Typicality of Prethermalization

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(Received 11 September 2018; revised manuscript received 13 December 2018; published 1 March 2019)

Prethermalization refers to the remarkable relaxation behavior which an integrable many-body system in the presence of a weak integrability-breaking perturbation may exhibit: After initial transients have died out, it stays for a long time close to some nonthermal steady state, but on even much larger time scales, it ultimately switches over to the proper thermal equilibrium behavior. By extending Deutsch's conceptual framework from Phys. Rev. A **43**, 2046 (1991), we analytically predict that prethermalization is a typical feature for a very general class of such weakly perturbed systems.

DOI: 10.1103/PhysRevLett.122.080603

Isolated many-body quantum systems are known to equilibrate; i.e., expectation values exhibit an initial relaxation and then spend most of their time close to a constant value, provided some rather weak preconditions are fulfilled. Furthermore, thermalization is expected for so-called nonintegrable systems; i.e., the long-time behavior is well approximated by a microcanonical ensemble. (Possible exceptions, e.g., due to many-body localization, are tacitly ignored here.) In contrast, integrable systems usually exhibit quite significant deviations from such a thermal long-time behavior. All these issues have been extensively explored in the literature, as reviewed, among others, in Refs. [1–4]. They are not the subject of our present Letter but, rather, will be taken for granted.

Our main issue is the question of how the temporal relaxation of an integrable system changes in response to a weak integrability-breaking perturbation. More specifically, we will derive a rigorous bound for the difference between unperturbed and perturbed expectation values, implying that those changes remain, over a long period of time, negligibly small for a very large class of weak perturbations. Our approach is conceptually akin to Deutsch's seminal work on thermalization, treating the perturbations along the lines of random matrix theory [5]. In particular, we will exploit Deutsch's result concerning the ultimate thermalization of the perturbed systems. With respect to the unperturbed (integrable) system, moreover, we will take for granted that its initial relaxation is not extremely slow, and that it exhibits clearly observable deviations from a thermal long-time behavior. Thus, altogether, we are left with a very large class of perturbations with the following quite remarkable property, henceforth named prethermalization: Initially, the perturbed system closely follows the unperturbed relaxation towards a nonthermal steady state, but on even much larger time scales, there must be a clearly visible transition to the ultimate thermal behavior.

Originally, the term prethermalization was introduced by Berges, Borsányi, and Wetterich [6] for matter under extreme conditions in a quasisteady state far from equilibrium, which, nevertheless, exhibits some genuine thermal properties, however, without any reference to the concept of integrability. Our present somewhat different notion of prethermalization has been independently established by Moeckel and Kehrein in Ref. [7]. During recent years, these, and further slightly differing guises of prethermalization, have been explored in numerous theoretical [8–11], as well as experimental [12], investigations, see also the recent reviews [4,13] and further references therein.

Incidentally, the particular examples in Moeckel and Kehrein's original work [7] and, also, in some subsequent studies [9] are beyond the above mentioned realm of our present approach: If the unperturbed system is initially at thermal equilibrium or in the energy ground state, as is the case in [7,9], then the unperturbed dynamics is trivial, and also, the signatures of prethermalization after adding a weak perturbation remain too small for our purposes.

Against our treating the perturbations as random matrices (in the unperturbed energy basis), one might object that the "true" perturbation in any concrete physical model is not a random matrix. In particular, the true matrix is often banded [5,14–16]; i.e., the typical magnitude of its entries decreases with increasing distance from the matrix diagonal. Furthermore, for noninteracting systems perturbed by few-body interactions, the matrix will be very sparse; i.e., only a small fraction of its entries is nonzero [3,16–18].

To overcome these concerns, we will consider ensembles of random matrices which can be tailored to emulate the basic features of many concrete models, such as sparsity, bandedness, and other statistical characteristics [3,5,14–18]. Thus, the true perturbation is expected to be contained as one specific matrix in such a properly tailored ensemble as well. (For simplicity, one may imagine matrices of large but finite dimension, whose entries can assume only a finite number of different possible values, as is the case in any

numerical investigation. If each possible value has nonzero probability, there is a finite chance of sampling the true matrix from the ensemble.) Hence, if one could prove that some property applies to all members of the ensemble, the property would also apply to the true model. Our main result consists in a slightly weaker statement, namely, that the property "prethermalization" at least applies with overwhelming probability when randomly sampling perturbations from the ensemble ("typicality of prethermalization"). Therefore, it is still very reasonable to expect that the true model is not one of the extremely unlikely exceptions. An illustrative example (spin chain model) is provided in the Supplemental Material [19]. Analogous arguments are routinely adopted in random matrix theory, which is well known to be extremely successful in practice [3,18], though its applicability has, to our knowledge, not been rigorously justified in any concrete physical example. Similar considerations also apply to many other "nonsystematic" but practically very well established approximations, such as density functional theory or Boltzmann equations beyond the validity limits of their derivation.

We will demonstrate typicality of prethermalization for a great variety of different ensembles. The resulting total set of all admitted perturbations is, therefore, extremely large. This seems to us a quite noteworthy finding in itself, independent of the question whether some particular model is covered or not. Moreover, to actually exclude some particular model, it would have to be untypical with respect to every one of those various ensembles. Finally, we remark that most applications of random matrix theory focus on the ensemble-averaged behavior and take for granted that most individual matrices behave very similarly to the average [3,18]. In our present approach, no such extra assumption will be needed.

The unperturbed system is described by a Hamiltonian H_0 with eigenvalues E_n^0 and eigenvectors $|n\rangle_0$. The unperturbed evolution of an arbitrary initial state $\rho(0)$ can, thus, be written as $\rho_0(t) = e^{-iH_0t/\hbar}\rho(0)e^{iH_0t/\hbar}$ and the expectation value of any given observable A as

$$\mathcal{A}_0(t) := \mathrm{Tr}\{\rho_0(t)A\} = \sum_{mn}\rho^0_{mn}(0)A^0_{nm}e^{i\frac{E^0_n-E^0_m}{\hbar}t}, \quad (1)$$

$$\rho^0_{mn}(t) := {}_0\langle m|\rho_0(t)|n\rangle_0, \qquad A^0_{nm} := {}_0\langle n|A|m\rangle_0, \quad (2)$$

where, depending on the specific system under consideration, the indices m and n run from 1 to infinity or to some finite upper limit.

Likewise, the perturbed system

$$H = H_0 + V \quad (3)$$

exhibits eigenvalues E_n and eigenvectors $|n\rangle$. Focusing on the same initial state $\rho(0)$ as before, the expectation value $\mathcal{A}(t)$ under the perturbed dynamics is then given by the same formulas as in (1) and (2), except that all indices "0" must be omitted. In terms of the unitary basis transformation matrix

$$U_{mn} := \langle m|n\rangle_0, \quad (4)$$

this expectation value can be further rewritten as

$$\mathcal{A}(t) = \sum_{mn}\sum_{\mu\nu\sigma\tau}U_{m\mu}U^*_{n\nu}U_{n\sigma}U^*_{m\tau}\rho^0_{\mu\nu}(0)A^0_{\sigma\tau}e^{i\frac{E_n-E_m}{\hbar}t}. \quad (5)$$

The quantity of foremost interest is the difference

$$\Delta(t) := \mathcal{A}(t) - \mathcal{A}_0(t) \quad (6)$$

between the perturbed and the unperturbed expectation values. Taking into account $\rho^0_{mn}(0)e^{i(E^0_n-E^0_m)t/\hbar} = \rho^0_{mn}(t)$ [see above (1)], it follows, with (1) and (5), that

$$\Delta(t) = \sum_{\mu\nu\sigma\tau}\rho^0_{\mu\nu}(t)A^0_{\sigma\tau}[\gamma_{\tau\mu}(t)\gamma^*_{\sigma\nu}(t) - \delta_{\tau\mu}\delta_{\sigma\nu}], \quad (7)$$

$$\gamma_{\tau\mu}(t) := \sum_m U^*_{m\tau}U_{m\mu}e^{i(E^0_\mu-E_m)t/\hbar}, \quad (8)$$

where δ_{mn} is the Kronecker delta.

Finally, instead of one particular perturbation V in (3), we consider a statistical ensemble of different V's, and we indicate averages over the ensemble by an overline. This randomization of V is inherited by the Hamiltonian H in (3) and, thus, by the eigenvalues E_n, the eigenvectors $|n\rangle$, the U_{mn} in (4), and the $\gamma_{\tau\mu}(t)$ in (8). On the other hand, H_0, $\rho(0)$, and A are considered as arbitrary but fixed (nonrandom); hence, the same must apply to E_n^0, $|n\rangle_0$, $\rho_0(t)$, and to the matrix elements in (2).

The first main result of our Letter consists in the general rigorous bound

$$\overline{|\Delta(t)|} \leq \frac{\Delta_A}{2}f(t), \quad (9)$$

$$f(t) := 3\sqrt{1-Y(t)} + \sqrt{1-Y(t)+W(t)}, \quad (10)$$

$$Y(t) := \sum_{\mu\nu\sigma}\rho^0_{\mu\nu}(t)\overline{\gamma_{\sigma\mu}(t)}[\overline{\gamma_{\sigma\nu}(t)}]^*, \quad (11)$$

$$W(t) := 4Y(t) - [Z(t) + Z^*(t)]^2, \quad (12)$$

$$Z(t) := \sum_{\mu\nu}\rho^0_{\mu\nu}(t)\overline{\gamma_{\nu\mu}(t)}, \quad (13)$$

where Δ_A is the measurement range of A (largest minus smallest eigenvalue). The quite tedious derivation has been relegated to the Supplemental Material [19].

Applying Markov's inequality to (9), it follows for any $\epsilon > 0$ that

$$\mathrm{Prob}(|\Delta(t)| \leq \epsilon\Delta_A) \geq 1 - f(t)/2\epsilon, \quad (14)$$

where the left hand side denotes the probability that $|\Delta(t)| \leq \epsilon\Delta_A$ when randomly sampling perturbations V. For sufficiently small $f(t)$, the difference $\Delta(t)$ in (6) will, thus, be negligible for the vast majority of all V's.

Our first assumption regarding the so far arbitrary ensemble of V's is as follows: Multiplying the unperturbed energy eigenvectors $|n\rangle_0$ by arbitrary factors $\sigma_n \in \{\pm 1\}$ leaves the V ensemble invariant. Hence, the statistical properties of $\gamma_{\mu\nu}$ in (18) also remain unchanged if all the matrix elements U_{mn} in (4) are multiplied by arbitrary factors $\sigma_n \in \{\pm 1\}$. As a consequence (see also [19]), the ensemble average of (8) must vanish unless $\tau = \mu$,

$$\overline{\gamma_{\tau\mu}(t)} = \delta_{\tau\mu}\overline{g_\mu(t)}, \tag{15}$$

$$g_\mu(t) := \sum_m |U_{m\mu}|^2 e^{i(E_\mu^0 - E_m)t/\hbar}. \tag{16}$$

To justify this assumption, we note that randomly flipping the signs of the $|n\rangle_0$ leaves all physical properties unchanged but randomizes the signs of the true perturbation matrix elements V_{mn}^0. Hence, it is appropriate to adopt a random matrix model with the above invariance property.

As stated in the introduction, the unperturbed system is assumed to exhibit equilibration but not thermalization. Implicitly, this requires a macroscopically well defined system energy; i.e., there must exist a microcanonical (MC) energy interval $I_{\mathrm{MC}} := [E - \delta E, E]$ so that only energies $E_n^0 \in I_{\mathrm{MC}}$ exhibit nonnegligible level populations $\rho_{nn}^0(0)$. The number of energies E_n^0 contained in I_{MC} is denoted by N and, without loss of generality, we assume that $n \in \{1, \ldots, N\}$ for all those E_n^0's. Furthermore, whenever $E_n^0 \notin I_{\mathrm{MC}}$, we adopt the idealization that $\rho_{nn}^0(0)$ is strictly zero [21]. The Cauchy-Schwarz inequality $|\rho_{mn}^0(0)|^2 \leq \rho_{mm}^0(0)\rho_{nn}^0(0)$ then implies that, in (1), only summands with $m, n \in \{1, \ldots, N\}$ actually contribute. As usual, we take for granted that N is huge (exponentially large in the system's degrees of freedom [22]), while the local level density remains close to $D := \delta E/N$ throughout the interval I_{MC}.

Given that only indices $m, n \in \{1, \ldots, N\}$ actually matter in (1), we can and will assume that their range is extended to arbitrary integer values and that the energies E_n^0 and the matrix elements $V_{mn}^0 := {}_0\langle m|V|n\rangle_0$ are (re-)defined for arbitrary integers $m, n \notin \{1, \ldots, N\}$ by way of "extrapolating" in a physically natural way their properties for $m, n \in \{1, \ldots, N\}$.

As a first example, we consider the particularly simple case that $E_{n+1}^0 - E_n^0 = D$ for all n, and that the statistical properties of the matrix elements V_{mn}^0 do not depend separately on m and n, but only on the difference $m - n$. As a consequence (see also [19]), the statistical properties of (16) remain invariant when simultaneously adding an arbitrary integer ν to all indices on the right hand side (but not on the left hand side). Thus, upon averaging, one can infer that $\overline{g_\mu(t)} = \overline{g_{\mu+\nu}(t)}$, hence,

$$g(t) := \overline{g_\mu(t)} \tag{17}$$

is a well-defined (μ-independent) function.

Under the additional assumption that all statistical properties of the diagonal matrix elements V_{nn}^0 are identical to those of $-V_{nn}^0$, one can finally show [19] that

$$g(t) = [g(t)]^*. \tag{18}$$

To justify this assumption, we note that the diagonal elements of the true V in (3) can always be readjusted to vanish on the average. A symmetrization procedure for the remaining distribution will be provided later.

Introducing (15)–(18) into (11)–(13), and taking into account that $\sum_\nu \rho_{\nu\nu}^0 = 1$ yields $Y(t) = [g(t)]^2$, $Z(t) = g(t)$, and $W(t) = 0$; hence, (10) takes the form

$$f(t) = 4\sqrt{1 - [g(t)]^2}. \tag{19}$$

One readily infers from (4), (16), and (17) that $g(0) = 1$ and that $|g(t)| \leq 1$ for all t. Furthermore, it is convenient to rewrite (16) as

$$g_\mu(t) = \int dE\; h_\mu(E) e^{-iEt/\hbar}, \tag{20}$$

$$h_\mu(E) := \sum_m |U_{m\mu}|^2 \delta(E - E_m + E_\mu^0). \tag{21}$$

The quantity $F_\mu(E) := h_\mu(E - E_\mu^0)$ plays a key role in random matrix theory under the name strength function or local spectral density of states [3,16]. Specifically, one finds that the ensemble average $\overline{h_\mu(E)}$ is very well approximated by the Breit-Wigner (BW) distribution

$$h_{\mathrm{BW}}(E) := \frac{1}{2\pi}\frac{\Gamma}{E^2 + \Gamma^2/4}, \tag{22}$$

under conditions which, together with the concomitant definition of Γ, will be discussed in more detail shortly. Introducing this result into (20) yields $\overline{g_\mu(t)} = e^{-\Gamma|t|/2\hbar}$, and with (17), (19) we obtain

$$f(t) = 4\sqrt{1 - e^{-\Gamma|t|/\hbar}} \leq 4\sqrt{\Gamma|t|/\hbar}. \tag{23}$$

Equations (14) and (23) represent our main results. In the remainder of the Letter, we focus—as usual in random matrix theory [3,16,18]—on the case that all V_{mn}^0 with $m \geq n$ are statistically independent of each other (those with $m < n$ follow from $V_{nm}^0 = [V_{mn}^0]^*$), that the statistics only depends on $m - n$ [see above (17)], and that V_{mn}^0 and $-V_{mn}^0$ are equally likely [see above Eqs. (15), (18) and [19]].

If all V_{mn}^0 are, furthermore, real and Gaussian distributed with variance σ_v^2, the result (22) with

$$\Gamma := 2\pi\sigma_v^2/D \tag{24}$$

was obtained by Deutsch [5]. Substantial generalizations have been worked out by Fyodorov *et al.* in Refs. [16], including distributions with a pronounced delta peak at zero, corresponding to sparse random matrices V^0_{mn}. In addition, they also admitted the possibility of banded matrices [23]. We have further extended their analytical supersymmetry approach and, moreover, performed extensive numerical explorations, showing that the key results (22)–(24) also remain valid for complex V^0_{mn}'s and under still considerably weaker assumptions regarding their statistics. A few illustrative examples are provided in the Supplemental Material [19].

Multiplying V in (3) by an extra factor λ (coupling strength) entails a factor λ^2 in (24); hence, the characteristic time scale in (23) decreases as λ^{-2}, in agreement with previous findings for the persistence of the prethermalized state [4,11]. However, strictly speaking, we note that our inequality (9) admits no conclusions regarding the actual appearance of nonsmall differences in (6).

In order to abandon the requirement of equally spaced energies E^0_n [see above Eq. (17)], let us consider an unperturbed Hamiltonian $\tilde{H}_0$ with the same eigenvectors $|n\rangle_0$ as the original H_0, but with modified energies $\tilde{E}^0_n = E^0_n + \epsilon_n$. In view of (1), one anticipates that the corresponding expectation value $\tilde{\mathcal{A}}_0(t)$ still remains close to $\mathcal{A}_0(t)$ for sufficiently small ϵ_n and not too large t. Indeed, it can be rigorously shown [19] that

$$|\tilde{\mathcal{A}}_0(t) - \mathcal{A}_0(t)| \leq \Delta_A |t| \max_{1\leq n\leq N} |\epsilon_n|/\hbar. \tag{25}$$

Taking for granted that the unperturbed Hamiltonian $\tilde{H}_0$ exhibits equilibration but not thermalization (see beginning of the Letter), we denote its relaxation time by t_{rel}; i.e., $\tilde{\mathcal{A}}_0(t)$ remains very close to some (nonthermal) equilibrium value $\mathcal{A}_{\mathrm{eq}}$ for (almost) all $t \geq t_{\mathrm{rel}}$. It follows with (25) that $\mathcal{A}_0(t)$ also exhibits practically the same initial relaxation behavior and then remains close to $\mathcal{A}_{\mathrm{eq}}$ for quite some time, provided $|\epsilon_n| \ll \hbar/t_{\mathrm{rel}}$ for all $n = 1, \ldots, N$. Recalling that the energy level density is exponentially large in the degrees of freedom [22], these conclusions must actually apply to rather general nonequidistant energies $\tilde{E}^0_n$.

Returning to our perturbed systems of the form (3), where the considered ensemble of V's satisfies the rather weak assumptions mentioned above, we can, thus, conclude from (6), (14), (23), and (24) that the perturbed expectation values $\mathcal{A}(t)$ also exhibit an initial relaxation and then remain close to $\mathcal{A}_{\mathrm{eq}}$ for quite some time [24], at least for the vast majority of perturbations V, and provided they are sufficiently weak so that

$$\sigma_v^2 \ll \frac{1}{32\pi}\frac{\hbar D}{t_{\mathrm{rel}}}. \tag{26}$$

On the other hand, ultimate thermalization for most such H's in (3) has been established in Refs. [5,26]. Thus, recalling the considerations at the beginning of our Letter, all those "typical" H's exhibit prethermalization.

Finally, upon defining modified perturbations $\tilde{V}$ via $\tilde{V}^0_{mn} := V^0_{mn} - \delta_{mn}\epsilon_n$, we can conclude, with Eq. (3), that $\tilde{H}_0 + \tilde{V} = H$. Hence, the vast majority of those perturbations $\tilde{V}$ of $\tilde{H}_0$ must, again, entail prethermalization. In doing so, the ϵ_n's are often expected to be so small that the resulting ensemble of $\tilde{V}$'s is almost identical to the original ensemble of V's (see below). More generally, since the modified energies $\tilde{E}^0_n$ need no longer be ordered by magnitude, even substantially more general ensembles of $\tilde{V}$'s than of V's are actually admitted, see also [19]. Along similar lines, a possible asymmetry of the V^0_{nn} distribution can also be removed, as announced below (18).

Next, we turn to the question of how far perturbations which satisfy (26) are "weak" in some physically meaningful sense. Quite obviously, such considerations are only possible in terms of nonrigorous arguments and rough estimates.

First of all, typicality of thermalization, as invoked below (26), trivially fails for vanishing perturbations and, hence, may possibly still fail for extremely weak perturbations [2,27]. Yet, a closer inspection of the nonperturbative approach from Refs. [5,26] suggests [28] that typicality of thermalization generally does apply provided $\Gamma \gg D$ [cf. Eq. (24)], and thus,

$$\sigma_v \gg D. \tag{27}$$

In particular, the diagonal matrix elements V^0_{nn} are then typically much larger than the level spacing D, thus, corroborating the claim below (26) that the ensembles of $\tilde{V}$'s and of V's are often quite similar.

Second, while the unperturbed system $\tilde{H}_0$ is assumed not to thermalize for the given initial condition $\rho(0)$, one still expects that it exhibits the usual thermodynamic properties when the system state happens to be the microcanonical ensemble $\rho^0_{\mathrm{MC}} := N^{-1}\sum_{n=1}^{N} |n\rangle_{00}\langle n|$ corresponding to the energy window $I_{\mathrm{MC}} := [E - \delta E, E]$ introduced below Eq. (16). Denoting by $\Omega(E)$ the number of energy levels $\tilde{E}^0_n$ below E, by k_B Boltzmann's constant, and by $S(E) := k_B \ln[\Omega(E)]$ the entropy, the temperature is, thus, given by $T(E) := 1/S'(E)$. Moreover, δE must not exceed $k_B T(E)$, otherwise, the level density would no longer be (approximately) constant throughout I_{MC} [as assumed below Eq. (16)]. Now, it seems reasonable to say that a perturbation is weak if it does not notably change the thermal equilibrium properties [$S(E)$, $T(E)$, heat capacity, state of matter, etc.] of the unperturbed system. Closer inspection of the approach from Refs. [5,26] implies that the perturbations are weak in this sense as long as $\Gamma \ll k_B T(E)$ (otherwise, regions with different level densities start to "interact" via the perturbation). According to (24), this amounts to $\sigma_v^2 \ll k_B T(E) D$. Focusing on the special (largest possible) choice $\delta E = k_B T(E)$ and exploiting

$D := \delta E/N$, we arrive at $\sigma_v \ll \sqrt{N}D$ and $\sigma_v^2 \ll \delta E\ D$. The first relation complements the lower bound from (27). Thus, since N is exponentially large in the degrees of freedom, the range of admitted σ_v values is still very large. The second relation agrees with (26) if t_{rel} is comparable to $\hbar/\delta E$. As shown in Ref. [29], this is, indeed, the case for a quite large class of Hamiltonians $\tilde{H}_0$, observables A, and initial conditions $\rho(0)$.

Alternatively, a perturbation may be considered as weak if the perturbed expectation value $\mathcal{A}(t)$ remains for (almost) all sufficiently large times t close to the expectation value $\mathrm{Tr}\{\rho_{\mathrm{MC}}^0 A\}$, which the unperturbed system would assume in thermal equilibrium. By similar arguments as above, one can see that this alternative weak perturbation criterion is essentially equivalent to the one from the previous paragraph *and* the condition (27).

Thus, altogether, Eq. (26) seems to be a physically very natural weak perturbation condition, and it appears reasonable to conjecture that prethermalization will, in general, be ruled out if (26) is violated. We plan to further pursue this issue in our future work.

In summary, prethermalization has been established for a very large class of integrable (nonthermalizing) Hamiltonians $\tilde{H}_0$ and weak perturbations $\tilde{V}$, which closely imitate the essential features of many particular examples of interest in this context. Thus, adopting the common lore of random matrix theory [3,5,16,18], the same conclusion is also expected to apply "typically" or "with overwhelming likelihood" to any such given example, unless there is some *a priori* reason (inappropriate choice of the ensemble, another nonthermalizing system very nearby, etc.) why the specific example at hand must be one of the very rare exceptions with respect to every admitted $\tilde{V}$ ensemble [22,25,26,29,30]. Remarkably, the same predictions also apply to any other $\tilde{H}_0$ which exhibits equilibration but not thermalization, for instance due to many-body localization effects [1,31,32].

This work was supported by the Deutsche Forschungsgemeinschaft (DFG) under Grant No. RE 1344/10-1 and within the Research Unit FOR 2692 under Grant No. 397303734.

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