulence is of fundamental importance to many fields of science, including astrophysics, solar physics, space weather, and nuclear fusion. The Reynolds numbers of such flows are typically very large, giving rise to plasma fluctuations with power-law spectra over a vast range of scales where both viscosity and resistivity are negligible. We call such a range "inertial inductive" since ideal dynamics dominate. There are several competing theories for the spectrum of strong MHD turbulence over the inertial-inductive range [1–5], all of which assume scale locality of the energy cascade, which has been shown to hold [6].

In a scale-local cascade, energy transfer across scale ℓ is predominantly due to scales within a moderate multiple of ℓ [7]. This gives rise to an inertial-inductive scale range over which the flow evolves without direct communication with the largest or smallest scales in the system.

In MHD turbulence, only the sum of magnetic and kinetic energy (ME and KE, respectively), i.e., total energy, is a global invariant of the inviscid unforced dynamics. Therefore, it has been justified to discuss only the cascade of total energy but not of KE or ME separately, which are coupled by magnetic field-line stretching. In principle, the process of magnetic field-line stretching can operate at *all* scales, giving rise to various phenomena such as Alfvén waves.

We shall show here that magnetic field-line stretching is a large-scale process, which operates over a "conversion range" of scales of limited extent and vanishes *on average* at intermediate and small scales in the inertial-inductive range [8]. Over the ensuing part of the inertial-inductive range, mean KE and ME cascade conservatively and at an equal rate to smaller scales despite not being separate invariants.

Our findings are important in subgrid scale modeling of systems such as accretion disks, whose evolution is controlled by magnetic flux through the disk [9–11]. The strength of the magnetic field is determined by a balance between (i) turbulent advection (or turbulent viscosity) which accretes the field radially inward and (ii) turbulent resistivity which diffuses it outward [12–15]. Other applications are outlined in the conclusion.

We start from the incompressible MHD equations with a constant density ρ:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{u} + \mathbf{f}, \tag{1}$$

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}. \tag{2}$$

Here, $\mathbf{u}$ is the velocity, and $\mathbf{B}$ is the magnetic field normalized by $\sqrt{4\pi\rho}$ to have Alfvén (velocity) units. Both fields are solenoidal: $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0$. The pressure is p, $\mathbf{J} = \nabla\times\mathbf{B}$ is the (normalized) current density, $\mathbf{f}$ is external forcing, ν is viscosity, and η is resistivity.

In a statistically steady state, the space-averaged KE and ME budgets are, respectively,

$$\langle S_{ij}B_iB_j\rangle = \epsilon^{\mathrm{inj}} - \nu\langle|\nabla\mathbf{u}|^2\rangle, \tag{3}$$

$$\langle S_{ij}B_iB_j\rangle = \eta\langle|\nabla\mathbf{B}|^2\rangle, \tag{4}$$

where $\langle\ldots\rangle$ is a spatial average, $S_{ij} = (\partial_j u_i + \partial_i u_j)/2$ is the strain rate tensor, and $\epsilon^{\mathrm{inj}} = \langle\mathbf{f}\cdot\mathbf{u}\rangle$ is the kinetic energy injection rate. It is clear from Eqs. (3) and (4) that mean KE-to-ME conversion due to magnetic field-line stretching is positive and bounded: $0 \le \langle\mathbf{B}\cdot\mathbf{S}\cdot\mathbf{B}\rangle \le \epsilon^{\mathrm{inj}}$. The bound holds in the presence of an arbitrarily strong uniform magnetic field $\mathbf{B}_0$ indicating significant cancellations. This can be understood by considering that in a turbulent flow, the strain $\mathbf{S}$ being a derivative is dominated by the small scales, whereas $\mathbf{B}$ is dominated by the large scales near the magnetic spectrum's peak leading to decorrelation effects.

To analyze how magnetic field-line stretching operates at different length scales, we utilize a coarse-graining approach for diagnosing multiscale dynamics [7,16]. A coarse-grained field which contains modes at length scales $> \ell$ is defined by $\overline{f}_\ell(\mathbf{x}) = \int d\mathbf{r} G_\ell(\mathbf{r}-\mathbf{x}) f(\mathbf{r})$, where $G_\ell(\mathbf{r}) \equiv \ell^{-3} G(\mathbf{r}/\ell)$ is a normalized kernel with its main weight in a ball of diameter ℓ. Coarse-grained MHD equations can then be written to describe $\overline{\mathbf{u}}_\ell$ and $\overline{\mathbf{B}}_\ell$, along with corresponding budgets for the quadratic invariants at scales $\geq \ell$, for *arbitrary* ℓ in contrast to the mean field approach [17,18] (see Ref. [16] and references therein). Hereafter, we drop subscript ℓ when possible.

The KE and ME density balance at scales $> \ell$ are

$$\partial_t \left(\frac{|\overline{\mathbf{u}}|^2}{2} \right) + \nabla \cdot [\cdots] = -\overline{\Pi}^u_\ell - \overline{S}_{ij} \overline{B}_i \overline{B}_j - \nu |\nabla \overline{\mathbf{u}}|^2 + \overline{\mathbf{f}} \cdot \overline{\mathbf{u}}, \quad (5)$$

$$\partial_t \left(\frac{|\overline{\mathbf{B}}|^2}{2} \right) + \nabla \cdot [\cdots] = -\overline{\Pi}^b_\ell + \overline{S}_{ij} \overline{B}_i \overline{B}_j - \eta |\nabla \overline{\mathbf{B}}|^2, \quad (6)$$

where $\nabla \cdot [\cdots]$ represents spatial transport terms. Dissipation terms $\nu |\nabla \overline{\mathbf{u}}|^2$ and $\eta |\nabla \overline{\mathbf{B}}|^2$ are mathematically guaranteed to be negligible [16,19] at scales $\ell \gg (\ell_\nu, \ell_\eta)$, with ℓ_ν and ℓ_η the viscous and resistive length scales, respectively.

The first term on the rhs of Eq. (5), $\overline{\Pi}^u_\ell$, appears as a sink in the KE budget of large scales $> \ell$ and as a source in the KE budget of small scales $< \ell$ [16]. It quantifies the KE transfer *across* scale ℓ and is defined as $\overline{\Pi}^u_\ell \equiv -\overline{S}_{ij} \overline{\tau}_{ij}$, where $\overline{\tau}_{ij} = \tau_\ell(u_i, u_j) - \tau_\ell(B_i, B_j)$ is the sum of both the Reynolds stress and the Maxwell stress generated by scales $< \ell$ acting against the large-scale strain, $\overline{S}_{ij}$. Subscale stress is defined as $\tau_\ell(f, g) = \overline{(fg)}_\ell - \overline{f}_\ell \overline{g}_\ell$ for any two fields f and g. Similarly, $\overline{\Pi}^b_\ell \equiv -\overline{\mathbf{J}}_\ell \cdot \overline{\boldsymbol{\varepsilon}}_\ell$ in Eq. (6) quantifies the ME transfer *across* scale ℓ, where $\overline{\boldsymbol{\varepsilon}}_\ell \equiv \overline{\mathbf{u} \times \mathbf{B}} - \overline{\mathbf{u}} \times \overline{\mathbf{B}}$ is (minus) the electric field generated by scales $< \ell$ acting on the large-scale current $\overline{\mathbf{J}} = \nabla \times \overline{\mathbf{B}}$ resulting in a "turbulent Ohmic dissipation" to the small scales.

Term $\overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell$ appears as a sink in Eq. (5) and a source in Eq. (6) representing KE expended by the large-scale flow to bend and stretch large-scale $\overline{\mathbf{B}}$ lines. Unlike the cascade terms $\overline{\Pi}^u_\ell$ and $\overline{\Pi}^b_\ell$, which involve large-scale fields acting against subscale terms ($\overline{\boldsymbol{\tau}}_\ell$ and $\overline{\boldsymbol{\varepsilon}}_\ell$), $\overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell$ is purely due to large-scale fields and does not participate in energy transfer *across* scale ℓ. A more refined scale-by-scale analysis in Ref. [6] showed how energy lost or gained from one field ($\mathbf{u}$ or $\mathbf{B}$) by line stretching reappeared in or disappeared from the other field at the *same* scale.

In a steady state, space-averaging Eqs. (5) and (6) at any scale ℓ in the inertial-inductive range $L \gg \ell \gg (\ell_\nu, \ell_\eta)$ yields

$$\langle \overline{\Pi}^u_\ell \rangle = \epsilon^{\text{inj}} - \mathcal{C}^{ub}(\ell), \quad (7)$$

$$\langle \overline{\Pi}^b_\ell \rangle = \mathcal{C}^{ub}(\ell), \quad (8)$$

where we have dropped the dissipation terms and assumed that forcing is due to modes at scales $\sim L \gg \ell$, such that $\overline{\mathbf{f}}_\ell = \mathbf{f}$. Mean conversion $\mathcal{C}^{ub}(\ell) \equiv \langle \overline{S}_{ij} \overline{B}_i \overline{B}_j \rangle$ in Eqs. (7) and (8) quantifies the cumulative KE-to-ME conversion at *all* scales $> \ell$.

Using scale locality of the cascade terms $\overline{\Pi}^u_\ell$ and $\overline{\Pi}^b_\ell$, which was proved in Ref. [6], we will now argue that mean magnetic field-line stretching is primarily a large-scale process which vanishes at intermediate and small scales within the inertial-inductive range. Note that the scale locality discussed in Refs. [6,7] is "diffuse" [20] and states that contributions from disparate scales decay only as a power law of the scale ratio.

Define ℓ_d as the largest scale at which nonideal microphysics becomes significant, $\ell_d = \max(\ell_\nu, \ell_\eta)$. Define the cumulative KE-to-ME conversion at scales $> \ell_d$ by $\mathcal{C}^{ub}_d \equiv \mathcal{C}^{ub}(\ell_d)$, which is not necessarily equal to the unfiltered $\langle \mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B} \rangle$ due to possible contributions from scales $< \ell_d$ [see discussion shortly after Eq. (10) below].

Define ℓ_s as the largest scale at which $\mathcal{C}^{ub}(\ell_s) = \mathcal{C}^{ub}_d$. We will argue that (i) $\ell_s \neq \ell_d$ and (ii) $\mathcal{C}^{ub}(\ell) = \mathcal{C}^{ub}_d$ for all scales $\ell_s > \ell \gg \ell_d$.

First, assume $\ell_s = \ell_d$. This implies that as functions of ℓ, $\mathcal{C}^{ub}(\ell) = \langle \overline{\Pi}^b_\ell \rangle = \epsilon^{\text{inj}} - \langle \overline{\Pi}^u_\ell \rangle$ depend on dissipative parameters ν or η. However, $\langle \overline{\Pi}^u_\ell \rangle$ and $\langle \overline{\Pi}^b_\ell \rangle$ are scale local in the inertial-inductive range [6] and are insensitive to the microphysics when $\ell \gg \ell_d$. Therefore, $\ell_s \neq \ell_d$. Second, if $\mathcal{C}^{ub}(\ell) \neq \mathcal{C}^{ub}_d$ over $\ell_s > \ell \gg \ell_d$, then $\mathcal{C}^{ub}(\ell)$, which we assume is continuous, will have an extremum at a scale ℓ_* within that range [since $\mathcal{C}^{ub}(\ell_s) = \mathcal{C}^{ub}(\ell_d) = \mathcal{C}^{ub}_d$]. Therefore, $\langle \overline{\Pi}^u_\ell \rangle$ and $\langle \overline{\Pi}^b_\ell \rangle$ will also have extrema, indicating the existence a special scale ℓ_* in the inertial-inductive range, in conflict with scale invariance of ideal MHD.

Therefore, $\mathcal{C}^{ub}(\ell) \to \mathcal{C}^{ub}_d$ within a conversion range $L > \ell > \ell_s$ and, over the ensuing range $\ell_s > \ell \gg \ell_d$, it saturates at $\mathcal{C}^{ub}(\ell) = \mathcal{C}^{ub}_d$. Since $\mathcal{C}^{ub}(\ell)$ measures the cumulative KE-to-ME conversion at all scales $> \ell$, saturation implies a zero contribution from $\ell_s > \ell \gg \ell_d$. We conclude that mean KE-to-ME conversion $\langle S_{ij} B_i B_j \rangle$ is a large-scale process within the inertial-inductive range, acting over a conversion range $L > \ell > \ell_s$ of limited extent; i.e., the scale-range does not increase asymptotically with the Reynolds number. Mean KE and ME budgets decouple in the absence of conversion over the "decoupled range" of scales $\ell_s > \ell \gg \ell_d$:

TABLE I. Each suite of runs was carried out at different Reynolds numbers at 256^3, 512^3, and 1024^3 resolutions. Run V was also conducted at 2048^3 resolution. $Pm = \nu/\eta$ is the magnetic Prandtl number. $B_k^{\max} = \sqrt{\max_k[E^b(k)]}$ is at the magnetic spectrum's $[E^b(k)]$ peak. Arn'old-Beltrami-Childress (ABC) (helical) and Taylor-Green (TG) (nonhelical) forcing were applied at wave number k_f. More details are in the Supplemental Material [25].

Run	Forcing	k_f	Pm	$\lvert\mathbf{B}_0\rvert/B_k^{\max}$
I	ABC	2	1	0
II	ABC	2	1	10
III	TG	1	1	0
IV	TG	1	2	0
V	ABC	2	1	2

$$\langle \overline{\Pi}^u_\ell \rangle = \epsilon^{\rm inj} - \mathcal{C}^{ub}_d, \tag{9}$$

$$\langle \overline{\Pi}^b_\ell \rangle = \mathcal{C}^{ub}_d. \tag{10}$$

With the rhs of Eqs. (9) and (10) being independent of scale ℓ, KE and ME each cascades conservatively after the mechanism coupling them halts. Scale locality suggests that the normalized KE and ME cascade rates $\langle \overline{\Pi}^u \rangle/\epsilon^{\rm inj}$ and $\langle \overline{\Pi}^b \rangle/\epsilon^{\rm inj}$ should have a universal value of order unity over $\ell_s > \ell \gg \ell_d$, regardless of the forcing, $Pm = \nu/\eta$, or $\mathbf{B}_0$. Note that scale ℓ_s at which the budgets decouple is within the inertial-inductive range, despite the well-known non-equipartition of KE and ME spectra in that range [21–24] (see Fig. 8 in the Supplemental Material [25]).

While the above argument suggests that $\mathcal{C}^{ub}(\ell)$ should become constant at scales smaller than the conversion range, it only applies within the inertial-inductive range, $L \gg \ell \gg \ell_d$. It is possible for $\mathcal{C}^{ub}(\ell)$ to vary again when transitioning to scales $\lesssim \ell_d$. An example is the viscous-inductive (Batchelor) range $\ell_\nu \gg \ell \gg \ell_\eta$ over which a scale-by-scale analysis in Ref. [6] showed that magnetic field-line stretching can act as a forcing term in the ME budget, consistent with our understanding of high Pm flows [44,45]. The above argument for saturation of $\mathcal{C}^{ub}(\ell)$ breaks down at scales $\lesssim \ell_d$, such as in the viscous-inductive range where scale locality does not hold due to a smooth velocity field [6].

Our conclusions are supported by a suite of pseudo-spectral direct numerical simulations up to 2048^3 in resolution with phase-shift dealiasing using hyperdiffusion and other parameters summarized in Table I.

Figure 1 shows results from the five flows we consider at the highest resolution (see Supplemental Material [25] for lower resolution runs and evidence of convergence). In all runs, total energy being a global invariant is transferred conservatively across scales $L \gg \ell \gg \ell_d$, as indicated by a scale-independent total energy flux, $\langle \overline{\Pi}_\ell \rangle = \langle \overline{\Pi}^u_\ell + \overline{\Pi}^b_\ell \rangle$.

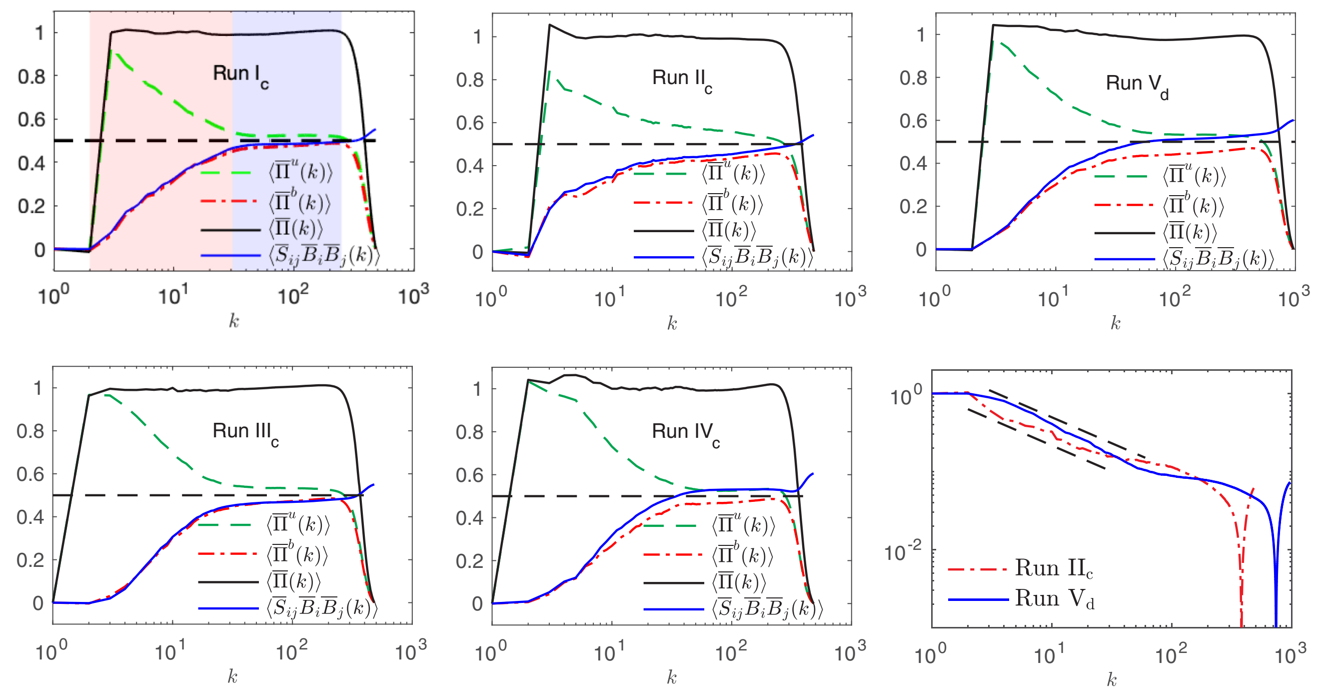

FIG. 1. The first five panels show $\langle \overline{\Pi} \rangle = \langle \overline{\Pi}^u + \overline{\Pi}^b \rangle$, $\langle \overline{\Pi}^u \rangle$, $\langle \overline{\Pi}^b \rangle$, and $\langle \overline{S}_{ij}\overline{B}_i\overline{B}_j \rangle$ as a function of $k \equiv 2\pi/\ell$ from our highest resolution runs (1024^3 for runs I to IV and 2048^3 for run V; see Supplemental Material [25] for lower resolutions). In the top-left panel, the conversion (decoupled) range is shaded red (blue). All plots are time averaged and normalized by $\epsilon^{\rm inj}$. The horizontal straight dashed line is at 0.5. Bottom-right panel shows a log-log plot of relative residual conversion $\mathcal{R}^{ub}(k)/\mathcal{C}^{ub}_d$ and a reference black-dashed line with a $-2/3$ slope, suggesting that KE-to-ME conversion saturates in a manner consistent with scale locality [6].

Both $\overline{\Pi}_\ell^u$ and $\overline{\Pi}_\ell^b$ decay to zero at scales $\lesssim \ell_d$ when the nonlinearities shut down in the dissipation range. Mean KE-to-ME conversion $\mathcal{C}^{ub}(\ell)$ increases from 0 at the largest scales to $\approx \mathcal{C}_d^{ub} \approx \epsilon^{\rm inj}/2$ at an intermediate scale ℓ_s within the inertial-inductive range. Over the ensuing range, $\ell_s > \ell \gg \ell_d$, $\mathcal{C}^{ub}(\ell)$ is scale independent, indicating a negligible contribution to magnetic field-line stretching at these scales. There is a slight increase in $\mathcal{C}^{ub}(\ell)$ in the dissipation range at scales $\lesssim \ell_d$ where our argument is not expected to hold due to a lack of scale locality. In all cases, $\langle \overline{\Pi}_\ell^b \rangle \approx \mathcal{C}^{ub}(\ell)$ and $\overline{\Pi}_\ell^u \approx \epsilon^{\rm inj} - \mathcal{C}^{ub}(\ell)$ over the inertial-inductive range, consistent with Eqs. (7) and (8). Beyond the conversion range, scale transfer becomes independent of ℓ, $\langle \overline{\Pi}_\ell^u \rangle \approx \epsilon^{\rm inj} - \mathcal{C}_d^{ub}$ and $\langle \overline{\Pi}_\ell^b \rangle \approx \mathcal{C}_d^{ub}$ over $\ell_s > \ell \gg \ell_d$, consistent with Eqs. (9) and (10), and indicative of a conservative cascade of KE and ME energy, respectively. In all runs, we observe that the KE and ME cascade rates become equal in magnitude $\langle \overline{\Pi}_\ell^u \rangle \approx \langle \overline{\Pi}_\ell^b \rangle$ over $\ell_s > \ell \gg \ell_d$, with magnetic field-line stretching channeling $\approx 1/2$ of the injected energy to the magnetic field, regardless of the forcing, Pm, or $\mathbf{B}_0$, consistent with scale locality.

Among the five cases in Fig. 1, the conversion range is widest in the presence of $|\mathbf{B}_0|/B_k^{\max} = 10$ (run II$_c$). However, according to our argument, its extent cannot increase indefinitely with an increasing dynamic range of scales (or Reynolds number, Re). After all, $\langle \mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B} \rangle$ is bounded even in the $|\mathbf{B}_0| \to \infty$ limit. Indeed, a plot of the relative residual conversion $\mathcal{R}^{ub}(\ell)/\mathcal{C}_d^{ub} \equiv \langle \overline{\mathbf{B}}_{\ell_d} \cdot \overline{\mathbf{S}}_{\ell_d} \cdot \overline{\mathbf{B}}_{\ell_d} - \overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell \rangle / \langle \overline{\mathbf{B}}_{\ell_d} \cdot \overline{\mathbf{S}}_{\ell_d} \cdot \overline{\mathbf{B}}_{\ell_d} \rangle$ in Fig. 1 (and Fig. 5 in the Supplemental Material [25]) decays at least as fast as a power law as $\ell \to \ell_d$, consistent with what is expected from scale locality (we take ℓ_d as the scale at which $\langle \overline{\Pi}_\ell \rangle = \epsilon^{\rm inj}/2$). Moreover, plots of $\mathcal{C}^{ub}(\ell)$ at increasing Re (Fig. 4 in the Supplemental Material [25]) show a clear convergence to $\mathcal{C}_d^{ub} \approx \epsilon^{\rm inj}/2$.

The negligible mean KE-to-ME conversion at small scales within the decoupled range might seem counterintuitive at first. After all, a hallmark of MHD turbulence is Alfvén waves which are fastest at small scales. The

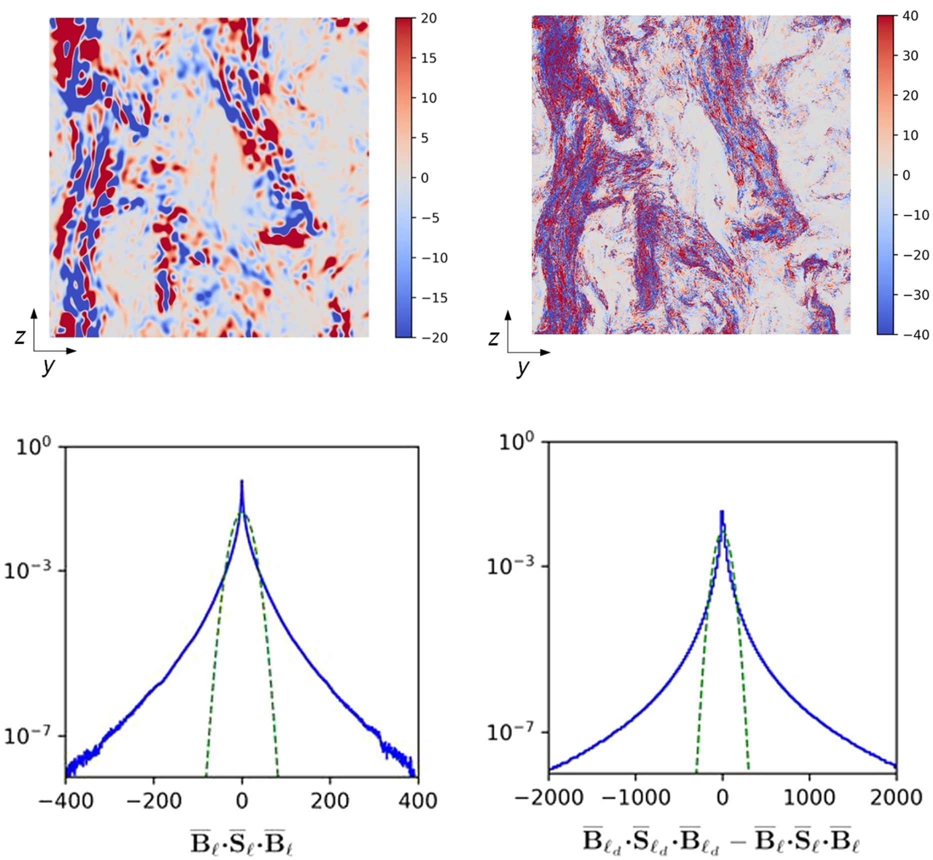

FIG. 2. For scale $\ell = 2\pi/30$ ($k = 30$) from run II$_c$ in Fig. 1 at one instant in time: Top two panels show a 2D slice from the 3D domain of pointwise conversion at large scales $\overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell(\mathbf{x})$ (top left) and small scales $\overline{\mathbf{B}}_{\ell_d} \cdot \overline{\mathbf{S}}_{\ell_d} \cdot \overline{\mathbf{B}}_{\ell_d}(\mathbf{x}) - \overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell(\mathbf{x})$ (top right). $\mathbf{B}_0$ is in the z direction. Bottom two panels show probability density function of conversion as a function of $\mathbf{x}$ at large scales (bottom left) and small scales (bottom right). The large-scale distribution has mean of 0.43 and variance of 223.54. The small-scale distribution has mean of 0.09 and variance of 3060.84. Quantities are normalized by energy injection rate $\epsilon^{\rm inj}$. Un-normalized Gaussians (green dashed lines) are added to both plots.

decoupling of ME and KE budgets poses no contradiction since it is only in the *mean*, which allows for decorrelation effects at small scales similar to those arising in compressible turbulence [46,47]. Utilizing the simultaneous information in both scale and space afforded by our coarse-graining approach, we analyze $\overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell(\mathbf{x})$ acting on scales $> \ell$ and the residual conversion within the inertial-inductive range $\overline{\mathbf{B}}_{\ell_d} \cdot \overline{\mathbf{S}}_{\ell_d} \cdot \overline{\mathbf{B}}_{\ell_d}(\mathbf{x}) - \overline{\mathbf{B}}_\ell \cdot \overline{\mathbf{S}}_\ell \cdot \overline{\mathbf{B}}_\ell(\mathbf{x})$ as a function of space $\mathbf{x}$ in Fig. 2. For an intermediate scale $\ell = 2\pi/30$ from run II_c (and run I_c in Fig. 7 of the Supplemental Material [25]), Fig. 2 shows how magnetic field-line stretching, which is concentrated in magnetic filaments, is an order of magnitude more intense at scales smaller than $\ell = 2\pi/30$ compared to larger scales. Yet, the small-scale contribution fluctuates vigorously in sign, yielding a mere 17% (10% in run I_c in Fig. 7 of the Supplemental Material [25]) to the space average. To illuminate the role of waves, we repeat in the Supplemental Material [25] the analysis above on two examples of noncolliding Alfvén waves, a monochromatic wave and a wave packet, which are exact solutions of the MHD equations and which lack energy transfer between scales.

In conclusion, small scales of the magnetic field in the decoupled scale range are maintained, on average, by turbulent Ohmic dissipation (the ME cascade), $\langle \overline{\Pi}^b \rangle = \langle \overline{\mathbf{J}} \cdot \overline{\boldsymbol{\varepsilon}} \rangle$. Magnetic field-line stretching acts on average as a large-scale driver of the ME cascade, justifying the inclusion of a low-mode forcing in the induction Eq. (2) when resolving the transitional conversion range is unimportant, such as in high-*Re* asymptotic scaling studies of MHD turbulence [48–50]. Our results will help in deriving relations equivalent to the Politano-Pouquet relations [51] but for the separate cascades of KE and ME, with potential implications on the scaling in MHD turbulence. This work can also help subgrid scale model development and testing in large eddy simulations of MHD turbulence [52,53]. For example, it provides a direct measure of the turbulent magnetic Prandtl number, which is unity within the decoupled range due to equipartition of the cascades $\langle \overline{\Pi}^u_\ell \rangle = \langle \overline{\Pi}^b_\ell \rangle$. This has important implications in astrophysical flows such as accretion disks [12,14,15]. Our findings are also relevant for turbulent magnetic reconnection [22,54,55] since they imply that the net bending and twisting of magnetic field lines at length scales in the decoupled range is driven by the effective electric field $-\overline{\boldsymbol{\varepsilon}}_\ell$ rather than by the flow's strain, giving independent support to previous studies [22,56]. Our framework for quantifying field-line stretching at various scales may also prove insightful in magnetic dynamo studies [57–60].

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Correction: The first term in Eq. (6) contained an error, the second inline equation in the tenth paragraph had a sign error, and the fourth entry in the "Forcing" column in Table I was erroneous. All three instances have been fixed and related information in the Supplemental Material has been revised.