Resource Theory of Entanglement with a Unique Multipartite Maximally Entangled State

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Entanglement theory is formulated as a quantum resource theory in which the free operations are local operations and classical communication (LOCC). This defines a partial order among bipartite pure states that makes it possible to identify a maximally entangled state, which turns out to be the most relevant state in applications. However, the situation changes drastically in the multipartite regime. Not only do there exist inequivalent forms of entanglement forbidding the existence of a unique maximally entangled state, but recent results have shown that LOCC induces a trivial ordering: almost all pure entangled multipartite states are incomparable (i.e., LOCC transformations among them are almost never possible). In order to cope with this problem we consider alternative resource theories in which we relax the class of LOCC to operations that do not create entanglement. We consider two possible theories depending on whether resources correspond to multipartite entangled or genuinely multipartite entangled (GME) states and we show that they are both nontrivial: no inequivalent forms of entanglement exist in them and they induce a meaningful partial order (i.e., every pure state is transformable to more weakly entangled pure states). Moreover, we prove that the resource theory of GME that we formulate here has a unique maximally entangled state, the generalized GHZ state, which can be transformed to any other state by the allowed free operations.

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Introduction.—Entanglement is a striking feature of quantum theory with no classical analogue. Although initially studied to address foundational issues [1], the development of quantum information theory [2] in the last few decades has elevated it to a resource that allows tasks to be implemented which are impossible in classical systems. The resource theory of entanglement [3] aims at providing a rigorous framework to qualify and quantify entanglement and, ultimately, to understand fully its capabilities and limitations within the realm of quantum technologies. However, this theory is much more firmly developed for bipartite than multipartite systems. In fact, although a few applications have been proposed within the latter setting such as secret sharing [4], the one-way quantum computer [5], and metrology [6]; a deeper understanding of the complex structure of multipartite entangled states might inspire further protocols in quantum information science and better tools for the study of condensed-matter systems.

The wide applicability of the formulation of entanglement theory as a resource theory has motivated an active line of work [7] that studies different quantum effects from this point of view such as coherence [8], reference frame alignment [9], thermodynamics [10], nonlocality [11] or steering [12]. The main question a resource theory addresses is to order the set of states and provide means to quantify their nature as a resource. The so-called free operations are crucial to this task. This is a subset of transformations, which the given scenario dictates can be implemented at no cost. Thus, all states that can be prepared with these operations are free states. Conversely, nonfree states acquire the status of a resource: granted such states, the limitations of the corresponding scenario might be overcome. Moreover, the concept of free operations allows an order relation to be defined. If a state ρ can be transformed into σ by some free operation, then ρ cannot be less resourceful than σ since any task achievable by σ is also achievable by ρ as the corresponding transformation can be freely implemented. However, the converse is not necessarily true. Furthermore, one can introduce resource quantifiers as functionals that preserve this order.

Since entanglement is a property of systems with many constituents which may be far away, the natural choice for free operations in this resource theory is local operations and classical communication (LOCC). Indeed, parties bound to LOCC can only prepare separable states, and entangled states become a resource to overcome the constraints imposed by LOCC manipulation. Nielsen characterized in Ref. [13] the possible LOCC conversions among pure bipartite states, which revealed that the LOCC ordering reduces to majorization [14] and, remarkably, that there is a unique maximally entangled state for fixed local dimension. This is because this state can be transformed by LOCC into any other state of that dimension but no other state of that dimension can be transformed into it. This state is then regarded as a gold standard to measure entanglement and, unsurprisingly, it turns out to be the most useful state for bipartite entanglement applications such as teleportation. Importantly, the situation changes drastically in the multipartite case. Here, Ref. [15] and subsequent work [16] have shown that there exist inequivalent forms of entanglement: the state space is divided into classes, the so-called stochastic LOCC (SLOCC) classes, of states which can be interconverted with nonzero probability by LOCC but cannot be transformed outside the class by LOCC, even probabilistically. This in particular shows that no maximally entangled state can exist for multipartite states. Still, one could, in principle, study the ordering induced by LOCC within each SLOCC class. Recent work [17] in this direction has revealed, however, an extreme feature that culminates with the result of Ref. [18]: almost all pure states of more than three parties are *isolated*; i.e., they cannot be obtained from nor transformed to another inequivalent pure state of the same local dimensions by LOCC. This means that almost all pure states are incomparable by LOCC, inducing a trivial ordering and a meaningless arbitrariness in the construction of entanglement measures. In this sense, one may say that the resource theory of multipartite entanglement with LOCC is generically trivial.

We believe this calls for a critical reexamination of the resource theory of entanglement and, in particular, for the notion of LOCC as the ordering-defining relation. Indeed, although LOCC transformations have a clear operational interpretation, this is not, in fact, the most general class of transformations that maps the set of separable states into itself. In other words, LOCC is strictly included in the class of nonentangling operations. Thus, from the abstract point of view of resource theories other consistent theories of entanglement (i.e., with separable states being the free states) are possible where the set of free operations is larger than LOCC. Hence, in principle, these could give a more meaningful ordering and revealing structure in the set of multipartite entangled states. To study such possibility is precisely the goal of this Letter. A similar approach has been taken to address other unsatisfying features of the resource theory of entanglement under LOCC such as irreversibility of state transformations for an arbitrarily large number of copies [19]. Remarkably, Ref. [20] has shown that shifting the paradigm from LOCC to asymptotic nonentangling operations provides a reversible theory of asymptotic entanglement interconversion with a unique entanglement measure and this result has been extended in Ref. [21] to arbitrary resource theories under asymptotic resource-non-generating operations [7]. Also, in the absence of a clear set of physical constraints determining the free operations, certain quantum resource theories have been constructed by first defining the set of free states and then considering classes of operations that preserve this set. This is the case of the resource theory of coherence [22], which has been found useful in, e.g., metrology applications [23] and quantum channel discrimination [24] and which has subsequently given rise to a fruitful research line considering an operational interpretation for the set of free operations (see Refs. [8,25] and references therein).

Since we seek whether a nontrivial theory is at all possible for single-copy manipulations, we consider here the resource theory of entanglement under the largest possible class of free operations in this regime: strictly nonentangling operations. However, multipartite entanglement comes in two different forms. We will call entangled those states that are not fully separable (FS), while we will call genuinely multipartite entangled (GME) those states which are not biseparable (BS). Thus, one can formulate two theories: one in which entangled states are considered a resource and where the free operations are full separability preserving (FSP) and the analogous with GME states and biseparability-preserving (BSP) operations. Interestingly, our first result is that both formalisms lead to nontrivial theories: no resource state is isolated in any of these scenarios. Moreover, we show that there are no inequivalent forms of entanglement. Then, we consider whether there exists a unique multipartite maximally entangled state in these theories like in the bipartite case. While we find a negative answer (at least in the simplest nontrivial case of 3-qubit states) for FSP operations, our main result is that the question is answered affirmatively in the resource theory of GME under BSP operations. The maximally GME state turns out to be the generalized Greenberger-Horne-Zeilinger (GHZ) state.

Definitions and preliminaries.—We will consider n-partite systems with local dimension d, i.e., states in the Hilbert space $H = H_1 \otimes \cdots \otimes H_n = (\mathbb{C}^d)^{\otimes n}$. Given a subset M of $[n] = \{1, \ldots, n\}$ and its complement $\bar{M}$, we denote by H_M the tensor product of the Hilbert spaces corresponding to the parties in M and analogously with $H_{\bar{M}}$. A pure state $|\psi\rangle \in H$ is FS (otherwise entangled) if $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ for some states $|\psi_i\rangle \in H_i \, \forall \, i$, while it is BS (otherwise GME) if $|\psi\rangle = |\psi_M\rangle \otimes |\psi_{\bar{M}}\rangle$ for some states $|\psi_M\rangle \in H_M$ and $|\psi_{\bar{M}}\rangle \in H_{\bar{M}}$ and $M \subsetneq [n]$. These notions are extended to mixed states by the convex hull and we define the sets of FS and BS states by

$$\mathcal{FS} = \mathrm{conv}\{\psi : |\psi\rangle \, \mathrm{is\, FS}\}, \qquad \mathcal{BS} = \mathrm{conv}\{\psi : |\psi\rangle \, \mathrm{is\, BS}\}, \tag{1}$$

where here and throughout the Letter we use the notation $\psi = |\psi\rangle\langle\psi|$ whenever a state is specified as pure. Transformations in quantum theory are given by completely positive and trace preserving (CPTP) maps and we say that such a map Λ (from and to operators on H) is FSP (BSP) if $\Lambda(\rho) \in \mathcal{FS} \, \forall \, \rho \in \mathcal{FS}$ $[\Lambda(\rho) \in \mathcal{BS} \, \forall \, \rho \in \mathcal{BS}]$. We will say that a functional E taking operators on H to non-negative real numbers is an FSP measure (BSP measure) if $E(\rho) \geq E[\Lambda(\rho)]$ for every state ρ and FSP (BSP) map Λ. This is completely analogous to entanglement measures, which are required to be non-increasing under LOCC maps. Although LOCC is a strict subset of the FSP and BSP maps, some well-known entanglement measures are still FSP or BSP measures and this will play an important role in assessing which transformations are possible within the two formalisms that we consider here. Indeed, measures of the form

$$E_{\mathcal{X}}(\rho) = \inf_{\sigma\in\mathcal{X}} E(\rho||\sigma), \tag{2}$$

where $\mathcal{X}$ stands for either $\mathcal{FS}$ or $\mathcal{BS}$, have the corresponding monotonicity property as long as the distinguishability measure $E(\rho||\sigma)$ is contractive, i.e., $E[\Lambda(\rho)||\Lambda(\sigma)] \leq E(\rho||\sigma)$ for every CPTP map Λ. This includes the relative entropy of entanglement [26] for $E(\rho||\sigma) = \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma)$ and the robustness ($R_{\mathcal{X}}$) [27] for

$$E(\rho||\sigma) = R(\rho||\sigma) = \min\{s\colon (\rho + s\sigma)/(1+s) \in \mathcal{X}\}. \tag{3}$$

If one uses the fidelity $E(\rho||\sigma) = 1 - F(\rho||\sigma) = 1 - \mathrm{tr}^2\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$, for pure states Eq. (2) boils down to the geometric measure [28], which we will denote by $G_{\mathcal{X}}$ and which is then seen to be a measure under maps that preserve $\mathcal{X}$. Notice, however, that, as has been recently shown in the bipartite case in Ref. [29], not all LOCC-measures remain monotonic under nonentangling maps since the latter formalism allows state conversions that the former does not. In the following, in order to understand the ordering of resources induced by these theories, we study which transformations are possible among pure states under FSP and BSP maps. However, first one should point out that whenever there exist maps Λ and Λ' in the corresponding class of free operations such that $\Lambda(\psi) = \phi$ and $\Lambda'(\phi) = \psi$, then the states ψ and ϕ are equally resourceful and should be regarded as equivalent in the corresponding theory. This is moreover necessary so as to have a well-defined partial order. Hence, although for simplicity we will talk about properties of states, one should have in mind that one is actually speaking about equivalence classes. Specifically, it is known that two pure states are interconvertible by LOCC if and only if they are related by local unitary transformations [30]. Interestingly, we will see that the equivalence classes are wider in the resource theory of GME under BSP. It should be stressed that, to our knowledge, this is the first time that a resource theory of GME is formulated. Notice that the restriction to LOCC can only have FS states as free states. Furthermore, allowing a strict subset of parties to act jointly and classical communication does not fit the bill either as $\mathcal{BS}$ is not closed under these operations.

Nontriviality of the theories.—Our first two results are valid in both the FSP and BSP regimes. Thus, following the notation above, the two possible classes of maps will be referred to as $\mathcal{X}$ preserving.

Theorem 1—collapse of the SLOCC classes: In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, all resource states are interconvertible with a nonzero probability; i.e., given any pure ψ_1, $\psi_2 \notin \mathcal{X}$, there exists a completely positive and trace non-increasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = p\psi_2$ with $p \in (0, 1]$.

Theorem 2—no isolation: In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, no resource state is isolated; i.e., given any pure $\psi_1 \notin \mathcal{X}$ on H, there exists an inequivalent pure $\psi_2 \notin \mathcal{X}$ on H and a CPTP $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = \psi_2$.

The full proof of these two results can be found in Ref. [31]. The proof of Theorem 1 is based on explicitly constructing a completely positive and trace nonincreasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = p\psi_2$ whenever it holds that

$$p \leq \frac{1}{R_{\mathcal{X}}(\psi_2)} \frac{G_{\mathcal{X}}(\psi_1)}{1 - G_{\mathcal{X}}(\psi_1)}. \tag{4}$$

Since it can be guaranteed that $R_{\mathcal{X}}(\psi_2) > 0$ and $0 < G_{\mathcal{X}}(\psi_1) < 1$ when ψ_1, $\psi_2 \notin \mathcal{X}$, there always exists $p \in (0, 1]$ such that Eq. (4) holds. Theorem 2 then arises as a corollary as, given any $\psi_1 \notin \mathcal{X}$, continuity arguments show that there always exists an inequivalent $\psi_2 \notin \mathcal{X}$ with $R_{\mathcal{X}}(\psi_2)$ small enough so that one can take $p = 1$ in Eq. (4) and construct a CPTP map.

Theorem 1 proves that in our case there are no inequivalent forms of entanglement. This is in sharp contrast to LOCC where, leaving aside the case $H = (\mathbb{C}^2)^{\otimes 3}$, the state space splits into a cumbersome zoology of infinitely many different SLOCC classes of unrelated entangled states. Theorem 2 provides the nontriviality of our theories. While almost all states turn out to be isolated under LOCC [18], our classes of free operations induce a meaningful partial order structure where, as in the case of bipartite entanglement, every pure state can be transformed into a more weakly entangled pure state. It is important to mention that the result of Ref. [18] proves generic isolation when transformations are restricted among GME states with the rank of all n single-particle reduced density matrices equal to d. However, Theorem 2 still holds under this restriction [31].

Existence of a maximally resourceful state.—Theorems 1 and 2 show that limitations of the resource theory of multipartite entanglement under LOCC can be overcome if one considers FSP or BSP operations instead. These positive results raise the question of whether the induced structure is powerful enough to have a unique multipartite maximally entangled state. If this were so, our theories would point to a relevant class of states that should be at the heart of the applications of multipartite entanglement in a similar fashion to the maximally entangled state in the bipartite case. In order to answer this question, we provide first an unambiguous definition of a maximally resourceful state which, on the analogy of the bipartite case, depends on the number of parties n and local dimension d: a state ψ on H is the maximally resourceful state on H if it can be transformed by means of the free operations into any other state on H [41]. We analyze first the case of FSP operations, where we find a negative answer to the above question.

Theorem 3: In the resource theory of entanglement where the free operations are FSP maps, there exists no maximally entangled state on $H = (\mathbb{C}^2)^{\otimes 3}$.

Although the details of the proof are given in Ref. [31], we outline here its structure. First, we use that if a maximally entangled state in this case existed, it would need to be the W state $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. This is because it has been shown in Ref. [42] that the W state is the unique state in this Hilbert space that achieves the maximal possible value of $G_{\mathcal{FS}}$, which we have shown above to be an FSP measure. Thus, if there existed a maximally entangled state, it would be necessary that the W state could be transformed by FSP into any other state. However, we show that there exists no FSP map transforming the W state into the GHZ state [$|\mathrm{GHZ}(3,2)\rangle$ in Eq. (5) below]. To verify this last claim, it suffices to find an FSP measure E such that $E(\mathrm{GHZ}) > E(W)$. However, as discussed above, not many FSP measures are known and, as with the geometric measure, it is also known that the relative entropy of entanglement of the W is larger than that of the GHZ state [43]. This leaves us then with the robustness measure $R_{\mathcal{FS}}$, for which we are able to show that $R_{\mathcal{FS}}(W) = R_{\mathcal{FS}}(\mathrm{GHZ}) = 2$. This alone does not forbid that $\mathrm{W} \rightarrow_{\mathrm{FSP}} \mathrm{GHZ}$, but from the insight developed in computing these quantities, an obstruction to such transformation can be found even though they are equally robust. It is worth mentioning that, to our knowledge, this is the first time that the robustness is computed for multipartite states and we have reasons to conjecture that the W and GHZ states attain its maximal value on H, being the only states that do so.

Theorem 3 forbids then the existence of a multipartite maximally entangled state under FSP in the simplest case of $H = (\mathbb{C}^2)^{\otimes 3}$. However, it is instructive to compare with the LOCC scenario since these values of n and d make up the only case where no state is isolated in the latter formalism (aside from the bipartite case). We show in Ref. [31] that the W and GHZ states can be transformed by FSP operations into states that are not obtainable from any other 3-qubit states by LOCC. These states might be chosen to lie in different SLOCC classes, so, additionally, this provides an explicit example of deterministic FSP conversions among states in different SLOCC classes.

Finally, we study the resource theory under BSP operations where, remarkably, we find a unique maximally GME state for any value of n and d, given by the generalized GHZ state

$$|\mathrm{GHZ}(n,d)\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle^{\otimes n}. \tag{5}$$

Theorem 4: In the resource theory of entanglement where the free operations are BSP maps, there exists a maximally GME state on every H. Namely, $\forall\, |\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, there exists a CPTP BSP map Λ such that $\Lambda[\mathrm{GHZ}(n,d)] = \psi$.

The complete proof of this result is given in Ref. [31]. The main idea behind it is to use again the construction of the proof of Theorems 1 and 2, which shows that there is a CPTP BSP map Λ such that $\Lambda[\mathrm{GHZ}(n,d)] = \psi$ if $R_{\mathcal{BS}}(\psi) \leq G_{\mathcal{BS}}[\mathrm{GHZ}(n,d)]/\{1 - G_{\mathcal{BS}}[\mathrm{GHZ}(n,d)]\}$ [cf. Eq. (4)]. However, unlike for the FS case, $G_{\mathcal{BS}}$ is straightforward to compute [44] in terms of the Schmidt decomposition across every possible bipartite splitting of the parties $M|\bar{M}$ ($|\psi\rangle = \sum_i \sqrt{\lambda_i^{M|\bar{M}}}\, |i\rangle_M |i\rangle_{\bar{M}}$) as

$$G_{\mathcal{BS}}(\psi) = 1 - \max_{M \subsetneq [n]} \lambda_1^{M|\bar{M}}, \tag{6}$$

where $\lambda_1^{M|\bar{M}}$ is the largest Schmidt coefficient of ψ in the corresponding splitting. This immediately shows that the generalized GHZ state has maximal value of the geometric measure, $G_{\mathcal{BS}}[\mathrm{GHZ}(n,d)] = (d-1)/d$. Finally, a simple estimate shows that $R_{\mathcal{BS}}(\psi) \leq d - 1\ \forall\, |\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$, which leads to the desired result.

It follows from the proof that it suffices to have maximal $G_{\mathcal{BS}}$ to be convertible to any other state by BSP operations. Thus, any state fulfilling that $G_{\mathcal{BS}} = (d-1)/d$ must automatically maximize any other BSP measure. More importantly, this also shows that any two states achieving this value of the geometric measure are deterministically interconvertible by BSP operations and, therefore, belong to the same GME-equivalence class despite potentially not being related by local unitary transformations. An example of such class when $d = 2$ are GME graph states for which it is known that $G_{\mathcal{BS}} = 1/2$ [45]. Hence, all graph states including the generalized GHZ state are in the equivalence class of the maximally GME state in this theory. It is remarkable to find that this very relevant family of states [46] in quantum computation and error correction has this feature in a resource theory of GME and we believe this is worth further research. Another previously considered family of states that belongs to this equivalence class is that of absolutely maximally entangled (AME) states [47], which is defined as those states for which all reduced density matrices are proportional to the identity in the maximum possible dimensions. It follows from Eq. (6) that for all AME states it holds that $G_{\mathcal{BS}} = (d-1)/d$ (for those values of n and d for which they exist). Equation (6) also tells us that a necessary condition for a state to be in the equivalence class of the maximally GME state is that all single-particle reduced density matrices must be proportional to the d-dimensional identity. However, this condition is not sufficient: the state in $(\mathbb{C}^2)^{\otimes 4}$ $|\phi\rangle = \sqrt{p}|\phi^+\rangle_{12}|\phi^+\rangle_{34} + \sqrt{1-p}|\phi^-\rangle_{12}|\phi^-\rangle_{34}$ $(|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2})$ is a GME state (if $p \neq 0, 1$) with this property but $G_{BS}(\phi) < 1/2$ (if $p \neq 1/2$).

Conclusions.—We have shown that nontrivial (i.e., without isolation) resource theories of multipartite entanglement are possible in which moreover inequivalent forms of entanglement do not exist. However, no resource theory of non-full separability can have a maximally entangled state for 3-qubit states since this is not possible under FSP transformations, the largest conceivable class of free operations (future work should study whether this no-go result generalizes to other values of n and d). On the other hand, the BSP paradigm induces a resource theory of GME with a maximally resourceful state. Given this positive result, it would be interesting to analyze further features of this theory and, in particular, whether an operational grounding to this conceptually satisfying structure can be found. We also note that GME does not fulfill Axiom 1 of Ref. [21], so it is open whether an asymptotically reversible theory of this resource is possible.

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