Two-Loop Five-Point Amplitude in $\mathcal{N}=4$ Super-Yang-Mills Theory

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(Received 3 January 2019; published 29 March 2019)

We compute the symbol of the two-loop five-point scattering amplitude in $\mathcal{N}=4$ supersymmetric Yang-Mills theory, including its full color dependence. This requires constructing the symbol of all two-loop five-point nonplanar massless master integrals, for which we give explicit results.

DOI: 10.1103/PhysRevLett.122.121603

A great deal of progress in calculating scattering amplitudes has been driven by the fruitful interplay between new formal ideas and the need for increasingly precise theoretical predictions at collider experiments. For instance, techniques such as generalized unitarity [1] and the symbol calculus [2] were first introduced in the realm of maximally supersymmetric Yang-Mills theory ($\mathcal{N}=4$ SYM) and went on to have a large impact on precision collider physics. In this Letter, we use cutting-edge techniques to take a first look at the analytic form of the two-loop five-point amplitude in $\mathcal{N}=4$ SYM theory beyond the planar, $N_c \to \infty$, limit of $SU(N_c)$ gauge theory.

Amplitudes in $\mathcal{N}=4$ SYM theory possess rigid analytic properties that make them easier to compute than their pure Yang-Mills counterparts, the state of the art being the three-loop four-gluon $\mathcal{N}=4$ SYM amplitude [3]. Historically, calculations in $\mathcal{N}=4$ SYM theory have therefore preceded analogous computations in QCD. The planar five-point amplitude at two loops in $\mathcal{N}=4$ SYM theory was first obtained numerically [4], confirming the prediction of [5]. In pure Yang-Mills theory, the first planar two-loop five-point amplitude, evaluated numerically, was for the all-plus helicity configuration [6]. Since then, a flurry of activity in *planar* multileg two-loop amplitudes has seen the analytic calculation of the all-plus amplitude [7], the numerical evaluation of all five-parton QCD amplitudes [8–11], and, recently, the computation of analytic expressions for all five-gluon scattering amplitudes [12,13]. These achievements were made possible by the development of efficient ways to reduce amplitudes to master integrals using integration-by-parts (IBP) relations [14,15], automated by Laporta's algorithm [16] or modern reformulations based on unitarity cuts and computational algebraic geometry [10,17–20], and to compute master integrals from their differential equations [21,22]. Indeed, all planar five-point master integrals have been computed [23,24], and substantial progress has been made in the nonplanar sectors as well [25–27].

In this work, we first discuss the *integrand* of the two-loop five-point amplitude in $\mathcal{N}=4$ SYM theory, and how it can be reduced to a form involving only so-called pure integrals (i.e., integrals satisfying a differential equation in canonical form [22]). We then use the aforementioned new techniques for integral reduction and differential equations (most notably the method introduced in [26]) to compute the *symbols* [2] (see also [28,29]) of *all* nonplanar massless two-loop five-point master integrals. From these integrals, we finally assemble the symbol of the complete two-loop five-point $\mathcal{N}=4$ SYM amplitude and discuss consistency checks of our result. Throughout, we work at the level of the symbol where transcendental constants are set to zero. While such contributions are important for the numerical evaluation of an amplitude, the symbol itself contains a major part of the nontrivial analytic structure of the amplitude.

Our result constitutes the first analytic investigation of two-loop five-point amplitudes in any gauge or gravity theory beyond the planar limit. Just as the one-loop five-gluon amplitude [30] did, our two-loop result should provide valuable theoretical data for further exploring the properties of structurally complex amplitudes, as well as the proposed duality between scattering amplitudes and Wilson loops at subleading color [31]. Furthermore, the methods will impact precision collider phenomenology: The master integrals are directly applicable to QCD amplitudes, opening the way to computing three-jet production at hadron colliders at next-to-next-to-leading order.

Construction of the amplitude.—In any $SU(N_c)$ gauge theory with all states in the adjoint representation, the trace-based color decomposition [32,33] of any two-loop five-point amplitude is [34]

$$\mathcal{A}_5^{(2)} = \sum_{S_5/(S_3\times Z_2)} \frac{\mathrm{tr}[15](\mathrm{tr}[234]-\mathrm{tr}[432])}{N_c} A^{\mathrm{DT}}[15|234] + \sum_{S_5/D_5}(\mathrm{tr}[12345]-\mathrm{tr}[54321]) \times \left(A^{\mathrm{ST}}[12345]+\frac{A^{\mathrm{SLST}}[12345]}{N_c^2}\right). \quad (1)$$

Here, single-trace (ST), subleading-color single-trace (SLST), and double-trace (DT) denote different *partial* amplitudes. Note that S_n (Z_n) is the (cyclic) permutation group, and D_n is the dihedral group.

It is a powerful fact about MHV scattering amplitudes in $\mathcal{N}=4$ SYM theory that all leading singularities [35] are given in terms of different permutations of Parke-Taylor tree-(super-)amplitudes [36,37]. This highly nontrivial result has been derived from a dual formulation of leading singularities in terms of the Grassmannian [38]. Furthermore, $\mathcal{N}=4$ SYM amplitudes are conjectured to be of uniform transcendental weight [5,39–41]. A representation of the four-dimensional integrand has been given in [42], where this Parke-Taylor structure, together with further special analytic properties of $\mathcal{N}=4$ SYM theory (logarithmic singularities and no residues at infinite loop momentum), is manifest. In this representation, the full, color-dressed amplitude splits into three distinct parts

$$\mathcal{A}_n^{(2)} = C\otimes \mathrm{PT}\otimes g_{\mathrm{pure}}, \quad (2)$$

where C schematically denotes the color structure of the gauge theory. For a five-point scattering amplitude, the space of Parke-Taylor factors is spanned by a set of 3! Kleiss-Kuijf (KK) independent elements [43] that we denote by $\mathrm{PT}[1\sigma_2\sigma_3\sigma_4 5]$, where

$$\mathrm{PT}[\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5] = \frac{\delta^8(Q)}{\langle\sigma_1\sigma_2\rangle\langle\sigma_2\sigma_3\rangle\langle\sigma_3\sigma_4\rangle\langle\sigma_4\sigma_5\rangle\langle\sigma_5\sigma_1\rangle}. \quad (3)$$

The super-momentum conserving delta function $\delta^8(Q)$ encodes the supersymmetric Ward identities relating the $(--+++)$-helicity five-gluon amplitude to all other five-particle amplitudes. The third part, g_{pure}, denotes a pure function of transcendental weight 4.

The goal of this section is to compute the partial amplitudes in (1). Our starting point is the integrand of [44], which is valid in $d=4-2\epsilon$ space-time dimensions and is given in terms of the six topologies in Fig. 1,

$$\mathcal{A}_5^{(2)} = \sum_{S_5}\left(\frac{I^{(a)}}{2}+\frac{I^{(b)}}{4}+\frac{I^{(c)}}{4}+\frac{I^{(d)}}{2}+\frac{I^{(e)}}{4}+\frac{I^{(f)}}{4}\right). \quad (4)$$

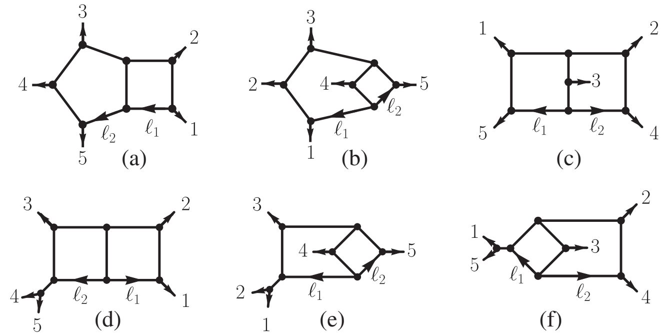


FIG. 1. Diagram topologies entering the local representation of the two-loop five-point integrand of $\mathcal{N}=4$ SYM theory [44]. Each diagram has an associated color structure and numerator which we suppress.

The sum is over all 5! permutations of external legs, and the rational numbers correspond to diagram-symmetry factors.

For each of the topologies in Fig. 1, we construct a basis of pure master integrals, on which the amplitude (4) can be decomposed, so the separation into color, rational, and transcendental parts (2) becomes manifest. Most required master integrals are already known in pure form [7,23,26,27,45]. The one missing topology, which we discuss momentarily, is the nonplanar double pentagon [diagram (c) of Fig. 1]. The integrals we are concerned with are functions of five Mandelstam invariants, s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, with $s_{ij}=(k_i+k_j)^2$. We also encounter the parity-odd ε-tensor contraction

$$\mathrm{tr}_5 = 4i\varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma = \mathrm{tr}(\gamma^5 \not{k}_1\not{k}_2\not{k}_3\not{k}_4). \quad (5)$$

To find a basis of pure master integrals for the top-level (eight-propagator) topology of Fig. 1(c), it is necessary to construct nine independent numerators. Specifically, we choose the following set of master integrals: 1. The parity-even part of the integral with numerator $N_1^{(a)}$ identified in [42], rewritten as spinor traces in Eq. (21) of [46]. By deleting γ^5 from the spinor traces, we obtain the parity-even parts in a form that is valid in d dimensions. Two more pure integrals are obtained from it by using the diagram's $Z_2\times Z_2$ symmetry. 2. $(6-2\epsilon)$-dimensional scalar integrals with any of the eight propagators squared, normalized by a factor of tr_5 and a homogeneous linear function of the s_{ij} variables. Six such integrals, which we have converted to integrals in $(4-2\epsilon)$ dimensions [47–50], are included in our basis. Explicit expressions for these new pure master integrals can be found in the Supplemental Material `masters.m` [51].

Next, we construct differential equations in canonical form [22] for the master integrals. The (iterated) branch-cut structure of the integrals is encoded in the *symbol letters*, which are algebraic functions of the kinematic invariants. It is convenient to parametrize the five-point kinematics in terms of variables that rationalize all letters of the *alphabet*. This can be accomplished via momentum twistors [52] and the x_i parametrization proposed in [6], see also [53]. For the nonplanar double-pentagon integral, we find that the complete system contains 108 masters and depends on the 31 W_α letters suggested in [25]:

$$\partial_{x_i}\mathcal{I}_a \equiv \frac{\partial \mathcal{I}_a}{\partial x_i} = \epsilon \sum_{\alpha=1}^{31} \frac{\partial \log W_\alpha}{\partial x_i} M_\alpha^{ab} \mathcal{I}_b, \quad 1 \le a, b \le 108. \qquad (6)$$

Ten of the letters ($\alpha \in \{1,\ldots,5\} \cup \{16,\ldots,20\}$) are simple Mandelstam invariants s_{ij}, 15 of the letters ($\alpha \in \{6,\ldots,15\} \cup \{21,\ldots,25\}$) are the differences of Mandelstam invariants $s_{ij} - s_{kl}$, the five parity-odd letters ($\alpha \in \{26,\ldots,30\}$) can be expressed as ratios of spinor brackets such as $(\langle 12\rangle[15]\langle 45\rangle[24])/([12]\langle 15\rangle[45]\langle 24\rangle)$, which invert under complex conjugation $\langle\cdot\rangle \leftrightarrow [\cdot]$ or $\mathrm{tr}_5 \to -\mathrm{tr}_5$, and the final parity-even letter ($\alpha = 31$) is tr_5. The 31 M_α matrices consist of simple rational numbers.

Computing the M_α matrices in (6) requires performing IBP reduction on differentials of the original masters $\partial_{x_i}\mathcal{I}_a$ with respect to the kinematic variables in order to reexpress them in terms of the original basis $\mathcal{I}_a$. We use the efficient approach introduced in [26], which builds on the modern formulation of IBP relations in terms of unitarity cuts and computational algebraic geometry [10,17–20]. The method requires IBP reduction at only 30 rational, numerical phase-space points to fix all the M_α, dramatically reducing the computation time compared to analytic IBP reduction. Combined with the first-entry condition [54], which restricts integrals to only have branch-cut singularities at physical thresholds, we obtain solutions to the differential equations at the symbol level for all master integrals. As a check, we verified that we reproduce (at symbol level) all known results for descendant integrals (≤7 propagators). The full results are included in the Supplemental Material `masters.m` [51].

Having established a basis and computed the master integrals required for massless two-loop five-point amplitudes, we can now write the $\mathcal{N} = 4$ SYM amplitude in that basis. As already stated, we use the d-dimensional representation of the integrand given in [44]. While this representation has the advantage of being in the so-called Bern-Carrasco-Johansson (BCJ) form [55], which allows for the immediate construction of the gravity integrand via the "double-copy" prescription, it obscures some of the simplicity of the final result. For instance, each individual diagram in Fig. 1 introduces spurious rational factors. Applying Fierz color identities [32] to decompose the integrand (4) into the partial amplitudes in (1) and using IBP reduction to rewrite those in our pure basis, we can obtain a representation that is manifestly in the form of (2). In particular, we find a simple rational kinematic dependence for all partial amplitudes via at most six KK-independent Parke-Taylor factors:

$$\begin{aligned} A^{\mathrm{ST}}[12345] &= \mathrm{PT}[12345]\mathrm{M}^{\mathrm{BDS}}_{(2)}, \\ A^{\mathrm{DT}}[15|234] &= \sum_{\sigma(234)\in S_3} \mathrm{PT}[1\sigma_2\sigma_3\sigma_4 5] g^{\mathrm{DT}}_{\sigma_2\sigma_3\sigma_4}, \\ A^{\mathrm{SLST}}[12345] &= \sum_{\sigma(234)\in S_3} \mathrm{PT}[1\sigma_2\sigma_3\sigma_4 5] g^{\mathrm{SLST}}_{\sigma_2\sigma_3\sigma_4}, \end{aligned} \qquad (7)$$

where $\mathrm{M}^{\mathrm{BDS}}_{(2)}$ is the two-loop BDS ansatz [5] and $g^{\mathrm{X}}_{\vec\sigma}$ are pure functions. Both $\mathrm{M}^{\mathrm{BDS}}_{(2)}$ and $g^{\mathrm{X}}_{\vec\sigma}$ can be written as $\mathbb{Q}$-linear combinations of our pure master integrals. The IBP reduction is done following the same strategy already discussed for the differential equations. Given the simple kinematic dependence of the result, it is sufficient to perform the reduction at six numerical kinematic points. Furthermore, we were able to achieve a computational speedup by performing all calculations in a finite field with a 10-digit cardinality, before reconstructing the simple rational numbers from their finite-field images using Wang's algorithm [56–58].

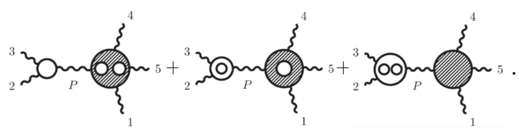
Inserting the symbol of the master integrals, we directly obtain the symbol of the two-loop five-point $\mathcal{N} = 4$ SYM amplitude. The amplitude is naturally decomposed into parity-even and parity-odd parts under a sign flip of "tr_5" defined in (5). At symbol level, the parity grading can be determined by counting the number of parity-odd letters, $W_{26},\ldots,W_{30}$, in a given symbol tensor. The parity-odd part of our result is highly constrained by the first- and second-entry conditions, as well as the integrability of the symbol [2], leading to a much simpler structure than the even part. It is important to note that in all collinear limits the parity-odd parts of the amplitude vanish since the external momenta span only a three-dimensional space and hence $\mathrm{tr}_5 = 0$. We attach the explicit symbol-level results for the partial amplitudes in the Supplemental Material `amplitudes.m` [51].

Validation.—In the previous section we described the assembly of the two-loop five-point amplitude in $\mathcal{N} = 4$ SYM theory in terms of pure master integrals. In this section we validate our final result by checking nontrivial identities between different terms and verifying universal behavior in kinematic limits. We focus our discussion on verifying collinear factorization when two external momenta become parallel [59]. Aside from this check, we also verified the following: (i) The planar amplitude matches the BDS ansatz [5] stating that four- and five-particle amplitudes in planar $\mathcal{N} = 4$ SYM theory are given to all orders by exponentiating the one-loop amplitude [30]. (ii) The partial amplitudes satisfy the group-theoretic Edison-Naculich relations [60], allowing us to write all subleading single-trace partial amplitudes A^{SLST} in terms of linear combinations of planar A^{ST} and double-trace A^{DT} amplitudes, e.g.,

$$\begin{aligned} A^{\mathrm{SLST}}[12345] = 5A^{\mathrm{ST}}[13524] &+ \sum_{\mathrm{cyclic}} \Big[A^{\mathrm{ST}}[12435] - 2A^{\mathrm{ST}}[12453] \\ &+ \frac{1}{2}(A^{\mathrm{DT}}[12|345] - A^{\mathrm{DT}}[13|245])\Big], \end{aligned} \qquad (8)$$

where the five cyclic permutations are generated by the relabeling $i \to i+1$ (mod 5). Thus, we need not discuss A^{SLST} further, and the amplitude is fully specified by two functions, $\mathrm{M}^{\mathrm{BDS}}_{(2)}$ and g^{DT}_{234}. (iii) The infrared poles of the amplitude match the universal pole structure predicted by Catani [61] (see also, e.g., Refs. [59,62]), where the poles of two-loop amplitudes can be computed in terms of known tree- and one-loop amplitudes. Several of these checks require the one-loop five-point amplitudes expanded through order ϵ^2. An exact expression for the integrand of this amplitude is known [63]. The box integrals are known to all orders in ϵ [48]. The only integral that is not known to all orders is the six-dimensional scalar pentagon whose symbol can either be computed to any order in ϵ from [64] or by direct evaluation of the integral with HyperInt [65]. We denote by $\mathcal{I}_5^{d=6-2\epsilon}$ the integral normalized by (minus) the tr_5 of (5), so that it is a pure parity-odd function, and give its symbol in the Supplemental Material `purePentagon6d.m` [51].

The test we discuss in more detail is the collinear limit of the double-trace partial amplitudes A^{DT}. As already stated, all parity-odd contributions of any partial amplitude vanish in this limit since $\mathrm{tr}_5 = 0$. For concreteness, in the rest of this section we focus on $A^{\mathrm{DT}}[15|234]$, which in our conventions is symmetric in the (15) indices and totally antisymmetric in (234). All other double-trace amplitudes are given by simple relabeling. Scattering amplitudes obey a universal collinear factorization equation [1,59]. Here, we discuss the five-point limit $2\|3$ where two momenta, k_2 and k_3, become collinear $k_2 = \tau P$, $k_3 = (1-\tau)P$ with collinear splitting fraction τ. The two-loop amplitude factorizes into $\sum_{\ell=0}^{2} \mathrm{Split}_{23}^{(\ell)}(\epsilon) \times \mathcal{A}_4^{(2-\ell)}(\epsilon)$:

$\mathcal{A}_5^{(2)} \xrightarrow{2\|3}$ 

(9)

The empty blobs on the left of each diagram denote the collinear splitting functions, and the filled blobs on the right are the four-point amplitudes depending only on P, k_1, k_4, and k_5. The color part of the splitting function is very simple: In the example above, it is directly proportional to f^{23P}. Kinematic expressions for the one- and two-loop splitting functions can be found in [1,59]. Furthermore, the one- and two-loop four-point amplitudes [1,33], and relevant integrals [66,67], are also known to the required order in the ϵ expansion. To approach the collinear limit, we map from the generic five-dimensional kinematic space (parametrized in terms of the x_i of [6]) to the collinear limit. This can be done via the following substitution [53]:

$$x_1 \mapsto s\tau, \qquad x_2 \mapsto c s\delta, \qquad x_3 \mapsto r_2 c s \delta,$$
$$x_4 \mapsto \delta, \qquad x_5 \mapsto -\frac{1}{c\delta}, \tag{10}$$

where s characterizes the overall scale of all Mandelstam invariants, $\delta \to 0$ corresponds to the collinear limit, τ is the aforementioned collinear splitting fraction, $r_2 = (s_{15}/s_{45})$ is the ratio of Mandelstam invariants of the underlying four-point process, and $c \sim ([23]/\langle 23\rangle)$ corresponds to an azimuthal phase. Expanding the 31-letter alphabet to leading order in δ, we find 14 multiplicatively independent letters in the collinear limit: 7 physical $\{\delta, s, \tau, 1-\tau, r_2, 1+r_2, c\}$ (in fact, this number reduces to 6 at leading power because c and δ always appear in the same combination, $c\delta^2$) and 7 spurious letters that cannot be part of the (leading power) limit. When comparing the collinear limit of our result to the factorization formula (9), we note that only Parke-Taylor factors where legs 2 and 3 are adjacent become singular. For instance, while $\mathrm{PT}[12345] \mapsto 1/(\sqrt{\tau(1-\tau)}\langle 23\rangle) \times \mathrm{PT}[1P45]$, $\mathrm{PT}[12435]$ has no collinear singularity in the $2\|3$ limit. We find that our result exactly matches the collinear factorization formula (9). Besides this limit, there are two further inequivalent collinear limits we can check for $A^{\mathrm{DT}}[15|234]$: when $1\|5$ and $1\|2$. When looking at the color factors of the appropriate relabeling of (9), it becomes clear that neither of them contains $\mathrm{tr}[15](\mathrm{tr}[234] - \mathrm{tr}[432])$ so $A^{\mathrm{DT}}[15|234]$ is forced to be nonsingular in these limits. We have checked that our result indeed reproduces this behavior.

TABLE I. Summary of vanishing (0) and nonvanishing (✓) terms in the ϵ expansion of the different partial amplitudes.

	$1/\epsilon^4\ w_0$	$1/\epsilon^3\ w_1$	$1/\epsilon^2\ w_2$	$1/\epsilon^1\ w_3$	$\epsilon^0\ w_4$
$A^{\mathrm{ST}}_{\mathrm{even}}$	✓	✓	✓	✓	✓
$A^{\mathrm{ST}}_{\mathrm{odd}}$	0	0	0	✓	✓
$A^{\mathrm{DT}}_{\mathrm{even}}$	0	✓	✓	✓	✓
$A^{\mathrm{DT}}_{\mathrm{odd}}$	0	0	0	✓	✓
$A^{\mathrm{SLST}}_{\mathrm{even}}$	0	0	✓	✓	✓
$A^{\mathrm{SLST}}_{\mathrm{odd}}$	0	0	0	0	✓

Discussion of the result and outlook.—After discussing various consistency checks of our answer for the two-loop five-point amplitude in $\mathcal{N}=4$ SYM, let us briefly summarize some of its analytic features. First, we highlight in Table I that a number of terms in the ϵ expansion vanish, which is of course predicted by the Catani formula. We note that some of the two-loop master integrals have weight-2 odd terms, but this contribution is absent from the amplitude.

We also note that our answers for the amplitude, as well as individual pure master integrals, are compatible with the empirical *second-entry conditions* first observed for individual integrals in [7,24,25,27]. It would be very interesting to understand the underlying physical reason for this property, perhaps from the point of view of a diagrammatic coaction principle [64,68,69].

Our full result is too lengthy to print in this Letter. However, it has very restricted analytic structure. For instance, the parity-odd transcendental part of *any* derivative of *any* weight-4 function in the amplitude belongs to a 12-dimensional subspace of the 111-dimensional space of weight-3 parity-odd functions that obey integrability and the second-entry condition of [25]. This 12-dimensional subspace is spanned by the 12 inequivalent permutations, Σ_j, of the $\mathcal{O}(\epsilon^0)$ part of the pure, parity-odd scalar pentagon in $d=6$, $\mathcal{I}_5^{d=6}(\Sigma_j)$. (Because of the dihedral D_5 invariance of the integral, there are only $5!/10=12$ inequivalent permutations.) The parity-odd part of the $1/\epsilon$ coefficient of $\mathrm{M}_{(2)}^{\mathrm{BDS}}$ is just $-5\mathcal{I}_5^{d=6}(\{12345\})$.

Let us recall that the amplitude is fully specified by g_{234}^{DT} and the previously known $\mathrm{M}_{(2)}^{\mathrm{BDS}}$. We may write the odd transcendental part of the derivative of the odd part of g_{234}^{DT} using this $\mathcal{I}_5^{d=6}$ basis, as

$$\partial_{x_i}[g_{234}^{\mathrm{DT,odd}}]|_{\mathrm{odd}} = \sum_{j,\gamma} \mathcal{I}_5^{d=6}(\Sigma_j) m_{j\gamma} \frac{\partial \log W_\gamma}{\partial x_i}, \quad (11)$$

where j labels the 12 inequivalent pentagon permutations $\{12543\}, \{12453\}, \{13524\}, \{12534\}, \{13254\}, \{12354\}, \{14325\}, \{13425\}, \{14235\}, \{12435\}, \{13245\}, \{12345\}$, and $\gamma \in \{1,\ldots,5\} \cup \{16,\ldots,20\} \cup \{31\}$ are the nonzero final entries. The matrix $m_{j\gamma}$ is

$$m_{j\gamma} = \begin{pmatrix} -\frac{17}{4} & -\frac{5}{4} & -6 & -\frac{17}{4} & -\frac{7}{2} & -\frac{17}{4} & -\frac{7}{4} & \frac{1}{2} & -1 & -\frac{17}{4} & 10 \\ \frac{17}{4} & \frac{5}{4} & \frac{5}{4} & \frac{17}{4} & 4 & \frac{17}{4} & \frac{11}{2} & \frac{17}{4} & \frac{1}{2} & \frac{1}{2} & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ -\frac{17}{4} & -6 & -\frac{5}{4} & -\frac{17}{4} & -\frac{7}{2} & \frac{1}{2} & -\frac{7}{4} & -\frac{17}{4} & -\frac{17}{4} & -1 & 10 \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & 0 & 1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{17}{4} & 6 & 6 & \frac{17}{4} & 9 & -\frac{1}{2} & 4 & -\frac{1}{2} & -\frac{5}{4} & -\frac{5}{4} & -10 \end{pmatrix}$$

which has rank 8, so only eight independent combinations of final entries appear. For concreteness, we give the symbol of $\mathcal{I}_5^{d=6}(\{12345\})$ in the Supplemental Material `purePentagon6d.m` [51].

While the first derivatives are quite constrained, the second derivatives (actually the $\{2,1,1\}$ coproducts) of the ϵ^0 terms of the amplitude span the entire 79-dimensional space identified in [25].

Building on this first analytic result for a nonplanar two-loop five-point amplitude, there are a number of avenues for future research. The upcoming work of [70] will explore the analytic structure of the factorization of the amplitude when one of the external gluons becomes soft. For this limit, there exists an eikonal semi-infinite Wilson line picture. Starting at two loops, the possibility of coupling three hard lines via nontrivial color connections opens up, which leads to an interesting parity-odd component of the soft-emission function which is compatible with the soft limit of our symbol-level result. Furthermore, it would be interesting to explore the subleading-in-color behavior of this scattering amplitude in multi-Regge kinematics [71–73]. With our result, it now also becomes possible to test the proposed relation between scattering amplitudes and Wilson loops beyond the leading term in the large N_c limit [31], and it would be interesting to match our result to a future near-collinear OPE computation on the Wilson-loop side.

Since we have now computed the symbol of all relevant Feynman integrals for massless two-loop five-point scattering, we can, in principle, discuss other theories, such as $\mathcal{N}<4$ SYM theory as well as $\mathcal{N}\geq 4$ supergravity. In particular, it would be interesting to investigate the uniform transcendentality (UT) property of two-loop five-point amplitudes in $\mathcal{N}=8$ supergravity. According to [74], this integrand only has logarithmic singularities and no poles at infinity, so one would expect a UT result. Finding such a result would lend further credence to the empirical relation between logarithmic poles of the integrand and transcendentality properties of amplitudes.

L. D. and E. H. thank Falko Dulat and Hua Xing Zhu for valuable discussions, as well as Huan-Hang Chi and Yang Zhang for initial collaboration on a related project. We thank Harald Ita for useful comments on the manuscript. The work of S. A. is supported by the Fonds de la Recherche Scientifique–FNRS, Belgium. The work of L. D. and E. H. is supported by the U.S. Department of Energy (DOE) under Contract No. DE-AC02-76SF00515. The work of M. Z. is supported by the Swiss National Science Foundation under Contract No. SNF200021179016 and the European Commission through the ERC grant pertQCD. The work of B. P. is supported by the French Agence Nationale pour la Recherche, under Grant No. ANR17CE31000101. We thank the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support. L. D. acknowledges support by a grant from the Simons Foundation (341344, LA).

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