All Sets of Incompatible Measurements give an Advantage in Quantum State Discrimination

Paul Skrzypczyk,[1,*] Ivan Šupić,[2,†] and Daniel Cavalcanti[3,‡]
[1]H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol, BS8 1TL, United Kingdom
[2]Département de Physique Appliquée, Université de Genève, 1211 Genève, Switzerland
[3]ICFO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain

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Some quantum measurements cannot be performed simultaneously; i.e., they are incompatible. Here we show that every set of incompatible measurements provides an advantage over compatible ones in a suitably chosen quantum state discrimination task. This is proven by showing that the robustness of incompatibility, a quantifier of how much noise a set of measurements tolerates before becoming compatible, has an operational interpretation as the advantage in an optimally chosen discrimination task. We also show that if we take a resource-theory perspective of measurement incompatibility, then the guessing probability in discrimination tasks of this type forms a complete set of monotones that completely characterize the partial order in the resource theory. Finally, we make use of previously known relations between measurement incompatibility and Einstein-Podolsky-Rosen steering to also relate the latter with quantum state discrimination.

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Introduction.—In quantum mechanics, observables described by noncommuting operators satisfy an uncertainty relation, which implies that we cannot acquire precise information about them simultaneously [1]. First thought to be a limitation, recent advances in quantum information theory have demonstrated that this feature is behind several applications, such as the security of quantum key distribution [2] and nonlocality based (or device-independent) applications [3].

Commutation is well defined for sharp (von Neumanm) measurements. However, a more refined notion of measurement incompatibility is needed for general measurements described by positive-operator-value measures (POVMs) [4]. This is captured by the idea of joint measurability [5]. Suppose a set of measurements $\{\mathbb{M}_x\}_x$ is labeled by $x = 1, \ldots, m$, each described by measurement operators $M_{a|x}$ ($M_{a|x} \geq 0$, $\sum_a M_{a|x} = \mathbb{1}\ \forall\ a, x$), where $a = 1, \ldots, o$ labels each of the measurement outcomes. This set is said to be jointly measurable (or compatible) if there exists a "parent" measurement $\mathbb{G}$ with measurement operators G_λ, and conditional probability distributions $p(a|x, \lambda)$, such that

$$M_{a|x} = \sum_\lambda p(a|x, \lambda) G_\lambda \quad \forall\ a, x. \tag{1}$$

Otherwise the set is said to be incompatible. This definition can be interpreted as follows: if Eq. (1) holds, all measurements $\mathbb{M}_x$ can be performed jointly, by the implementation of the single measurement $\mathbb{G}$ and a probabilistic classical postprocessing defined by the weights $p(a|x, \lambda)$.

A number of previous results have sought to understand the power of incompatible measurements from the perspective of nonlocality. For example, it was shown in Ref. [6] that every pair of dichotomic incompatible measurements can be used to violate the CHSH Bell inequality. In Refs. [7,8] it was further shown that every set of incompatible measurements leads to a weaker form of nonlocality, namely, Einstein-Podolsky-Rosen steering [9]. However, more recently it was shown in Refs. [10,11] that there exist sets of incompatible measurements that will never lead to violations of any Bell inequality, showing that such a general result does not hold for the strongest form of nonlocality.

In this Letter we wish to understand the power of incompatible measurements more directly. We do so, by finding an operational interpretation of measurement incompatibility in terms of quantum state discrimination: we show that a set of measurements is incompatible if and only if they provide an advantage over compatible ones in a quantum state discrimination (QSD) task with multiple ensembles of states. Moreover, we also show that the advantage of an optimally chosen QSD task is exactly quantified by the robustness of incompatibility of the set, a previously proposed quantifier of measurement incompatibility [12]. This result fits within a number of results recently obtained that have linked robustness-based quantifiers with advantages in suitably chosen discrimination games [13–17]. It can also be seen as an analogue for the measurement incompatibility of Ref. [18], which showed that a state is entangled if and only if it provides an advantage in a channel discrimination task.

Incompatibility and advantage in quantum state discrimination.—We consider the following two-party QSD task [19–21]: Bob can prepare different ensembles $\{\mathcal{E}_y\}_y$ $(y = 1, \ldots, n)$ of quantum states $\mathcal{E}_y = \{\rho_{b|y}, q(b|y)\}_b$, for $b = 1, \ldots, p$. At each round of the protocol, Bob chooses one of the ensembles y with probability $q(y)$ and sends Alice his choice y, and the state prepared $\rho_{b|y}$, which occurs with probability $q(b|y)$. Upon receiving y and $\rho_{b|y}$, Alice's goal is to identify which state she was sent, i.e., to correctly identify b.

We will consider playing this game in two different scenarios. In the first scenario, Alice has access to a fixed set of incompatible measurements $\{\mathbb{M}_x\}_x$ in order to play. We consider the most general probabilistic strategies assuming that the only way Alice can interact with the system is through her fixed measuring device. In particular, we allow any strategy consisting of the following [22]: After receiving the state and the value of y, Alice makes use of a random variable μ to perform the measurement $\mathbb{M}_x$, with probability $p(x|y,\mu)$. After receiving outcome a she makes a guess of the value of b, according to $p(g|a,y,\mu)$. Optimizing over all strategies, we can quantify how well Alice does in this game by evaluating the average probability of correctly identifying b, i.e.,

$$P_g(\{\mathcal{E}_y\},\{\mathbb{M}_x\}) = \max_{\mathcal{S}} \sum_{byaxg\mu} q(b,y)p(\mu)p(x|y,\mu) \times \mathrm{tr}[\rho_{b|y}M_{a|x}]p(g|a,y,\mu)\delta_{g,b}, \tag{2}$$

where the maximization is over strategies $\mathcal{S} = \{p(\mu), p(x|y,\mu), p(g|a,y,\mu)\}$, and we have written $q(b,y) = q(y)q(b|y)$.

We will contrast this to a scenario where in any given run of the game Alice can only perform a single measurement (although we will allow once again the possibility of using randomness to mix over different fixed measurements in different runs of the game). In particular, we consider measurements $\mathbb{G}_\nu = \{G_{a|\nu}\}_a$, and allow for the most general strategy using any such measurements. Crucially now, since Alice can only perform a single measurement, the side information of y can only be used to implement a classical postprocessing of this measurement. The net effect is equivalent to Alice only being able to perform a set of compatible measurements, those achieved by the parent measurements $\mathbb{G}_\nu$. In this case the success probability is given by

$$P_g^{\mathrm{C}}(\{\mathcal{E}_y\}) = \max_{\mathcal{T}} \sum_{bya\nu g} q(b,y)p(\nu) \times \mathrm{tr}[\rho_{b|y}G_{a|\nu}]p(g|a,y,\nu)\delta_{g,b}, \tag{3}$$

where the maximization is over all strategies $\mathcal{T} = \{p(\nu), \mathbb{G}_\nu, p(g|a,y,\nu)\}$.

We are primarily interested in the *advantage* that is offered by a set of incompatible measurements $\{\mathbb{M}_x\}_x$ in any such QSD game. In particular, we are interested in the biggest relative increase in guessing probability that can be obtained by the set of measurements $\{\mathbb{M}_x\}_x$ compared to having access to only single measurements, among all possible ensembles, i.e.,

$$\max_{\{\mathcal{E}_y\}} \frac{P_g(\{\mathcal{E}_y\},\{\mathbb{M}_x\})}{P_g^{\mathrm{C}}(\{\mathcal{E}_y\})}. \tag{4}$$

The main result of this Letter is to show that this quantity is completely characterized by the robustness of incompatibility (ROI) of the measurements $I_R(\{\mathbb{M}_x\})$ as

$$1 + I_R(\{\mathbb{M}_x\}) = \max_{\{\mathcal{E}_y\}} \frac{P_g(\{\mathcal{E}_y\},\{\mathbb{M}_x\})}{P_g^{\mathrm{C}}(\{\mathcal{E}_y\})}. \tag{5}$$

The robustness of incompability $I_R(\{\mathbb{M}_x\})$ is defined as the minimal amount of "noise" that needs to be added to the set of measurements $\{\mathbb{M}_x\}_x$ before they become compatible [12]. Here, by noise, we mean that we mix the set of measurements with another, arbitrary, set of measurements $\{\mathbb{N}_x\}_x$, (of the same size, and with the same number of outcomes), in order to make the mixture compatible. Formally,

$$\begin{aligned} I_R(\{\mathbb{M}_x\}) = \min\ & r \\ \text{s.t.}\ & \frac{M_{a|x} + rN_{a|x}}{1+r} = \sum_\lambda p(a|x,\lambda)G_\lambda \\ & N_{a|x} \geq 0, \qquad \sum_a N_{a|x} = \mathbb{1}, \\ & p(a|x,\lambda) \geq 0, \qquad \sum_a p(a|x,\lambda) = 1, \\ & G_\lambda \geq 0, \qquad \sum_\lambda G_\lambda = \mathbb{1}, \end{aligned} \tag{6}$$

where the minimization is over r, $\{\mathbb{N}_x\}_x$ (where $\mathbb{N}_x = \{N_{a|x}\}_a$), $\mathbb{G} = \{G_\lambda\}_\lambda$ and $\{p(a|x,\lambda)\}_{a,x,\lambda}$, and all constraints are understood to hold for all values of a, x, or λ, as appropriate.

The ROI has a number of desirable properties: (i) It is faithful: $I_R(\{\mathbb{M}_x\}) = 0$ if and only if the set of measurements $\{M_x\}_x$ is incompatible; (2) It is convex: If the set of measurements $\{\mathbb{M}_x\}_x$ is a convex combination of two other sets of measurements, i.e., for all x, $\mathbb{M}_x = p\mathbb{M}_x^{(1)} + (1-p)\mathbb{M}_x^{(2)}$, for some $p > 0$, and for valid sets of measurements $\{\mathbb{M}_x^{(1)}\}_x$ and $\{\mathbb{M}_x^{(2)}\}_x$, then

$$I_R(\{\mathbb{M}_x\}) \leq pI_R(\{\mathbb{M}_x^{(1)}\}) + (1-p)I_R(\{\mathbb{M}_x^{(2)}\}); \tag{7}$$

(iii) It is nonincreasing under postprocessing of the measurements. That is, if we simulate a new set of measurements $\{\mathbb{M}'_y\}_y$ using $\{\mathbb{M}_x\}_x$, such that [23]

$$M'_{b|y} = \sum_{a,x,\mu} p(\mu)p(x|y,\mu)p(b|a,y,\mu)M_{a|x}, \tag{8}$$

where $p(\mu)$, $p(x|y,\mu)$, and $p(b|a,y,\mu)$ are arbitrary sets of probability distributions, then

$$I_R(\{\mathbb{M}'_y\}) \leq I_R(\{\mathbb{M}_x\}). \tag{9}$$

Because of Eq. (5), the properties (i)–(iii) are also satisfied by the advantage (4). In particular, due to (i), a set of measurements $\{\mathbb{M}_x\}_x$ provides an advantage over compatible measurements if and only if $I_R(\{\mathbb{M}_x\}) > 0$.

Another interesting consequence of Eq. (5) is that it gives an efficient way of computing the advantage (4). This is because the ROI can be shown to be expressed explicitly as the following semi-definite program (SDP):

$$\begin{aligned} 1 + I_R(\{\mathbb{M}_x\}) = \min_{s,\{\tilde{G}_\mathbf{a}\}} & \; s \\ \text{s.t.} & \sum_\mathbf{a} D_\mathbf{a}(a|x)\tilde{G}_\mathbf{a} \geq M_{a|x} \\ & \sum_\mathbf{a} \tilde{G}_\mathbf{a} = s\mathbb{1}, \qquad \tilde{G}_\mathbf{a} \geq 0, \end{aligned} \tag{10}$$

where $a = a_1 a_2 \cdots a_n$ is a string, which can be thought of as a list of "results," one for each measurement, $D_\mathbf{a}(a|x) = \delta_{a,a_x}$ are deterministic probability distributions, whereby $a = a_x$ with certainty, and $\tilde{\mathbb{G}} = \{\tilde{G}_\mathbf{a}\}_\mathbf{a}$ is a supernormalized parent POVM. The derivation of this SDP formulation can be found in the Supplemental Material [24].

Let us now sketch the proof of our main result (we leave the full proof for the Supplemental Material [24]). Consider that the solution of Eq. (6) is attained by $N^*_{a|x}$, G^*_λ, and $p^*(a|x,\lambda)$, which means that

$$\frac{M_{a|x} + I_R(\{\mathbb{M}_x\})N^*_{a|x}}{1 + I_R(\{\mathbb{M}_x\})} = \sum_\lambda p^*(a|x,\lambda)G^*_\lambda. \tag{11}$$

Since $I_R(\{\mathbb{M}_x\}) \geq 0$ and $N^*_{a|x} \geq 0$, we have that

$$[1 + I_R(\{\mathbb{M}_x\})]\sum_\lambda p^*(a|x,\lambda)G^*_\lambda \geq M_{a|x}, \quad \forall \; a, x.$$

Multiplying both sides of this expression by $\rho_{b|y}$, the probabilities appearing in the QSD game, taking the trace and applying the correct maximizations, we end up proving that

$$1 + I_R(\{\mathbb{M}_x\}) \geq \frac{P_g(\{\mathcal{E}_y\}, \{\mathbb{M}_x\})}{P_g^\mathrm{C}(\{\mathcal{E}_y\})}. \tag{12}$$

This expression is interesting by itself: it states that the ROI of a set of measurements provides an upper bound on the advantage that set provides in *any* QSD game (of the type considered here), defined by the ensembles $\{\mathcal{E}_y\}_y$. It also demonstrates that the ROI can readily be estimated experimentally. Indeed, the right-hand side can be experimentally estimated by carrying out a QSD game, and every such experiment places a lower bound on the ROI.

The second part of the proof consists in explicitly showing that for any set $\{\mathbb{M}_x\}_x$ there exists a choice $\{\mathcal{E}^*_y\}_y$ saturating the bound (12). Such a collection of ensembles can be constructed by using the duality theory of semidefinite programming [26]. In particular, in the Supplemental Material [24] we show that an equivalent formulation of the ROI (the dual formulation) is

$$\begin{aligned} 1 + I_R(\{\mathbb{M}_x\}) = \max_{\{\omega_{ax}\},X} & \; \mathrm{tr}\sum_{a,x}\omega_{ax}M_{a|x} \\ \text{s.t.} & \; X \geq \sum_{a,x}\omega_{ax}D_\mathbf{a}(a|x), \\ & \; \omega_{ax} \geq 0, \qquad \mathrm{tr}X = 1. \end{aligned} \tag{13}$$

Assuming that the maximum is attained by $\{\omega^*_{ax}\}_{ax}$, we can interpret these as unnormalized quantum states, which can be appropriately normalized, and from which we can then define a game through $\{\mathcal{E}^*_x\}_x$. We show in the Supplemental Material [24] that the advantage that $\{\mathbb{M}_x\}_x$ provide in playing this game is precisely $1 + I_R(\{\mathbb{M}_x\})$, which completes the proof.

To summarize, the above shows that the ROI, which was introduced as a purely geometrical quantifier of incompatibility, in fact has an operational interpretation as the advantage that a set of measurements provides in an optimally chosen QSD game. Moreover, since the ROI is faithful [property (i) above], every set of incompatible measurements gives an advantage in at least one QSD task, and thus this task captures the utility of incompatible measurements.

Resource theory of incompatibility.—We now turn to the next result of this Letter, and consider a resource theory of measurement incompatibility. We will see that this allows us to connect the notion of simulability of one set of measurements by another one, as given in Eq. (8), with the success probability of these sets in any QSD games considered here.

In any resource theoretic setting, there are 3 main ingredients [27]: (i) a set of free or resourceless objects, (ii) a set of expensive or resourceful objects, and (iii) a set of allowed transformations between objects, which should not be able to create resourceful objects from free objects. In the present setting, a resource theory of incompatible measurements can easily be formalized as (i) the free objects are the set of all compatible measurements, (ii) the resourceful objects are the set of all incompatible measurements, and (iii) the set of allowed transformations consist of all simulations [23]; i.e., we think of the simulation protocol of Eq. (8) as "transforming" the set of measurements $\{\mathbb{M}_x\}_x$ into the set $\{\mathbb{M}'_y\}_y$. From properties (i) and (iii) of the ROI, we see that any set of compatible measurements cannot be transformed into a set of incompatible ones by measurement simulation, and hence this is a consistent set of allowed transformations.

Within any resource theory, there is a natural partial order that arises between the objects of the theory: if one object can be transformed into another, then it is "before" it in the partial order. A basic question in any resource theory is then to understand the partial order—i.e., to find necessary and sufficient conditions that characterize whether one object can be transformed into another or not. Intuitively, objects can only be transformed into other objects that are not more resourceful than themselves, i.e., generalizing the idea that the allowed transformations not only cannot create resources from nothing, but cannot increase resources.

Any function of an object that cannot increase under an allowed transformation is known as a resource *monotone*, and acts as a witness that one object cannot be transformed into another object. In the present setting, property (iii) of the ROI shows that it is a monotone for the resource theory of incompatibility. It is, however, only a single monotone, and $I_R(\{\mathbb{M}_x\}) > I_R(\{\mathbb{M}_y\})$ does not in general imply that $\{\mathbb{M}_x\}_x$ can simulate $\{\mathbb{M}'_y\}_y$.

In the Supplemental Material [24], inspired by the connection between the ROI and QSD, we prove that Eq. (8) holds, which we will denote simply by $\{\mathbb{M}_x\}\succ\{\mathbb{M}'_y\}$, if and only if $\{\mathbb{M}_x\}$ outperforms $\{\mathbb{M}'_y\}$ in every single QSD game of the type considered above, i.e.,

$$P_g(\{\mathcal{E}_y\},\{\mathbb{M}_x\}) \geq P_g(\{\mathcal{E}_y\},\{\mathbb{M}'_y\}) \quad \forall\ \{\mathcal{E}_y\}_y \iff \{\mathbb{M}_x\}\succ\{\mathbb{M}'_y\}. \tag{14}$$

Notice that the backward implication ($\Leftarrow$) is natural: if $\{\mathbb{M}_x\}$ can simulate $\{\mathbb{M}'_y\}$, then it is obviously contradictory that there is a game where $\{\mathbb{M}'_y\}$ can outperform $\{\mathbb{M}_x\}$. Interestingly, the forward implication ($\Rightarrow$) holds, which proves that the QSD games studied here constitutes a complete set of operational monotones that determine if a set of measurements can simulate another. This, in particular, indicates that they capture the resource of incompatibility.

EPR steering and entanglement-based QSD.—Let us finally describe a connection between the present results and the notion of Einstein-Podolsky-Rosen (EPR) steering [9]. In the EPR steering scenario Alice and Bob share a bipartite quantum state ρ_{AB}, onto which Alice applies measurements $\mathbb{M}_x$, leaving Bob's state in the (unnormalized) postmeasurement states $\sigma_{a|x} = \mathrm{tr}_A[(M_{a|x}\otimes\mathbb{1})\rho_{AB}]$. The set of states $\{\sigma_{a|x}\}_{a,x}$—referred to as an assemblage [28]—is said to demonstrate EPR steering if they do not admit a local-hidden-state (LHS) decomposition of the type $\sigma_{a|x} = \sum_\lambda p(a|x,\lambda)\sigma_\lambda$, where $p(a|x,\lambda)$ are conditional probability distributions and σ_λ (unnormalized) quantum states [9]. Similarly to the case of incompatibility, the robustness of steerability $S_R(\{\sigma_{a|x}\})$ of $\{\sigma_{a|x}\}_{a,x}$ can be defined as the minimum amount of noise that has to be mixed with each state $\sigma_{a|x}$ from the assemblage, such that it admits a LHS decomposition [29]. It is straightforward to see that if $\{\mathbb{M}_x\}_x$ are a compatible set of measurements, then no matter which state ρ_{AB} is used in a steering experiment, all resulting assemblages $\{\sigma_{a|x}\}_{a,x}$ have a LHS decomposition. In the other direction, it also turns out that every set of incompatible measurements has the potential of generating steering [7,8]. That is, for every set of incompatible measurements there exists bipartite states that demonstrate steering if Alice uses them.

In what follows we make use of the connection between measurement incompatibility and EPR steering to also connect the latter with QSD and to show that the advantage in the QSD game here can be estimated in the so-called one-sided device-independent (1SDI) paradigm [30] where the set of measurements $\{\mathbb{M}_x\}$ are treated as a black box, such that we do not know the specific measurements made, or the dimension of system they act upon.

In order to accommodate the steering scenario let us describe an entanglement-based variation of the QSD scenario discussed before. Suppose that Bob tells Alice that he is going to measure his part of ρ_{AB} with the measurement $\mathbb{M}_y = \{M_{b|y}\}_b$ (such a measurement can be thought of as performing remote state preparation [31] of the states $\rho_{b|y}$ of Alice). Once again, Alice's goal is to make a measurement on her system in order to best guess Bob's outcome b (which is equivalent to guessing which state she will receive).

It was shown in Ref. [32] that a 1SDI lower bound can be placed on the ROI,

$$S_R^c(\{\sigma_{a|x}\}) \leq I_R(\{\mathbb{M}_x\}), \tag{15}$$

where $\{\sigma_{a|x}\}_{a,x}$ is an assemblage created by performing the measurements $\{\mathbb{M}_x\}_x$ on any state ρ_{AB}, and $S_R^c(\{\sigma_{a|x}\})$ is the *consistent* steering robustness, given by

$$\begin{aligned} S_R^c(\{\sigma_{a|x}\}) = \min\ & s \\ \text{s.t.}\ & \frac{\sigma_{a|x} + s\omega_{a|x}}{1+s} = \sum_\lambda p(a|x,\lambda)\sigma_\lambda, \\ & \omega_{a|x} \geq 0, \qquad \sigma_\lambda \geq 0, \\ & p(a|x,\lambda) \geq 0, \qquad \sum_a p(a|x,\lambda) = 1, \\ & \sum_a \omega_{a|x} = \sum_a \sigma_{a|x} = \sum_\lambda \sigma_\lambda, \end{aligned} \tag{16}$$

which can be seen as a modification of the steering robustness, with the additional constraint that the noise must have the same reduced state as the input assemblage [32]. Moreover, when ρ_{AB} is a pure entangled state (of full

Schmidt-rank), then $S_R^c(\{\sigma_{a|x}\}) = I_R(\{\mathbb{M}_x\})$; i.e., the bound is in fact tight.

This means that $1 + S_R^c(\{\sigma_{a|x}\})$ provides a 1SDI lower bound on the best advantage that Alice has in guessing b if she measures a set of incompatible measurements instead of a compatible one, and that if Alice and Bob share a pure entangled state, that this bound is in fact tight.

Conclusions.—In this Letter we have shown that measurement incompatibility, one of the most fundamental features of quantum mechanics, is intrinsically connected to the task of discriminating quantum states from collections of ensembles. Our results thus provide an operational interpretation of measurement incompatibility. Moreover, it shows that the robustness of incompatibility of a set of measurements is directly related to their usefulness for a natural quantum information game. Finally, we considered a resource theory of measurement incompatibility, and showed that the very same game is intimately related to the simulability of one set of measurements by another, providing (an infinite number of) criteria—often referred to as monotones—that collectively constitute necessary and sufficient conditions that must be met for one set of measurements to simulate another. This is similar to a number of other resource theories, where guessing probabilities in all discrimination games of a given type have also been shown to constitute complete criteria for transforming among objects in the theory [17,33,34].

There are a number of natural questions and extensions that we leave for future work. For example, it is interesting to consider partial notions of incompatibility (i.e., sets of measurements which are pairwise compatible, but not compatible as a complete set), and to ask whether there exist QSD games which characterize the usefulness of such sets. One can also consider generalizations of incompatibility in the other direction, where multiple parent measurements are allowed, and ask similar questions.

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Note added.—Recently, we became aware of related Letters [35,36]. In both works the authors also prove that all sets of incompatible measurements lead to an advantage in state discrimination (although in Ref. [35] without making a quantitative connection to the ROI). Reference [36] uses conic programming to prove more generally that robustness-based measures can be seen as quantifiers of performance in some discrimination tasks. Only our work shows that performance in all discrimination tasks constitutes necessary and sufficient conditions for measurement simulation.

*paul.skrzypczyk@bristol.ac.uk
†ivan.supic@unige.ch
‡daniel.cavalcanti@icfo.eu

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SDP formulation, deriving the upper bound on the advantage of QSD using the primal formulation, deriving the lower bound on the advantage of QSD using the dual formulation, and derivation of a complete set of monotones for measurement simulation. The Supplemental Material includes Ref. [25].
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Correction: Missing information in Refs. [15–17] has been inserted.