# Computationally Universal Phase of Quantum Matter

Robert Raussendorf,[1,2] Cihan Okay,[1,2] Dong-Sheng Wang,[1,2] David T. Stephen,[3] and Hendrik Poulsen Nautrup[4]
[1]*Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1*
[2]*Stewart Blusson Quantum Matter Institute, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z4*
[3]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany*
[4]*Institut für Theoretische Physik, Universität Innsbruck, Technikerstraße 21a, A-6020 Innsbruck, Austria*

(Received 26 May 2018; published 4 March 2019)

We provide the first example of a symmetry protected quantum phase that has universal computational power. This two-dimensional phase is protected by one-dimensional linelike symmetries that can be understood in terms of the local symmetries of a tensor network. These local symmetries imply that every ground state in the phase is a universal resource for measurement-based quantum computation.

DOI: 10.1103/PhysRevLett.122.090501

In the presence of symmetry, quantum phases of matter can have computational power. This was first conjectured in Refs. [1–3] and has been proven [4–6] or numerically supported [7,8] in several instances. The important property is that the computational power is uniform. It does not depend on the precise choice of the state within the phase, and it is thus a property of the phase itself. In this way, phases of quantum matter acquire a computational characterization and computational value.

The quantum computational power of physical phases is utilized by measurement-based quantum computation (MBQC) [9], in which the process of computation is driven by local measurements on an initial entangled state. Here, we consider initial states that originate from symmetry protected topological (SPT) phases [10–12].

Proofs of the existence of such "computational phases of quantum matter" have so far been confined to spatial dimension 1. After it was shown that computational wire—the ability to shuttle quantum information from one end of a spin chain to the other—is a property of certain SPT phases [3], the first phase permitting quantum computations on a single logical qubit was described in Ref. [4]. In fact, uniform computational power is ubiquitous in one-dimensional SPT phases [5,6].

Computationally, physical phases in dimension 2 and higher are more interesting than in dimension 1. The reason is that, in MBQC, one spatial dimension plays the role of the circuit model time. Therefore, MBQC in dimension $D$ corresponds to the circuit model in dimension $D-1$, and universal MBQC is possible only in $D \geq 2$.

Yet, to date, the evidence for quantum computational phases of matter is much more scant for $D \geq 2$ than for $D = 1$. Numerical evidence exists for deformed Affleck-Kennedy-Lieb-Tasaki Hamiltonians on the honeycomb lattice [7,8,13]. In addition, extended regions of constant computational power have also been observed in SPT phases with $\mathbb{Z}_2$ symmetry [14].

Numerous computationally universal resources states for MBQC have been constructed [15–18] using the tools of group cohomology that also form the basis for the classification of SPT order [11,12]. From the starting point of these special states, it remains open what happens to the computational power as one probes deeper into the SPT phases surrounding them.

For the cluster phase in $D = 2$, which is a symmetry protected phase that contains the cluster state, it was shown analytically that universal computational power persists throughout a finite region around the cluster state [19].

Here, we prove the existence of a computationally universal phase of quantum matter in spatial dimension 2; see Fig. 1. As in Ref. [19], the phase we consider is protected by one-dimensional linelike symmetries, generalizing the conventional notion of symmetry protected topological order defined by global on-site symmetries. As in the case of global symmetries, these line symmetries can be built from the local symmetries of a tensor network that persist throughout the

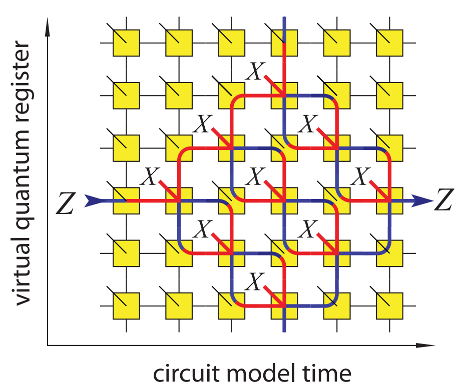



FIG. 1. Symmetry protected quantum correlations enable uniform computational power throughout the two-dimensional (2D) cluster phase. The long-range symmetry shown is composed of the symmetries of local projected entangled pair state (PEPS) tensors.

phase. Using this, we establish that computational universality persists throughout the entire phase. The backbone of the computational scheme is symmetry protected correlations in a virtual quantum register; see Fig. 1.

*Setting and result.*—We consider a two-dimensional simple square spin lattice that is, for simplicity of boundary conditions, embedded in a torus of a small circumference $n$ and a large circumference $nN$, with $n, N \in \mathbb{N}$, $N \gg n$, and $n$ even. Its Hamiltonian is invariant under all lattice translations and the symmetries

$$U_{c,+} = \bigotimes_{x=0}^{nN-1} X_{x,c+x} \quad \text{and} \quad U_{c,-} = \bigotimes_{x=0}^{nN-1} X_{x,c-x} \qquad (1)$$

for all $c \in \mathbb{Z}_n$. Therein, $X \equiv \sigma_x$, and the addition in the second index of $X$ is mod $n$. A graphical rendering of these symmetries is provided in Fig. 2(a). These symmetries were previously considered in Ref. [19]. We consider phases in which the ground state is unique, and thus shares the symmetries.

As the Hamiltonian is varied while respecting the symmetries [Eq. (1)], the respective ground states arrange into phases. The central object of interest is the "2D cluster phase," i.e., the physical phase that respects the symmetries [Eq. (1)] and which contains the 2D cluster state.

The main result of this Letter is the following:

*Theorem 1.*—For a spin-1/2 lattice on a torus with circumferences $n$ and $Nn$, where $N \gg n$ and $n$ is even, all ground states in the cluster phase, except a possible set of measure 0, are universal resources for measurement-based quantum computations on $n/2$ logical qubits.

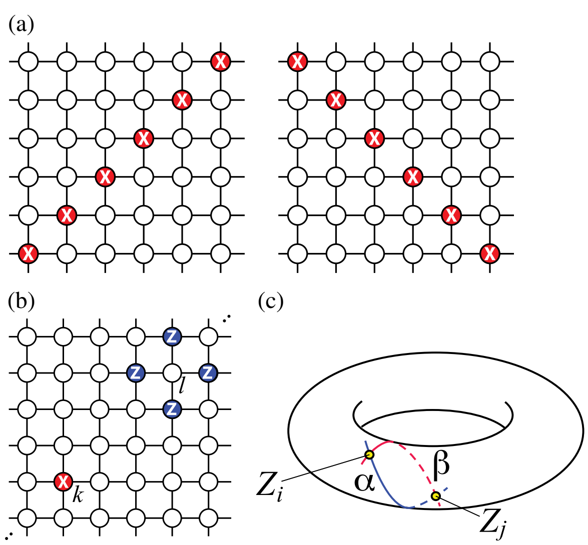
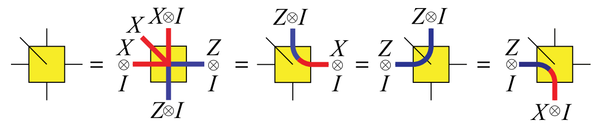


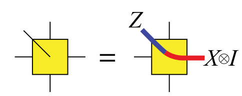

FIG. 2. (a) Linelike symmetry of Eq. (1). All translations are also symmetries. (b), (c) Generators of Pauli operators that commute with the symmetries [Eq. (1)]. (b) Local operators $X_k$ and $\text{Star}_l$, for all sites $k$ and $l$. (c) Geometrically nonlocal operators $Z_i \otimes Z_j$. Locations $i$ and $j$ are consecutive intersections of supports $\alpha$ and $\beta$ of two symmetries.

To facilitate the proof of Theorem 1, we introduce the notion of "clusterlike" states $|\Phi\rangle$. For a square grid in dimension 2, we represent states as projected entangled pair states (PEPSs) [20] with local tensors $A_\Phi$, such that contracting virtual legs on a torus as in Fig. 1 describes the wave function of $|\Phi\rangle$. The clusterlike states are those for which the PEPS tensors have the symmetries



(2)

Therein, red (blue) legs indicate Pauli operators $X$ ($Z$). This notation means that, for example, acting on the physical leg (diagonally directed) of the PEPS tensor with a Pauli $X$ is equivalent to a corresponding action of Pauli operators on the virtual legs. The reason for calling states satisfying Eq. (2) clusterlike is that the cluster state itself satisfies Eq. (2). Furthermore, if we add




to those symmetries, then we obtain cluster states as the only solution of the joint symmetry constraints.

The proof of Theorem 1 splits into two parts. First, we show that all states in the 2D cluster phase are clusterlike, and then we demonstrate that clusterlikeness implies universal computational power.

*A 2D physical phase of clusterlike states.*—Here, we prove the following result:

*Proposition 1.*—Every ground state $|\Phi\rangle$ in the 2D cluster phase has a description in terms of a local tensor $A_\Phi$ that has the symmetries of Eq. (2).

Our starting point is the characterization of SPT phases in terms of symmetric quantum circuits. A symmetric quantum circuit is a sequence of unitary gates $U = \prod_{i=1}^{l} U_i$, where each gate $U_i$ is invariant under the symmetry group $G$ of Eq. (1), $[U_i, U(g)] = 0$, for all $g \in G$. In any such circuit which is also local, each gate $U_i$ acts only on a bounded number of qubits [21]. We then have the following result [11]:

*Lemma 1.*—Symmetric gapped ground states in the same SPT phase are connected by symmetric local quantum circuits of constant depth.

To prove Proposition 1, we analyze the structure of the symmetry-respecting gates $U_{\Phi,i}$ of the circuit $U_\Phi$ mapping the cluster state $|C\rangle$ to a given state $|\Phi\rangle$ in the cluster phase $|\Phi\rangle = U_\Phi|C\rangle$. Writing

$$U_{\Phi,i} = \sum_j d_j P_j, \quad \text{with} \quad d_j \in \mathbb{C}, \quad \forall\ j, \qquad (3)$$

only symmetry-respecting $n$-qubit Pauli operators $P_j$, $P_j \in \mathcal{P}_n$, can appear on the rhs. The generators of such

Pauli operators are displayed in Figs. 2(b) and 2(c). Furthermore, the operators shown in Fig. 2(c) do not contribute because they are geometrically nonlocal. Thus, the Pauli operators appearing on the rhs of Eq. (3) are generated by local operators $X_k$ and $Z$-type star operators $\mathrm{Star}_l$ for all sites $k$ and $l$ of the lattice (see Sec. I A of the Supplementary Material (SM), Lemma 3 [22]).

Now, expanding the entire circuit $U_\Phi$ into a sum of Pauli operators, every Pauli operator in this expansion is also a product of $X_k$ and star operators. We further observe that, by the form of the cluster state stabilizer,

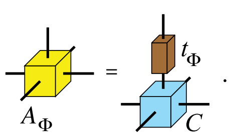
$$\mathrm{Star}_k|C\rangle = X_k|C\rangle, \tag{4}$$

for all lattice sites $k$. Using relation (4), all star operators in the expansion of $U_\Phi$ can be eliminated. We thereby obtain a transformation $T_\Phi$ that satisfies the relation $T_\Phi|C\rangle = U_\Phi|C\rangle = |\Phi\rangle$, and it is composed of Pauli-$X$ operators only,

$$T_\Phi = \sum_{\mathbf{j}} c_{\mathbf{j}} X(\mathbf{j}). \tag{5}$$

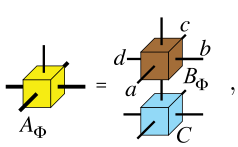
Therein, $X(\mathbf{j}) \coloneqq \bigotimes_k (X_k)^{j_k}$ is an $X$-type Pauli operator with support on the $n \times nN$ torus; i.e., $\mathbf{j}$ is a binary vector with $n^2N$ components.

*Proof of Proposition 1.*—To illustrate the idea of the proof, we first discuss the special case where the map $T_\Phi$ is a tensor product of local factors $T_\Phi = \bigotimes_k t_{\Phi,k}$. Then, to obtain a local tensor $A_\Phi$ representing $|\Phi\rangle$, we apply $T_\Phi$ sitewise to the local tensor $C$ representing the cluster state. Graphically,



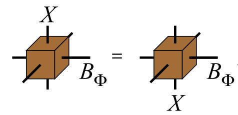

Because, by Eq. (5), $t_\Phi$ is a linear combination of $I$ and $X$, it commutes with $X$. Hence, the symmetries [Eq. (2)] of the cluster state tensors $C$ are also symmetries of the tensors $A_\Phi$ representing $|\Phi\rangle$.

Now, turning to the general case, the action of $T_\Phi$ on $|C\rangle$ results in local tensors $A_\Phi$ of the form



(6)

where the "junk tensor" $B_\Phi$ [3] forms a tensor network representation of the map $T_\Phi$, and it emerges as a consequence of the nonlocality of the map $T_\Phi$. It inherits from $T_\Phi$ the property that, on the physical leg of $C$ (pointing upwards), it acts as $I$ or $X$, depending on the state of the virtual links $a, \ldots, d$ (for details, see the SM, Sec. I B [22]). The junk tensor $B_\Phi$ thus commutes with the action of the local Pauli-$X$ operator,




As a result, the symmetries [Eq. (2)] hold for all tensors $A_\Phi$ describing a state $|\Phi\rangle$ in the cluster phase. □

*Cluster symmetries and computation.*—We now show that the symmetries [Eq. (2)] of the PEPS tensors imply MBQC universality of the corresponding quantum state. This proceeds in two steps. We establish (i) the computational wire, i.e., the ability to shuttle quantum information across the torus; and (ii) a universal set of quantum gates.

Computational wire: We now map to a quasi-one-dimensional (quasi-1D) setting by grouping spins into blocks of size $n \times n$. If we block $n \times n$ copies of the tensor $A_\Phi$, as in Fig. 1, we obtain the block tensor $\mathcal{A}_\Phi$, which forms a matrix product state (MPS) representation [23] of the quasi-1D system. Contracting the physical legs of this tensor with local $X$ eigenstates labeled by the $n^2$-component binary vector $\mathbb{i}$ gives the tensor component $\mathcal{A}_\Phi(\mathbb{i})$. We can now use the symmetries in Eq. (2) to constrain these tensor components:

*Lemma 2.*—Consider a torus of size $n \times nN$ and $n \in 2\mathbb{N}$. For all ground states $|\Phi\rangle$ in the 2D cluster phase, the corresponding block tensors $\mathcal{A}_\Phi(\mathbb{i})$ satisfy

$$\mathcal{A}_\Phi(\mathbb{i}) = \mathcal{C}(\mathbb{i}) \otimes \mathcal{B}_\Phi(\mathbb{i}). \tag{7}$$

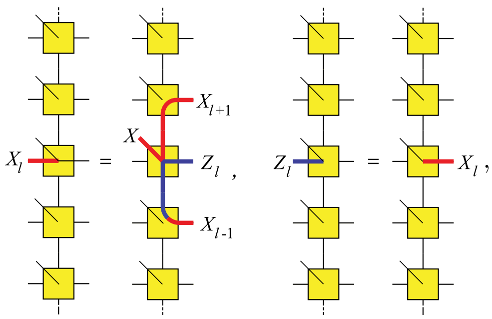
The logical tensors $\mathcal{C}(\mathbb{i})$ are constant throughout the phase, and

$$\mathcal{C}(\mathbb{i}) \in \mathcal{P}_n, \quad \forall\ \mathbb{i}. \tag{8}$$

Lemma 2 establishes the primitive of computational wire, similar to Theorem 1 in Ref. [3]. The Hilbert space on which the tensor components $\mathcal{A}_\Phi(\mathbb{i})$ act is the so-called virtual space, which decomposes into a "logical subsystem" and "junk subsystem" [3]. Upon measurement in the $X$ basis of all spins in a block, the logical subsystem is acted on by the operators $\mathcal{C}(\mathbb{i})$, which are uniform across the cluster phase. Conversely, the operators $\mathcal{B}_\Phi(\mathbb{i})$ acting on the junk space vary uncontrollably across the phase. Thus, to achieve computation, the logical subspace is used to encode and process information. The operators $\mathcal{C}(\mathbb{i})$ become the usual outcome-dependent byproduct operators of MBQC. They are of computational use, as described below under the "quantum gates" heading.

Two points are worth noting: one technical, and one physical. (i) With Lemma 2, we have mapped the original two-dimensional system to an effectively one-dimensional system composed of blocks. A wealth of techniques established for 1D SPT order thereby becomes available [3–6], [10–12,24]. (ii) The blocking notwithstanding, the basis $\{|\mathfrak{i}\rangle\}$ in which Eq. (8) holds is local *at the level of individual spins*, and not only at block level. (It is the local $X$ eigenbasis.) Because MBQC uses 1-spin local measurements, we require this stronger notion of locality.

Finally, we explain why Lemma 2 is a consequence of the symmetries of the local tensors $A_\Phi$ in the cluster phase. The local symmetries [Eq. (2)] can be combined in such a way that they map Pauli operators on the virtual logical register one column farther to the right,




(9)

for all $l$. (The tensor factors "$I$" for the action of the symmetries on the junk systems have been omitted.) Iterating these propagation relations $n$ times ($n$ is the circumference of the torus), we find that, upon measurement of the physical qubits in the local $X$ basis, each virtual local Pauli operator $Z$ is mapped onto itself up to sign. See Fig. 1 for illustration ($n = 6$ is shown). The same is true for Pauli operators $X$, cf. Fig. 4 in the SM [22]. Thus, every virtual Pauli operator is mapped to itself up to sign after one clock cycle of duration $n$. Therefore, the action of $\mathcal{A}_\Phi$ on the logical subsystem is indeed by Pauli operators, as stated by Lemma 2. As a technical remark, we note that the following construction requires that Lemma 2 holds also when $\mathcal{A}_\Phi$ is put into the so-called canonical MPS form [23]. Details of this condition as well as the proof of its veracity are given in the SM, Sec. III A [22].

Quantum gates: The subsequent construction significantly differs from the standard mapping to the circuit model [9]. Specifically, the technique of "cutting out coupled wires" by local $Z$ measurements is not available throughout the cluster phase, and it is therefore replaced.

As a first step, we observe that the byproduct operators $\mathcal{C}(\mathfrak{i})$ are of the form

$$\mathcal{C}(\mathfrak{i}) \sim \prod_{k\in K} \mathcal{C}[k]^{i_k}, \tag{10}$$

where "$\sim$" is equality up to phase, $K$ is the $n \times n$ block of spins, and $i_k$ is the measurement outcome at location $k$.

Equation (10) means that every site $k$ in the block has its own byproduct operator $\mathcal{C}[k]$. This is known to hold for the cluster state [9] and, by Lemma 2, it extends to the entire cluster phase.

Next, we find the precise form of the byproduct operators $\mathcal{C}[k]$ for certain sites $k \in K$. Namely, for the sites of $k = (1,l)$, $(2,l)$, and $(n,l)$ in the first, second, and last columns of each block, the operators $\mathcal{C}[k]$ are

$$\begin{aligned} \mathcal{C}[(1,l)] &= Z_l, \\ \mathcal{C}[(2,l)] &= Z_{l-1}X_lZ_{l+1}, \\ \mathcal{C}[(n,l)] &= X_l. \end{aligned} \tag{11}$$

They can be understood as follows. For the last column in the block, $n$, the operator $\mathcal{C}[(n,l)]$ is the standard byproduct operator for cluster states. By Lemma 2, it holds in the entire cluster phase. (See the SM, Sec. III B [22], for the result in canonical form.) The $\mathcal{C}[(r,l)]$ for earlier columns $r$ are also $X$ operators, inserted at position $(r,l)$. They are then propagated forward to the right boundary of the block using Eq. (9), resulting in Eq. (11).

If the resource is a 2D cluster state, which is the special state in the phase of interest, then on-site measurements in the $X/Y$ plane of the Bloch sphere are universal [25]. Because of the product form of the byproduct operators [Eq. (10)], every local measurement implements one logical gate. Suppose the measurement at site $k$ is in the basis spanned by $|0,\alpha\rangle_k = \cos(\alpha)|0\rangle_k - i\sin(\alpha)|1\rangle_k$ and $|1,\alpha\rangle_k = -i\sin(\alpha)|0\rangle_k + \cos(\alpha)|1\rangle_k$, with $|0\rangle$ and $|1\rangle$ referring to eigenstates of $X$. The resulting gate is $U_\alpha(i_k) = {}_k\langle i_k,\alpha|0\rangle_k I + {}_k\langle i_k,\alpha|1\rangle_k \mathcal{C}[k]$, hence

$$U_\alpha(i_k) = \mathcal{C}[k]^{i_k}\exp(i\alpha\mathcal{C}[k]).$$

Here, the operators $\mathcal{C}[k]$ of Eq. (8) become computational tools because they specify the unitary gate implemented. The outcome-dependent byproduct operator can be compensated for by classical side processing and adaptive measurement bases [9]. With Eq. (11), the gate set

$$\mathcal{U} = \{e^{i\alpha Z_{l-1}X_lZ_{l+1}}, e^{i\alpha Z_l}, e^{i\alpha X_l}, \quad \forall\ \alpha \in \mathbb{R}\} \tag{12}$$

can be realized. $\mathcal{U}$ is a universal set [26]; also see Sec. IV B of the SM [22].

When moving away from the cluster state into the cluster phase, nontrivial tensors $\mathcal{B}_\Phi$ appear; and measurement in a local basis away from the symmetry-respecting $X$ basis becomes nontrivial. If unaccounted for, the logical subsystem becomes entangled with the junk subsystem through such measurement [3], which introduces decoherence into the logical processing. However, this undesirable effect can be prevented by the techniques of Ref. [6]. By virtue of Lemma 2, we mapped to a quasi-1D setting to which we can apply Theorem 2 of Ref. [6]. (The essentials of Ref. [6] are reviewed in Sec. IV A of the SM [22].) As a result, the universal gate set $\mathcal{U}$ can be implemented in the whole cluster phase, and not only on the cluster state.

To summarize, the argument for computational universality of the 2D cluster phase splits into two parts. First, we have shown in Proposition 1 that all ground states in the 2D cluster phase are clusterlike, i.e., they satisfy the symmetry constraints [Eq. (2)]. Second, by mapping to a quasi-one-dimensional system, we showed that the symmetries [Eq. (2)] lead to universal computational power. Taken together, these two results yield Theorem 1.

*Conclusion*.—We have described the first symmetry protected topological phase in which every ground state (up to a possible set of measure 0) has universal power for measurement-based quantum computation. Our phase is protected by symmetries acting on a lower-dimensional subsystem, and it is associated with a set of local symmetries of tensor networks; see Eq. (2). These symmetries are sufficient to guarantee computational universality of the corresponding tensor network. What implications these symmetries have on the physics of this phase and others like it remains an interesting question.

As for the implications on the computational side, we ask the following: Can the computational power of quantum phases of matter be classified? In the spirit of this question, we conclude with three more specific ones: (i) How broadly can the present construction be generalized? (ii) The line-like symmetries we consider are neither global symmetries, which are typically used to define SPT phases, nor are they local like in a lattice gauge theory. Indeed, they are more closely related to the “higher-form” symmetries considered in Refs. [27–29], which act on lower-dimensional submanifolds of the whole lattice. Is this type of symmetry necessary for a computationally universal phase, or can other structurally different symmetries lead to similar results? (iii) As one-dimensional computational phases [5,6] build on symmetry protected computational wire [3], the present construction builds on a symmetry protected quantum cellular automaton. In particular, Eq. (9) defines the transition function of a quantum cellular automaton. Quantum cellular automata have been classified [30–33]. What is the relation between this classification and computational phases of quantum matter?

This work is supported by the NSERC (C. O., D.-S. W., D. T. S., R. R.), Cifar (R. R.), the Stewart Blusson Quantum Matter Institute (C. O. and D.-S. W.), ERC Grant WASCOSYS, Grant No. 636201 (D. T. S.), and the Austrian Science Fund FWF within the DK-ALM: W1259-N27 (H. P. N.). D.-S. W. thanks Z. C. Gu for discussions.

D. T. S. and H. P. N. contributed equally to this work.

[1] A. C. Doherty and S. D. Bartlett, Phys. Rev. Lett. **103**, 020506 (2009).
[2] A. Miyake, Phys. Rev. Lett. **105**, 040501 (2010).
[3] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty, Phys. Rev. Lett. **108**, 240505 (2012).
[4] J. Miller and A. Miyake, Phys. Rev. Lett. **114**, 120506 (2015).
[5] D. T. Stephen, D.-S. Wang, A. Prakash, T.-C. Wei, and R. Raussendorf, Phys. Rev. Lett. **119**, 010504 (2017).
[6] R. Raussendorf, D.-S. Wang, A. Prakash, T.-C. Wei, and D. T. Stephen, Phys. Rev. A **96**, 012302 (2017).
[7] A. S. Darmawan, G. K. Brennen, and S. D. Bartlett, New J. Phys. **14**, 013023 (2012).
[8] C.-Y. Huang, M. A. Wagner, and T.-C. Wei, Phys. Rev. B **94**, 165130 (2016).
[9] R. Raussendorf and H.-J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[10] Z. C. Gu and X. G. Wen, Phys. Rev. B **80**, 155131 (2009).
[11] X. Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B **82**, 155138 (2010).
[12] X. Chen, Z. C. Gu, Z. X. Liu, and X. G. Wen, Phys. Rev. B **87**, 155114 (2013).
[13] H. Niggemann, A. Klümper, and J. Zittartz, Z. Phys. B **104**, 103 (1997).
[14] T.-C. Wei and C.-Y. Huang, Phys. Rev. A **96**, 032317 (2017).
[15] H. Poulsen Nautrup and T. C. Wei, Phys. Rev. A **92**, 052309 (2015).
[16] J. Miller and A. Miyake, npj Quantum Inf. **2**, 16036 (2016).
[17] Y. Chen, A. Prakash, and T. C. Wei, Phys. Rev. A **97**, 022305 (2018).
[18] J. Miller and A. Miyake, Phys. Rev. Lett. **120**, 170503 (2018).
[19] D. V. Else, S. D. Bartlett, and A. C. Doherty, New J. Phys. **14**, 113016 (2012).
[20] F. Verstraete, J. I. Cirac, and V. Murg, Adv. Phys. **57**, 143 (2008).
[21] Y. Huang and X. Chen, Phys. Rev. B **91**, 195143 (2015).
[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.090501 for Lemma 3 and review of MBQC in 1D SPT phases.
[23] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum Inf. Comput. **7**, 401 (2007).
[24] D. Gross and J. Eisert, Phys. Rev. Lett. **98**, 220503 (2007).
[25] A. Mantri, T. F. Demarie, and J. F. Fitzsimons, Sci Rep. **7**, 42861 (2017).
[26] R. Raussendorf, Phys. Rev. A **72**, 052301 (2005).
[27] B. Yoshida, Phys. Rev. B **93**, 155131 (2016).
[28] S. Roberts, B. Yoshida, A. Kubica, and S. D. Bartlett, Phys. Rev. A **96**, 022306 (2017).
[29] A. Kapustin and R. Thorngren, *Algebra, Geometry, and Physics in the 21st Century*, Progress in Mathematics Vol. 324 (Birkhäuser Verlag, Cham, 2017), pp. 177–202.
[30] D. M. Schlingemann, H. Vogts, and R. F. Werner, J. Math. Phys. **49**, 112104 (2008).
[31] D. Gross, V. Nesme, H. Vogts, and R. F. Werner, Commun. Math. Phys. **310**, 419 (2012).
[32] C. Cedzich, T. Geib, F. A. Grünbaum, C. Stahl, L. Velazquez, A. H. Werner, and R. F. Werner, Ann. Henri Poincaré **19**, 325 (2018).
[33] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, J. Stat. Mech. (2017) 083105.