Pair-Breaking Collective Branch in BCS Superconductors and Superfluid Fermi Gases

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We demonstrate the existence of a collective excitation branch in the pair-breaking continuum of superfluid Fermi gases and BCS superconductors. At zero temperature, we analytically continue the equation on the collective mode energy in Anderson's Random Phase Approximation or Gaussian fluctuations through its branch cut associated with the continuum, and obtain the full complex dispersion relation, including in the strong coupling regime. The branch exists as long as the chemical potential μ is positive and the wave number below $\sqrt{2m\mu}/\hbar$ (with m the fermion mass). In the long wavelength limit, the branch varies quadratically with the wave number, with a complex effective mass that we compute analytically for an arbitrary interaction strength.

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Introduction.—Systems with a macroscopic coherence between pairs of fermions exhibit in their excitation spectrum a pair-breaking continuum, whose energy is greater than twice the order-parameter Δ. This is particularly the case of superconductors and cold gases of spin-$1/2$ fermionic atoms. The collective behavior of the neutral gases at energies below 2Δ is known: it is characterized by a bosonic excitation branch of phononic start [1]. The dispersion relation of this branch was calculated [2,3] and its existence experimentally confirmed [4–6].

Conversely, the existence of a collective mode *inside* the pair-breaking continuum remains a debated question that attracts much interest because of an analogy often suggested with Higgs modes in field theory [7]. The challenge is to understand whether the response of the continuum to an excitation is flat in frequency or presents a nontrivial structure like a resonance. We identify two major shortcomings in the existing theoretical treatment [8–12]: (i) it neglects the coupling between the amplitude and phase of the order-parameter, which restricts it to the weak coupling regime, (ii) it is limited to long wavelengths. These shortcomings are prejudicial as they maintain doubts about the very existence of this second collective mode [13], notably at zero wave vector [14].

Here, we clarify the description of the pair-breaking collective modes. By analytically continuing the pair propagator, we reveal a pole below the branch cut associated with the continuum, for positive chemical potential $\mu > 0$ and nonzero wave number only. We obtain the full dispersion relation of this mode completely accounting for amplitude-phase coupling. This allows us to deal with the strong coupling regime. Remarkably, the real part of the branch is wholly below 2Δ when $\Delta > 1.210\mu$ (yet the branch remains separated from the band gap $[0, 2\Delta]$ on the real axis by a branch cut). In the weak coupling and long wavelength limit, we agree with the result of [10] but disagree sharply with the prediction commonly accepted in the literature [12], notably for the damping rate that, we find, has a quadratic start at low wave number, rather than a linear one. All our predictions are based on Anderson's random phase approximation (RPA) or Gaussian approximation for contact interactions. This theory describes qualitatively well both cold Fermi gases in the BEC-BCS crossover and BCS superconductors [Coulomb interaction has no effect on amplitude modes at frequencies $O(\Delta/\hbar)$ [10]], and is a prerequisite for any more realistic description of interactions.

The branch we find describes the collective behavior of the pairs following an excitation of their internal degrees of freedom; its frequency is thus not simply the continuum threshold $2\Delta/\hbar$, as for the "Higgs oscillations" predicted and observed [14–23] at zero wave vector. It is observable in superfluid Fermi gases as a broadened peak at energies above 2Δ in the order-parameter-amplitude response function.

Fluctuations of the order parameter.—We consider a homogeneous system of spin-$1/2$ fermions of mass m and chemical potential μ, with contact interactions. At zero temperature, the fluctuations of the order-parameter Δ around its equilibrium value admit eigenmodes: the collective modes of the system. Expanding to second order in amplitude $\delta\lambda$ and phase $\delta\theta$ fluctuations yields the Gaussian action [24,25]

$$\mathcal{S} = \mathcal{S}_0 + \int d\omega \int d^3q \begin{pmatrix} -i\Delta\delta\theta^* & \delta\lambda^* \end{pmatrix} M(\omega, \mathbf{q}) \begin{pmatrix} i\Delta\delta\theta \\ \delta\lambda \end{pmatrix}. \tag{1}$$

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The symmetric fluctuation matrix M gives access to the propagator of Δ through a mere inversion. The equation on the collective mode energy $z_{\mathbf{q}}$ with wave vector $\mathbf{q}$ is then

$$\det M(z_{\mathbf{q}}, \mathbf{q}) = 0. \tag{2}$$

Since the order-parameter Δ describes pair condensation, the coefficients of its fluctuation matrix contain an integral over the internal wave vector $\mathbf{k}$ of the pairs, involving $\xi_{\mathbf{k}} = \hbar^2 k^2/2m - \mu$ and $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$, the dispersion relations of free fermions and BCS quasiparticles respectively, as well as the energy $E_{\mathbf{kq}} = E_{\mathbf{k}+\mathbf{q}/2} + E_{\mathbf{k}-\mathbf{q}/2}$ of a pair of quasiparticles of total wave vector $\mathbf{q}$:

$$M_{\pm\pm}(z, \mathbf{q}) = \int \frac{d^3k}{2} \left(\frac{(W_{\mathbf{kq}}^{\pm})^2}{z - E_{\mathbf{kq}}} - \frac{(W_{\mathbf{kq}}^{\pm})^2}{z + E_{\mathbf{kq}}} + \frac{1}{E_{\mathbf{k}}} \right), \tag{3}$$

$$M_{+-}(z, \mathbf{q}) = \int \frac{d^3k}{2} W_{\mathbf{kq}}^{+} W_{\mathbf{kq}}^{-} \left(\frac{1}{z - E_{\mathbf{kq}}} + \frac{1}{z + E_{\mathbf{kq}}} \right), \tag{4}$$

where the indices $+$ and $-$ refer to phase and amplitude fluctuations and we introduce the notation $(W_{\mathbf{kq}}^{\pm})^2 = (E_{\mathbf{k}+\mathbf{q}/2}E_{\mathbf{k}-\mathbf{q}/2} + \xi_{\mathbf{k}+\mathbf{q}/2}\xi_{\mathbf{k}-\mathbf{q}/2} \pm \Delta^2)/(2E_{\mathbf{k}+\mathbf{q}/2}E_{\mathbf{k}-\mathbf{q}/2})$ [26]. Equations (2)–(4) are found also with RPA [1,27,28], diagrammatic resummations [3] or linearized time-dependent BCS equations [29].

Since Eq. (2) is invariant under the change of z to $-z$, we impose $\mathrm{Re}\, z \geq 0$. The matrix M then has a branch cut for $z \in \mathcal{C}_{\mathbf{q}} = \{E_{\mathbf{kq}}, \mathbf{k} \in \mathbb{R}^3\}$, originating in the denominator $z - E_{\mathbf{kq}}$ in Eqs. (3)–(4). As such, Eq. (2) has at most one solution for fixed $\mathbf{q}$: it is real, below the continuum, and corresponds to the bosonic Anderson-Bogoliubov branch [3]. Conversely, the collective modes we want to characterize are *inside* the continuum, that is, *a priori* for $\mathrm{Re}\, z_{\mathbf{q}} > \min \mathcal{C}_{\mathbf{q}}$. As in the textbook problem of one atom coupled to the electromagnetic field [30], the correct way to solve Eq. (2) in the presence of the continuum is to analytically continue the matrix M through its branch cut [8]. This is an opportunity to recall the procedure of Nozières [31] to analytically continue a function of the form

$$f(z) = \int_{-\infty}^{+\infty} d\omega \frac{\rho(\omega)}{z - \omega}, \tag{5}$$

analytic for $\mathrm{Im}\, z \neq 0$, but exhibiting a branch cut on the real axis, wherever the spectral density ρ is nonzero. The nonanalytic contribution to $M_{\sigma\sigma'}$, with $\sigma, \sigma' = \pm$, is naturally cast into this form with the spectral densities

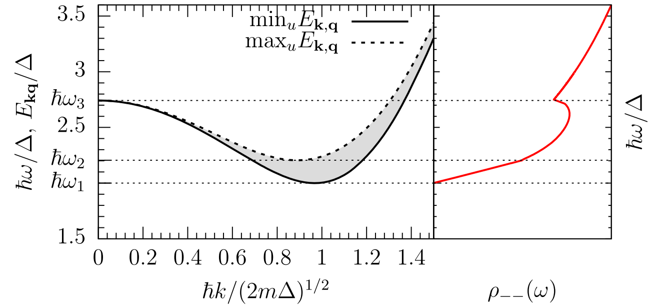

FIG. 1. Left: As a function of k, the interval between $\min_u E_{\mathbf{kq}}$ (reached for $u = 0$, solid line) and $\max_u E_{\mathbf{kq}}$ (reached for $u = \pm 1$, dashed line) determines an energy band (gray area) in which the resonance $\hbar\omega = E_{\mathbf{kq}}$ occurs for at least one value of $u = \cos(\widehat{\mathbf{k}, \mathbf{q}})$ in $[-1, 1]$. For fixed ω, the integration interval over k in Eq. (6) is read horizontally; as a function of ω, its structure undergoes 3 transitions in ω_1, ω_2, and ω_3, which results in angular points in the spectral density. Right: Example of ρ_{--} (solid line). Here, $\mu/\Delta = 1$ and $\hbar q/\sqrt{2m\Delta} = 0.5$.

$$\rho_{\sigma\sigma'}(\omega, \mathbf{q}) = \int \frac{d^3k}{2} W_{\mathbf{kq}}^{\sigma} W_{\mathbf{kq}}^{\sigma'} \delta(\hbar\omega - E_{\mathbf{kq}}) \tag{6}$$

The analytic continuation of f from the upper to lower half-plane, through an interval $[\omega_1, \omega_2]$ of the branch cut where ρ is analytic, is simply

$$f_{\downarrow}(z) = \begin{cases} f(z) & \text{if } \mathrm{Im}\, z > 0 \\ f(z) - 2i\pi\rho(z) & \text{if } \mathrm{Im}\, z \leq 0, \end{cases} \tag{7}$$

where $\rho(z)$ is the analytic continuation of ρ for $\mathrm{Im}\, z \neq 0$. This is readily demonstrated by writing $\rho(\omega) = [\rho(\omega) - \rho(z)] + \rho(z)$ in Eq. (5) with an energy cutoff.

To carry out the analytic continuation of M, we study the function $\omega \mapsto \rho_{\sigma\sigma'}$ on the real axis, and search for singularities. For that, we integrate over $\mathbf{k}$ in Eq. (6) in a spherical frame of axis $\mathbf{q}$ and use the Dirac-δ to perform the angular integration over $u = \mathbf{k} \cdot \mathbf{q}/kq$. The remaining integral over k is restricted to a domain represented on Fig. 1, whose form depends on ω. When $\mu > 0$, the BCS excitation branch has its minimum in $k_0 = \sqrt{2m\mu/\hbar^2}$; then, for $q > 0$ small enough [32] the function $\omega \mapsto \rho_{\sigma\sigma'}$ has three angular points related to a configuration change of the integration domain, which divides the real axis in four distinct sectors (see Fig. 1): (i) for $\omega < \omega_1 = 2\Delta/\hbar$, the resonance condition $\hbar\omega = E_{\mathbf{kq}}$ is never satisfied, so that $\rho_{\sigma\sigma'}(\omega < \omega_1) = 0$, (ii) for $\omega_1 < \omega < \omega_2$ it is reached on an interval $[k_1, k_2]$, (iii) for $\omega_2 < \omega < \omega_3$, it occurs on disjoint intervals $[k_1, k_1']$ and $[k_2', k_2]$, and (iv) for $\omega > \omega_3$, it occurs again on an interval $[k_2', k_2]$.

Numerical study at arbitrary q.—We find a solution $z_{\mathbf{q}} = \hbar\omega_{\mathbf{q}} - i\hbar\Gamma_{\mathbf{q}}/2$ to Eq. (2) in the analytic continuation through the sector $[\omega_1, \omega_2]$ (see the schematic on Fig. 2), which we identify as the energy of the sought collective

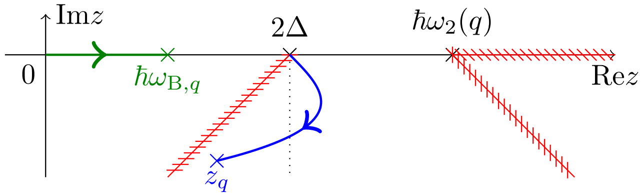

FIG. 2. Trajectories of the pair-breaking collective branch (blue curve) and Bogoliubov-Anderson branch (green line) as functions of q in the complex plane. The first one is revealed only after analytic continuation, hence the deformed branch cut (striped red lines) in the lower half-plane.

mode. In this sector, we express the spectral functions in terms of first and second kind complete elliptic integrals [33].

The dispersion relation $q \mapsto \omega_{\mathbf{q}}$ is represented on Fig. 3 for pairing strengths $\mu/\Delta = 1/10$, 5, and 100 ($1/k_F a \simeq 0.5$, -1.1 and -3.0 in Fermi gases with Fermi wave number k_F and scattering length a). Departing quadratically from its limit 2Δ in $q = 0$, the branch goes through a maximum of height proportional to Δ and location of order the inverse of the pair radius $\xi \approx \hbar^2 k_0/m\Delta$ at weak coupling $\Delta \ll \mu$, then dips below 2Δ. In the strong coupling regime $\Delta > \mu$, the domain where the energy of the branch is greater than 2Δ shrinks, until its disappearance for $\mu/\Delta \simeq 0.8267$. Conversely, the damping rate $\Gamma_{\mathbf{q}}$ is a strictly increasing function of q, also starting quadratically from its zero limit in $q = 0$. This is in direct contrast with the commonly accepted prediction in the literature of a damping rate linear in q [12]. The fact that our solution travels far away from the initial branch cut underlines the nonperturbative nature of our analytic continuation: there is no unperturbed solution on the real

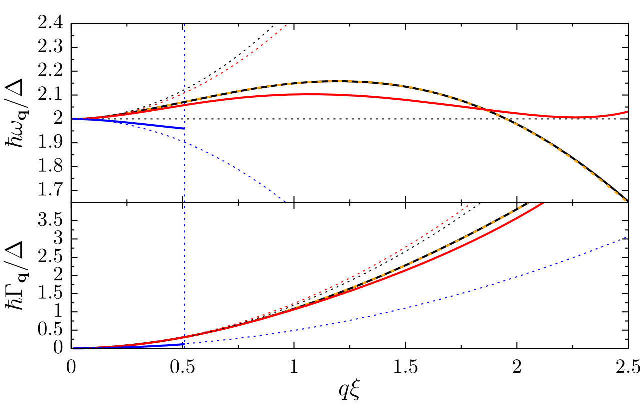

FIG. 3. Frequency (top) and damping rate (bottom) of the pair-breaking collective mode as functions of q for $\mu/\Delta = 100$ (black solid curve), $\mu/\Delta = 5$ (red solid curve), and $\mu/\Delta = 0.1$ (blue solid curve, disappears in $2k_0\xi \simeq 0.51$) as functions of q in units of the inverse pair size ξ [2]. Dashed curves: the same for $\mu/\Delta = 100$, omitting the amplitude-phase coupling M_{+-}. Dotted curves: low-q quadratic behavior obtained analytically from Eqs. (10)–(14).

axis from which $\mathrm{Im}\, z_{\mathbf{q}}$ could be deduced from Fermi's golden rule.

The branch disappears in $q = 2k_0$ (hence before the Bogoliubov-Anderson branch hits the continuum [3]) when the interval $[\omega_1, \omega_2]$ through which our analytic continuation passes reduces to a point. Last, we exclude the existence of a branch of energy above $2\sqrt{\Delta^2 + \mu^2}$ (twice the gap) in the BEC regime where $\mu < 0$ and where the three singularities ω_i of $\rho_{\sigma\sigma'}$ gather.

Long wavelength limit.—In this limit, we obtain several analytical results that corroborate our numerical study. We deal separately with the singular case $q = 0$, where the matrix $M(z, \mathbf{q} = 0)$ is expressible in terms of the complete elliptic integrals of the first and third kind $K(k)$ and $\Pi(n, k)$ [34,35]:

$$\mathrm{th}s\tilde{M}_{++}(z,0) = \frac{\tilde{M}_{--}(z,0)}{\mathrm{th}s} = -\pi(2e^l)^{1/2}[F(s) - F(-s)],$$
$$\tilde{M}_{+-}(z,0) = -\pi(2e^l)^{1/2}[F(s) + F(-s)], \tag{8}$$

with $l = \mathrm{argsh}(\mu/\Delta)$, $s = \mathrm{argch}(z/2\Delta)$, and

$$F(s) = (\mathrm{sh}l + \mathrm{sh}s)[\Pi(e^{l+s}, ie^l) - \Pi(-\mathrm{e}^{l-s}, ie^l)] + K(ie^l)\mathrm{ch}s. \tag{9}$$

Equation (2) then reads simply $F(s)F(-s) = 0$. Even after analytic continuation [36] this equation has no solution besides $s = i\pi/2$ ($z = 0$, the starting point of the Anderson-Bogoliubov branch); in particular, $F(s)$ has a finite nonzero limit when $z \to 2\Delta$ ($s \to 0$) with $\mathrm{Im}\, s > 0$. Thus, the threshold of the pair-breaking continuum $\omega = 2\Delta/\hbar$ is not a solution of the RPA equation (2) in $q = 0$ [37], and not a pole of the response functions. This is why, as understood by Refs. [9,14–18], the "Higgs" oscillations at this frequency are not sinusoidal as $\cos(2\Delta t/\hbar + \phi)$ but subject to a power-law damping as $\cos(2\Delta t/\hbar + \phi)/t^\alpha$, $\alpha > 0$.

For small but nonzero q, and $\mu > 0$, the resonance sector between $\hbar\omega_1 = 2\Delta$ and $\hbar\omega_2 = 2\Delta + \mu\hbar^2 q^2/2m\Delta + O(q^4)$ in Fig. 1 has a width $O(q^2)$ in energy, and $O(q)$ in the wave number k around the minimum location k_0 of the BCS branch. We then set

$$z_{\mathbf{q}} = 2\Delta + \zeta\frac{\hbar^2 q^2}{4m^*} + O(q^3) \quad \text{and} \quad k = k_0 + Kq, \tag{10}$$

with $m^* = m\Delta/2\mu$ the effective mass of the BCS branch minimum. We thus focus on the wave vector domain where the denominator in Eqs. (3), (4) is of order q^2:

$$z - E_{\mathbf{kq}} = z - 2\Delta - \frac{\hbar^2 q^2}{m^*}(K^2 + u^2/4) + O(q^3). \tag{11}$$

Now, using the expansions of the numerator amplitudes $W^+_{\mathbf{kq}} \sim 1$ and $W^-_{\mathbf{kq}} \sim \hbar^2 k_0 qK/m\Delta$, and performing the integration over the angular variable u before that over K we obtain the analytic expressions for $\mathrm{Im}\, z > 0$:

$$\tilde{M}_{++}(z,\mathbf{q}) \underset{q\to 0}{\sim} -\frac{i\pi^2(2m\Delta)^{1/2}}{\hbar q}\,\mathrm{asin}\frac{1}{\sqrt{\zeta}}, \tag{12}$$

$$\tilde{M}_{--}(z,\mathbf{q}) \underset{q\to 0}{\sim} -\frac{i\pi^2\mu\hbar q}{(8m\Delta^3)^{1/2}}\left[\sqrt{\zeta-1}+\zeta\,\mathrm{asin}\frac{1}{\sqrt{\zeta}}\right]. \tag{13}$$

Since the divergence of M_{++} of order $1/q$ is compensated by the suppression of M_{--} linear in q, the finite nonzero limit (8) of M_{+-} in $q=0, \hbar\omega = 2\Delta$ suffices. Inserting expressions (8), (12), (13) in the RPA equation (2) and analytically continuing the product $M_{++}M_{--}$ through its branch cut [0, 1] in ζ (corresponding to the segment $[\hbar\omega_1, \hbar\omega_2]$ in z) with the substitutions $\mathrm{asin}\,1/\sqrt{\zeta} \to \pi - \mathrm{asin}\,1/\sqrt{\zeta}$ and $\sqrt{\zeta-1} \to -\sqrt{\zeta-1}$, we obtain an explicit yet transcendental equation on ζ:

$$\left[\pi - \mathrm{asin}\frac{1}{\sqrt{\zeta}}\right]\left[\left(\pi - \mathrm{asin}\frac{1}{\sqrt{\zeta}}\right)\zeta - \sqrt{\zeta-1}\right] + \frac{2}{\pi^4\mu}\left(\frac{\hbar^2}{2m}\right)^3 M_{+-}^2(2\Delta,0) = 0. \tag{14}$$

The continuation is for the entire lower half-plane, including $\mathrm{Re}\, z < 2\Delta$ ($\mathrm{Re}\,\zeta < 0$). The unique solution of Eq. (14) shown in Fig. 4 faithfully reproduces the coefficient of q^2 in Fig. 3. The real part changes sign for $\mu/\Delta \simeq 0.8267$, which confirms that the branch is below 2Δ at strong coupling.

To understand the disappearance of the branch at $q=0$, we calculate the matrix residue of the analytically continued propagator $M_\downarrow(z,\mathbf{q})^{-1}$ at $z_\mathbf{q}$ and find that it vanishes linearly: it becomes proportional to the amplitude-channel projector $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ with a factor

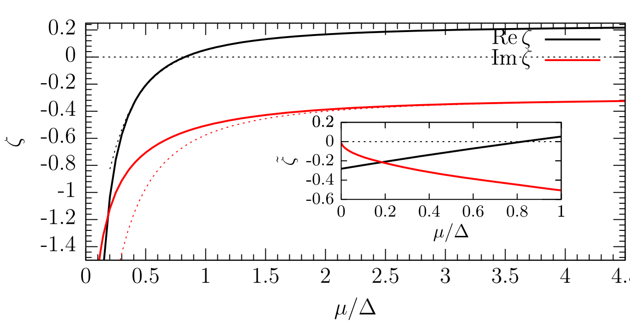

FIG. 4. Real and imaginary parts (black and red solid curves) of the dimensionless coefficient ζ of q^2 in the energy $z_\mathbf{q}$ of the pair-breaking collective mode as functions of μ/Δ. Dashed curves: weak coupling expansion $\zeta = \zeta_0 - [2\zeta_0^2/(\zeta_0-1)](\Delta/\pi\mu)^2\ln^2(\Delta/8\mu e) + \cdots$ with $\zeta_0 \simeq 0.2369 - 0.2956i$. Inset: rescaled coefficient $\tilde{\zeta} = \zeta\mu/\Delta = \zeta m/2m^*$ admitting the finite real limit $\tilde{\zeta}_\infty = -16K^2(i)/\pi^4 \simeq -0.2823$ at strong coupling $\mu/\Delta \to 0^+$, its imaginary part tending to zero like $-12K(i)(\mu/\Delta)^{1/2}/\pi^3$.

$$Z_\mathbf{q} \underset{q\to 0}{\sim} \frac{i\hbar^4 q}{2m^2\pi^2}\frac{\pi - \mathrm{asin}\frac{1}{\sqrt{\zeta}}}{(\pi - \mathrm{asin}\frac{1}{\sqrt{\zeta}})^2 + \frac{(\pi-\mathrm{asin}\frac{1}{\sqrt{\zeta}})\zeta - \sqrt{\zeta-1}}{2\zeta\sqrt{\zeta-1}}}. \tag{15}$$

This results from applying $(d/dz) \propto q^{-2}(d/d\zeta)$ to Eqs. (12), (13). $Z_\mathbf{q}$ is the weight of the collective mode above the continuum background; its suppression in $q=0$ means that the many-body response function can no longer be interpreted in terms of a quasiparticle on an incoherent background.

At weak coupling ($\mu/\Delta \to +\infty$), M_{+-} tends to zero because of the antisymmetry $k \leftrightarrow 2k_0 - k$ about the Fermi surface, valid for $(k-k_0)\xi = O(1)$. The RPA equation reduces to $M_{++}M_{--} = 0$ for $q\xi = O(1)$, and Eq. (14) to its ζ-dependent first line. While the phononic phase mode solves $M_{++} = 0$, the pair-breaking collective mode is then a pure amplitude mode (a root of $M_{\downarrow--}$) [38]. Its quadratic dispersion relation,

$$z_\mathbf{q} \underset{\mu/\Delta\to+\infty}{\overset{q\to 0}{\simeq}} 2\Delta + (0.2369 - 0.2956i)\frac{\hbar^2 q^2}{4m^*} \tag{16}$$

contradicts Ref. [12] (even $\mathrm{Re}\,\zeta$ differs from the value 1/3 of Ref. [12]), but confirms Ref. [10].

Our calculation shows the limits of the analogy with Higgs modes in field theory: although it is also a gapped amplitude mode at weak coupling, the collective mode, here immersed in a continuum, is obtained only after a nonperturbative treatment of the coupling to fermionic degrees of freedom; impossible therefore to obtain it reliably from a low-energy ($\hbar\omega \ll 2\Delta$) effective action as suggested sometimes [7,39].

Observability in response functions.—At low q, the pair-breaking collective mode is weakly damped, a favorable condition. At weak coupling, as shown in Fig. 5, there indeed appears in the response function of the order-parameter amplitude a smooth peak, whose position, width, and height are remarkably predicted by the branch obtained in the analytic continuation. At strong enough coupling [blue curve in Fig. 5(b)], the smooth resonance peak disappears and there remains a sharp one (with a vertical tangent), whose maximum is at $\omega = 2\Delta/\hbar$ even for $q \neq 0$. Qualitatively, this indicates that the collective frequency $\omega_\mathbf{q}$ is below $2\Delta/\hbar$ such that there is no complex resonance in the interval $[2\Delta/\hbar, \omega_2]$ where our analytic continuation is meaningful.

The amplitude response function $[|M_{++}/\det M(\omega + i0^+, \mathbf{q})|^2$, or $1/|M_{--}(\omega + i0^+, \mathbf{q})|^2$ at weak coupling], unlike the more commonly measured density-density response [6], is sensitive to the pair-breaking collective mode even at weak coupling. In cold gases, the order-parameter amplitude can be excited by Feshbach modulation of the interaction strength, and measured by spatially resolved interferometry [40]. Physically, Fig. 5 shows that the system absorbs energy from modulations of

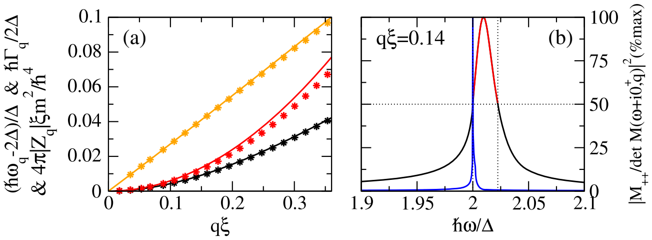

FIG. 5. (*a*) At weak coupling (Δ/μ tends to 0), frequency displacement $\omega_{\mathbf{q}} - 2\Delta/\hbar$, damping rate $\Gamma_{\mathbf{q}}$ and residue $Z_{\mathbf{q}}$ of the pair-breaking collective mode (black, red, orange solid lines) compared to the values (stars) extracted by fitting the amplitude response function $1/|M_{--}(\omega + i0^+, \mathbf{q})|^2$ [black curve of (*b*)] by the function $|C + Z_{\mathbf{q}}{}^{\text{fit}}/(\omega - \omega_{\mathbf{q}}^{\text{fit}} + i\Gamma_{\mathbf{q}}^{\text{fit}}/2)|^2$ [red curve of (*b*)] describing a resonance on a flat background C. Blue curve of (*b*): amplitude response function $|M_{++}/\det M(\omega + i0^+, \mathbf{q})|^2$ at strong coupling ($\Delta/\mu = 10$) exhibiting only a sharp peak at $2\Delta/\hbar$.

the pairing strength $|\Delta|$ at frequencies $\omega > 2\Delta/\hbar$ more efficiently when ω is close to $\omega_{\mathbf{q}}$. This resonance is broadened because the absorbed energy is dissipated by breaking pairs into unpaired fermions of wave vectors $\mathbf{q}/2 \pm \mathbf{k}$.

Conclusion.—We have established on solid theoretical foundations the existence of a collective branch inside the pair-breaking continuum of BCS superconductors and superfluid Fermi gases, and we have fully characterized its dispersion relation and damping rate, including in the strong coupling regime where it is a mixture of amplitude and phase fluctuations. We thus give a complete answer to an old condensed-matter problem. The branch appears clearly in the order-parameter response function which can be measured in cold atomic gases.

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[36] The branch cut $[2\Delta, +\infty[$ in z for $\mu > 0$ translates into a branch cut $[0, +\infty[$ in s. Thus $F(s) - F(-s)$ has the nonzero limit $i\pi(1 + \sqrt{1 + \Delta^2/\mu^2})^{-1/2}/\sqrt{2}$ when $s \to 0$ with $\mathrm{Im}\, s > 0$.

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