Resource Quantification for the No-Programing Theorem

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(Received 7 June 2018; published 28 February 2019)

The no-programing theorem prohibits the existence of a universal programmable quantum processor. This statement has several implications in relation to quantum computation but also to other tasks of quantum information processing, making this construction a central notion in this context. Nonetheless, it is well known that, even when the strict model is not implementable, it is possible to conceive of it in an approximate sense. Unfortunately, the minimal resources necessary for this aim are still not completely understood. Here, we investigate quantitative statements of the theorem, improving exponentially previous bounds on the resources required by such a hypothetical machine. The proofs exploit a new connection between quantum channels and embeddings between Banach spaces which allows us to use classical tools from geometric Banach space theory in a clean and simple way.

DOI: 10.1103/PhysRevLett.122.080505

Since the early days of quantum information theory, no-go theorems have served as a guideline in the search of a deeper understanding of quantum theory as well as for the development of applications of quantum mechanics to cryptography and computation. They shed light on those aspects of quantum information which make it so different from its classical counterpart. Some renowned examples are the no-cloning [1], no-deleting [2], and no-programing [3] theorems.

The no-programing theorem concerns the so-called universal programmable quantum processor (UPQP) [4]. A UPQP is a universal machine able to perform any quantum operation on an arbitrary input state of fixed size, programing the desired action in a quantum register inside the machine (a quantum memory). It can be understood as the quantum version of a stored-program computer. For the sake of simplicity, we will consider the programmability of unitary operations, although this is not really a restrictive assumption [5]. With this figure of merit, the no-programing theorem is stated as the nonexistence of a UPQP using finite-dimensional resources. The key observation made in Ref. [3] is that in order to program two different unitaries we need two orthogonal program states. Then, the infinite cardinality of the set of unitary operators, even in the case of a qubit, leads immediately to the requirement of an infinite-dimensional memory. Similar consequences follow for the related concept of universal programmable quantum measurements [6–8], which are machines with the capability to be programmed to implement arbitrary quantum measurements.

From a conceptual point of view, the no-programing theorem points out severe limitations in how universal quantum computation can be conceived. However, these limitations can be surpassed by relaxing the requirements on the model of UPQP. In particular, one can consider programmable devices working noisily or probabilistically. Indeed, in the past two decades, several proposals of such approximate UPQPs have appeared in the literature [3,9,10]. Thus, it is interesting to look for more quantitative statements about *quantum programmability*. To put it in explicit words, we worry here about the relation between the memory size of an approximate UPQP, m, and both the accuracy of the scheme, ε, and the size of the input register in which we want to implement the program, d. Despite their relevance, these relations are still poorly understood. Existing results are summarized in Table I.

In this Letter, we provide new upper and lower bounds which substantially clarify the ultimate resources required by approximate UPQPs. Indeed, the results in this work entail exponential improvements over previously known results. Our bounds show the optimal dependence of m with ε and d separately. In fact, the lower bound of Theorem 3 is nearly saturated for fixed ε by the performance of port-based teleportation, which was originally conceived as a UPQP [10]. On the other hand, we deduce an upper bound, (2), saturating almost optimally the scaling with ε of the bound from Ref. [11].

The proofs presented in this Letter are based on a connection with geometric functional analysis that we uncover. The use of techniques from this branch of functional analysis, in particular, from Banach space theory and operator spaces—as is the case in this work—have proven to be very fruitful in the study of different topics of quantum information theory such as entanglement theory, quantum nonlocality, and quantum channel theory (see [15,16], and references therein). We find the path to put forward this mathematical technology to the framework

TABLE I. Best known bounds for the optimal memory size of UPQPs in comparison with the results presented here. K denotes universal constants, not necessarily equal between them. Let us point out that the bound from Ref. [11] was deduced for programmable measurements instead of UPQPs. However, since a UPQP always can be turned into a universal programmable quantum measurement, this lower bound also applies for the case studied here. Notice that the alluded bound, although it enforces a strong scaling of m with ε, becomes trivial for large input dimension d. It is in this regime where the bound from Ref. [12] is more informative but still exponentially weaker than the bound provided by Theorem 3.

	Previous results		This work	
Lower bounds	$m \geq \mathsf{K}(1/d)^{(d+1)/2}(1/\varepsilon)^{(d-1)/2}$	[11]	$m \geq 2^{[(1-\varepsilon)/\mathsf{K}]d-(2/3)\log d}$	[Theorem 3]
	$m \geq \mathsf{K}(d/\varepsilon)^2$	[12]		
Upper bounds	$m \leq 2^{(4d^2 \log d)/\varepsilon^2}$	[10,13,14]	$m \leq (\mathsf{K}/\varepsilon)^{d^2}$	[Eq. (2)]

studied here. More precisely, we *characterize* UPQPs as isometric embeddings between concrete Banach spaces which are, in addition, complete contractions (considering some operator space structure). Once this characterization is established, the results about UPQPs are deduced by using classical tools from local Banach space theory in a simple and clean way. We think that the general ideas presented here and potential generalizations of them can provide further insights in other contexts related with quantum computation and cryptography.

In this main text, we limit ourselves to explain the results, stressing the ideas behind them [17].

Preliminaries.—Given a finite dimensional Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ or $\mathcal{S}_1(\mathcal{H})$ the space of (bounded) operators on $\mathcal{H}$ with the operator or the trace norm, respectively. We will also denote by $\mathcal{U}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ the subset of unitary and density operators. The set of quantum channels (that is, completely positive and trace preserving maps $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$) will be denoted by CPTP($\mathcal{H}$). In the particular case of unitary channels we will use the notation $\mathcal{C}_U(x) = UxU^\dagger$ for $x \in \mathcal{B}(\mathcal{H})$, $U \in \mathcal{U}(\mathcal{H})$. We will usually consider a d-dimensional complex Hilbert space $\mathcal{H}_d$ as the input state space and an ancillary m-dimensional complex Hilbert space $\mathcal{H}_m$ as the memory of the programmable device under consideration. When logarithms are used, they are generically considered in base 2.

Definition 1.—A quantum operation $\mathcal{P} \in \mathrm{CPTP}(\mathcal{H}_d \otimes \mathcal{H}_M)$ is a d-dimensional universal programmable quantum processor, UPQP$_d$, if for every $U \in \mathcal{U}(\mathcal{H}_d)$ there exists a unit vector $|\phi_U\rangle \in \mathcal{H}_m$ such that

$$\mathrm{Tr}_{\mathcal{H}_m}[\mathcal{P}(\rho \otimes |\phi_U\rangle\langle\phi_U|)] = U\rho U^\dagger, \quad \text{for every } \rho \in \mathcal{D}(\mathcal{H}_d).$$

Essentially, this is the concept of the universal quantum gate array introduced in Ref. [3] and whose impossibility is the content of the no-programing theorem discovered also there. As we said in the previous section, the no-programing theorem does not apply if one considers a relaxation of the previous definition, that is, in the case of approximate UPQPs. Two notions of approximate UPQPs have been considered in the literature: probabilistic settings [3,22], which implement exactly the desired unitary with some probability of failure, obtaining information about the success or failure of the procedure; and deterministic UPQPs [23], which always implement an operation which is close to the desired one. Notice that both notions are related, since probabilistic UPQPs can be also understood as deterministic ones by just ignoring the information about the success or failure of the computation. A natural way to express these notions of approximation is through the distance induced by the diamond norm [24]:

Definition 2.—We say that $\mathcal{P} \in \mathrm{CPTP}(\mathcal{H}_d \otimes \mathcal{H}_m)$ is a d-dimensional ε-universal programmable quantum processor, ε-UPQP$_d$, if for every $U \in \mathcal{U}(\mathcal{H}_d)$ there exists a unit vector $|\phi_U\rangle \in \mathcal{H}_m$ such that

$$\frac{1}{2}\|\mathrm{Tr}_{\mathcal{H}_m}[\mathcal{P}(\cdot \otimes |\phi_U\rangle\langle\phi_U|)] - \mathcal{C}_U(\cdot)\|_\diamond \leq \varepsilon,$$

where $\|\cdot\|_\diamond$ denotes the *diamond norm*.

Two relevant examples of approximate UPQPs are (i) the one built based on standard quantum teleportation [3], which can be understood as a probabilistic ε-UPQP$_d$ with $\varepsilon = 1 - (1/d^2)$ and memory dimension $m = d^2$, and (ii) the protocol of port-based teleportation itself [10], which can be arranged as a probabilistic or deterministic ε-UPQP$_d$ with memory dimension m, scaling as $\sim 2^{(4d^2 \log d)/\varepsilon^2}$ [13,14].

Notice that, in the first case, the resources used are remarkably efficient. The counterpart is that the success probability (accuracy of the setting) is rather low. In contrast, in the second example, the accuracy can be arbitrarily improved at the price of increasing the dimension of the resource state. These examples show the rich landscape of behaviors displayed by UPQPs, which makes the understanding of these objects challenging. The results presented here shed new light on them.

The connection.—In this section, we explain, omitting proofs, the key connection between ε-UPQP$_d$ and isometric embeddings between Banach spaces at the heart of the proofs of our main results.

The crucial ingredient is the characterization of UPQP_d as isometric embeddings $\Phi\colon \mathcal{S}_1(\mathcal{H}_d) \hookrightarrow \mathcal{B}(\mathcal{H}_m)$ with completely bounded norm $\|\Phi\|_{cb} \leq 1$, i.e., complete contractions. For ε-UPQP_d, the characterization holds, distorting the isometric property of the embedding with some disturbance $\delta(\varepsilon)$. Turning to the completely bounded norm of Φ, it can be understood in this particular case as follows. Let us put each $\mathcal{V} \in \mathcal{B}(\mathcal{H}_d \otimes \mathcal{H}_m)$ in one-to-one correspondence with the linear map

$$\Phi_{\mathcal{V}}\colon \mathcal{S}_1(\mathcal{H}_d) \hookrightarrow \mathcal{B}(\mathcal{H}_m), \quad \sigma \mapsto \Phi_{\mathcal{V}}(\sigma) \coloneqq \mathrm{Tr}_{\mathcal{H}_d} \mathcal{V}(\sigma^T \otimes \mathrm{Id}_M). \tag{1}$$

Given this correspondence, the completely bounded norm of $\Phi_{\mathcal{V}}$ can be simply regarded as $\|\Phi_{\mathcal{V}}\|_{cb} = \|\mathcal{V}\|_{\mathcal{B}(\mathcal{H}_d \otimes \mathcal{H}_m)}$.

More generally, within the theory of operator spaces, $\mathcal{S}_1(\mathcal{H}_d)$ and $\mathcal{B}(\mathcal{H}_m)$ are endowed naturally with a sequence of norms when tensorized with the set of $k \times k$ complex matrices M_k. Then, $\|\Phi\|_{cb}$ is nothing but $\sup_k \|\mathrm{Id}_{M_k} \otimes \Phi\|$. The equivalence between this more profound definition of the completely bounded norm and the one given before is provided by a well-known result in operator space theory (see [25], Proposition 8.1.2). We emplace the interested reader to Ref. [17].

We are now in a position to state formally the results of this section.

Theorem 1.—Every unitary ε-UPQP_d, $\mathcal{C}_{\mathcal{V}} \in \mathrm{CPTP}(\mathcal{H}_d \otimes \mathcal{H}_m)$, defines a completely contractive map $\Phi_{\mathcal{V}}\colon S_1(\mathcal{H}_d) \to \mathcal{B}(\mathcal{H}_m)$ such that

$$\|\sigma\|_{S_1(\mathcal{H}_d)} \geq \|\Phi_{\mathcal{V}}(\sigma)\|_{\mathcal{B}(\mathcal{H}_m)} \geq (1-\varepsilon)^{1/2} \|\sigma\|_{S_1(\mathcal{H}_d)}$$

for every $\sigma \in S_1(\mathcal{H}_d)$. Such a map is called a completely contractive ε embedding.

We also found true a converse of this statement.

Theorem 2.—Every completely contractive map $\Phi\colon \mathcal{S}_1(\mathcal{H}_d) \to \mathcal{B}(\mathcal{H}_m)$, such that

$$\|\sigma\|_{\mathcal{S}_1(\mathcal{H}_d)} \geq \|\Phi(\sigma)\|_{\mathcal{B}(\mathcal{H}_m)} \geq (1-\delta)\|\sigma\|_{\mathcal{S}_1(\mathcal{H}_d)}$$

for every $\sigma \in S_1(\mathcal{H}_d)$, defines a ε-UPQP_d with $\varepsilon = \sqrt{2\delta}$ and a memory dimension at most $dm^3 = \dim(\mathcal{H}_d \otimes \mathcal{H}_m^{\otimes 3})$.

This establishes a *characterization* rather than a simple relation between the objects considered.

Even when the proofs of these theorems were left outside the main text, let us finish this section noting that the starting point to establish this characterization is precisely the correspondence considered above, (1).

Results about UPQPs.—The characterization given in the preceding section leads to a better understanding of UPQPs, which is summarized in the last column of Table I. These results, presented in the remainder of the Letter, reduce drastically the existing gaps between the previous lower and upper bounds in the study of UPQPs. We now sketch the proof of them.

Let us begin with the upper bound:

$$m \leq \left(\frac{\tilde{C}}{\varepsilon}\right)^{d^2}, \tag{2}$$

where $\tilde{C}$ is a constant. Although this bound follows easily from a ε-net argument, we find it instructive to follow the lines of the proof of Theorem 2 in this simplified case.

First, we think at the level of embeddings between Banach spaces and consider the following mapping:

$$\Phi\colon \mathcal{S}_1(\mathcal{H}_d) \hookrightarrow \ell_\infty\{\mathsf{ball}[\mathcal{B}(\mathcal{H}_d)]\}, \quad \sigma \mapsto (\mathrm{Tr}[A\sigma^T])_{A \in \mathsf{ball}[\mathcal{B}(\mathcal{H}_d)]}, \tag{3}$$

where $\mathsf{ball}(X)$ denotes the unit ball of a Banach space X and, for a given set $\mathcal{X}$, $\ell_\infty(\mathcal{X})$ denotes the space of bounded functions from $\mathcal{X}$ to $\mathbb{C}$ endowed with the supremum norm. Then, it is straightforward to see that this embedding is isometric. Indeed, noting that $\mathcal{B}(\mathcal{H}_d)$ is the Banach dual of $\mathcal{S}_1(\mathcal{H}_d)$, the embedding considered is usually recognized as a standard consequence of the Hahn-Banach theorem [26].

In addition, the fact that $\ell_\infty(\mathcal{X})$ can be understood as a commutative C^* algebra guarantees that the bounded and completely bounded norms of any map $\Phi\colon E \to \ell_\infty(\mathcal{X})$ coincide (see [25], Proposition 2.2.6). This also allows us to drop out the awkward transposition in (3).

In order to obtain a finite-dimensional version of the embedding (3), we discretize the image by means of a ε net on $\mathcal{U}(\mathcal{H}_d)$. That is, we consider a finite sequence $\{U_i\}_{i=1}^{|\mathcal{I}|} \subset \mathcal{U}(\mathcal{H}_d)$ such that for every $U \in \mathcal{U}(\mathcal{H}_d)$ there exists an index $i \in \mathcal{I}$ verifying $\|U - U_i\|_{\mathcal{B}(\mathcal{H}_d)} \leq \varepsilon$. Then, we define the embedding

$$\tilde{\Phi}\colon \mathcal{S}_1(\mathcal{H}_d) \hookrightarrow \ell_\infty(\mathcal{I}) \hookrightarrow \mathcal{B}(\mathcal{H}_{\mathcal{I}}), \quad \sigma \mapsto (\mathrm{Tr}[U_i\sigma])_{i=1}^{|\mathcal{I}|} \mapsto \sum_{i \in \mathcal{I}} \mathrm{Tr}[U_i\sigma] |i\rangle\langle i|,$$

$\mathcal{H}_{\mathcal{I}}$ being a complex Hilbert space of dimension $|\mathcal{I}|$.

Now, it is an easy exercise to see that

$$\|\sigma\|_{\mathcal{S}_1(\mathcal{H}_d)} \geq \|\tilde{\Phi}(\sigma)\|_{\mathcal{B}(\mathcal{H}_{\mathcal{I}})} \geq \left(1 - \frac{\epsilon^2}{2}\right) \|\sigma\|_{\mathcal{S}_1(\mathcal{H}_d)},$$

for every $\sigma \in \mathcal{S}_1(\mathcal{H}_d)$. Then, $\tilde{\Phi}$ is a particular instance of a map in the conditions of Theorem 2, but its very simple structure allows us to get to the conclusion of the theorem very easily in this case, as we show now.

The embedding $\tilde{\Phi}$ suggests considering the unitary channel $\mathcal{C}_{\mathcal{V}}$, with $\mathcal{V} \in \mathcal{U}(\mathcal{H}_d \otimes \mathcal{H}_{\mathcal{I}})$ being the controlled unitary:

$$\mathcal{V} = \sum_i U_i \otimes |i\rangle\langle i|,$$

where the register $\mathcal{H}_\mathcal{I}$ plays the role of a memory. Then, according to Definition 2, let us compute the diamond distance of this channel (with a suitable memory state) with any unitary $U \in \mathcal{U}(\mathcal{H}_d)$. Since the action of the considered channel on the input state is unitary, the problem reduces in this case to computing the usual trace distance

$$\begin{aligned}\min_{i\in\mathcal{I}} \max_{|\psi\rangle\in ball(\mathcal{H}_d)} & \frac{1}{2}\| U_i|\psi\rangle\langle\psi|U_i^\dagger - U|\psi\rangle\langle\psi|U^\dagger\|_1 \\ &= \min_{i\in\mathcal{I}} \max_{|\psi\rangle\in ball(\mathcal{H}_d)} \sqrt{1-|\langle\psi|U_i^\dagger U|\psi\rangle|^2} \\ &\le \max_{|\psi\rangle\in ball(\mathcal{H}_d)} \sqrt{1-\left(1-\frac{\epsilon^2}{2}\right)^2} \le \epsilon.\end{aligned}$$

Therefore, the considered channel $\mathcal{C}_\mathcal{V}$ is a ε-UPQP$_d$ with memory dimension $|\mathcal{I}|$, the cardinality of the ε net considered. This cardinal can be taken lower than $(\tilde{C}/\varepsilon)^{d^2}$ for some constant $\tilde{C}$ (see [15], Theorem 5.11), which is the announced bound.

Because of the particular structure of the constructed ε-UPQP$_d$, we notice that the program states encoding different unitaries of the ε net $\{U_i\}_{i=1}^{|\mathcal{I}|}$ are indeed orthogonal. This is in consonance with the fact discovered by Nielsen and Chuang that for a UPQP$_d$ ($\varepsilon = 0$) any two program states encoding different unitaries must be orthogonal [3]. Then, given a ε-UPQP$_d$, it is tempting to try to reverse the previous ε-net argument to find $|\mathcal{I}|$ mutually orthogonal program states, lower bounding in this way the dimension m with the cardinality $|\mathcal{I}|$. However, in general ($\varepsilon > 0$), the orthogonality between program states is no longer true (one can consider, e.g., the case of port-based teleportation [10]). Moreover, previous lower bounds in Refs. [11,12] (see Table I) were based precisely on direct ε-net arguments which, in the end, essentially reduce to rough volume estimations. It turns out that the type constants (defined below) of the Banach spaces involved in Theorem 1 give more refined information of their geometry. This, together with the key property that type constants are preserved by subspaces, allows us to conclude from Theorem 1 the following exponential improvement over previous lower bounds on m.

Theorem 3.—Let $\mathcal{P}\in\mathrm{CPTP}(\mathcal{H}_d\otimes\mathcal{H}_m)$ be a ε-UPQP$_d$, then

$$\dim\mathcal{H}_m \equiv m \ge 2^{[(1-\varepsilon)/3C]d-(2/3)\log d}$$

for some constant C. Furthermore, if $\mathcal{P}$ is a unitary channel, one has $m \ge 2^{[(1-\varepsilon)/C]d}$.

Let us sketch how Theorem 3 can be obtained from Theorem 1. For simplicity, we restrict to the case where the considered UPQP is a unitary channel. The general case can be handled by means of a Stinesprings dilation of the channel under consideration.

The basic idea consists in studying ε embeddings between $\mathcal{S}_1(\mathcal{H}_d)$ and $\mathcal{B}(\mathcal{H}_m)$. These two spaces are extremely far apart from each other as Banach spaces, and it is this intuition which leads us to Theorem 3. A quick argument to study necessary conditions on the dimensions of the spaces involved is provided considering their type-2 constants. Given a Banach space X, its type-2 constant $T_2(X)$ is the infimum of the constants T satisfying the inequality

$$\left(\mathbb{E}\left[\|\sum_i \varepsilon_i x_i\|_X^2\right]\right)^{1/2} \le T\left(\sum_{i=1}^n \|x_i\|_X^2\right)^{1/2}$$

for every sequence $\{x_i\}_{i=1}^n \subset X$. There, $\mathbb{E}[\cdot]$ is the expected value over any combination of signs $\{\varepsilon_i\}_{i=1}^n \in \{-1,1\}^n$ with uniform weight $1/2^n$, that is, independent identically distributed Rademacher random variables [17]. Let us point out that, despite the great impact of the notion of type and cotype on Banach space theory in the past decades, in the context of quantum information it appeared only very recently in Ref. [27].

Since $\Phi_\mathcal{V}$ in Theorem 1 maps $\mathcal{S}_1(\mathcal{H}_d)$ into a subspace of $\mathcal{B}(\mathcal{H}_m)$–with distortion $(1-\varepsilon)^{1/2}$–the following relation between type constants of these spaces is enforced:

$$\begin{aligned}T_2[S_1(\mathcal{H}_d)] &\le \frac{1}{(1-\varepsilon)^{1/2}} T_2\{\Phi_\mathcal{V}[S_1(\mathcal{H}_d)]\} \\ &\le \frac{1}{(1-\varepsilon)^{1/2}} T_2[\mathcal{B}(\mathcal{H}_m)].\end{aligned}$$

The first inequality follows from $\Phi_\mathcal{V}$ being a ε embedding (in the sense of Theorem 1), while the second inequality follows from the property of type constants being preserved by subspaces. Introducing in those inequalities the following known estimates for type constants of the spaces involved:

$$\sqrt{d} \le T_2[S_1(\mathcal{H}_d)], \qquad T_2[\mathcal{B}(\mathcal{H}_m)] \le \sqrt{C\log m},$$

we obtain the desired bound:

$$d \le \frac{C}{(1-\varepsilon)}\log m \Rightarrow m \ge 2^{[(1-\varepsilon)/C]d}.$$

The constant here, as well as in the general case of nonunitary channels, can be taken equal to 4.

To finish, let us mention that the type argument sketched above can be made more explicit, obtaining bounds for the memory size necessary to program specific families of unitaries [17].

Discussion.—In this work, we have studied the minimal conditions, in terms of resources, that have to be satisfied by approximate UPQPs. The bounds presented here have clarified several questions about the optimality of this conceptual construction. In fact, we have almost closed the gaps in the optimal scaling of the memory size of UPQPs with the accuracy ε and input dimension d, when considered separately.

First, we have deduced the upper bound (2) giving a construction based on an ε net on $\mathcal{U}(\mathcal{H}_d)$. In this sense, this construction can be seen as a generalization to the case of UPQPs of the programmable measurement introduced in Ref. [8]. As in that case, our proposal improves exponentially the memory resources consumed by other known constructions (see Table I). In fact, the bound (2) exponentially improves the scaling with the accuracy ε of port-based teleportation and nearly saturates the lower bound deduced in Ref. [11] in the context of universal programmable *measurements*. This shows that, indeed, this is the optimal dependence on that parameter also in the case of UPQPs. More generally, it also outperforms port-based teleportation whenever $\tilde{C}/\varepsilon \leq d^{4/\varepsilon^2}$. Obviously, the drawback is that the optimal ε-UPQP$_d$ constructed here cannot be used to achieve any kind of teleportation.

On the other hand, the main result obtained is the lower bound expressed by Theorem 3. The first and most obvious consequence of this result is that, for any fixed value of ε, the dimension of the memory of a ε-UPQP$_d$ must scale exponentially in d. Indeed, in this case the dependence with d in the stated lower bound is exponentially stronger than all known previous results. Furthermore, this bound is saturated in this sense by the performance of port-based teleportation, referred to in Table I as the best upper bound for m.

Notwithstanding, more difficult relations ε-d can be considered, the general scaling being in this case still an open question. However, we also contribute to this point, giving an upper bound for the achievable accuracy by UPQPs with a memory of size $\mathsf{poly}(d)$. As a straightforward consequence of Theorem 3, we obtain the following.

Corollary 1.—For any ε-UPQP$_d$ with memory size $m \leq kd^s$ for some constants k and s, the following inequality is satisfied:

$$\varepsilon \geq 1 - C'_{k,s} \frac{\log d}{d},$$

where $C'_{k,s} = 3C(s + \log k + 2/3)$.

This severely restricts the accuracy achievable by ε-UPQP$_d$ with polynomially sized memories.

Moreover, due to the relation between UPQPs and other tasks, such as quantum teleportation [3,10], state discrimination [6,28], parameter estimation [29], secret and blind computation [30], homomorphic encryption [31], quantum learning of unitary transformations [32], etc., we believe that the knowledge about them could also be relevant in a wide variety of topics. For example, as a direct application of the results presented here, we also obtain a lower bound for the dimension of the resource space necessary to implement deterministic port-based teleportation. There exist more accurate bounds for this particular case (see [13]), but notice that we did not use in any way the many symmetries presented in that protocol, and our bound is generic for any protocol implementing, in some sense, a UPQP. Furthermore, it is deduced from our results that the unavoidable exponential scaling with ε^{-1} in the case of port-based teleportation comes entirely from the signaling restrictions imposed in this protocol and cannot be deduced from the programing properties of it.

Finally, some interesting questions related with the work presented here arise. The most direct one is whether it is possible to deduce a lower bound on m unifying the bound from Ref. [11] and the bound from Theorem 3. This could give more information about the optimality of UPQPs in cases beyond the scope of this work. In relation with that, it would be desirable to improve the exponents in the bounds to match exactly lower and upper bounds, though this will not affect qualitatively the consequences presented here. Further on, it would be also very interesting to look for relations between memory requirements on UPQPs and circuit complexity problems. A way to explore this line could consist of looking for correspondences between circuits and memory states in UPQPs.

We thank K. Chakraborty and C. Majenz for valuable discussions during the preparation of this work. We acknowledge financial support from Spanish MICINN (Grants No. MTM2014-54240-P and No. MTM2017-88385-P and Severo Ochoa Project No. SEV-2015-556), Comunidad de Madrid (Grant QUITEMAD+CM, Reference No. S2013/ICE-2801). A. M. K. is supported by the Spanish MICINN-FSE Project No. MTM2014-57838-C2-2-P, is partially supported by the "Ramón y Cajal program" (RYC-2012-10449). This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 648913).

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[1] N. Herbert, Found. Phys. **12**, 1171 (1982); D. Dieks, Phys. Lett. **92A**, 271 (1982); P. Milonni and M. Hardies, Phys. Lett. **92A**, 321 (1982); W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982); L. Mandel, Nature (London) **304**, 188 (1983).

[2] A. K. Pati and S. L. Braunstein, Nature (London) **404**, 164 (2000).

[3] M. A. Nielsen and I. L. Chuang, Phys. Rev. Lett. **79**, 321 (1997).

[4] Originally called a programmable quantum gate array [3].

[5] Even in this case, we could program general quantum channels, implementing a unitary first and tracing out a part of the output.

[6] M. Dušek and V. Bužek, Phys. Rev. A **66**, 022112 (2002).

[7] J. Fiurášek, M. Dušek, and R. Filip, Phys. Rev. Lett. **89**, 190401 (2002).

[8] G. M. D'Ariano and P. Perinotti, Phys. Rev. Lett. **94**, 090401 (2005).

[9] J. Kim, Y. Cheong, J.-S. Lee, and S. Lee, Phys. Rev. A **65**, 012302 (2001); M. Hillery, V. Bužek, and M. Ziman, Fortschr. Phys. **49**, 987 (2001); A. Brazier, V. Bužek, and P. L. Knight, Phys. Rev. A **71**, 032306 (2005); D. Bera, S. Fenner, F. Green, and S. Homer, in *Computing and Combinatorics*, edited by H. Q. Ngo (Springer, Berlin, 2009), pp. 418–428; G. Vidal, L. Masanes, and J. I. Cirac, Phys. Rev. Lett. **88**, 047905 (2002).

[10] S. Ishizaka and T. Hiroshima, Phys. Rev. Lett. **101**, 240501 (2008).

[11] D. Pérez-García, Phys. Rev. A **73**, 052315 (2006).

[12] C. Majenz, Ph.D. thesis, Faculty of Science, University of Copenhagen, 2017.

[13] M. Chrisstandl, F. Leditzky, C. Majenz, G. Smith, F. Speelman, and M. Walter, arXiv:1809.10751.

[14] S. Beigi and R. König, New J. Phys. **13**, 093036 (2011).

[15] G. Aubrun and S. J. Szarek, *Alice and Bob Meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory* (American Mathematical Society, Providence, Rhode Island, 2017).

[16] C. Palazuelos and T. Vidick, J. Math. Phys. (N.Y.) **57**, 015220 (2016).

[17] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.080505 for further details, which includes Refs. [18–21].

[18] J. Watrous, *The Theory of Quantum Information*, 1st ed. (Cambridge University Press, Cambridge, England, 2018).

[19] N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-Dimensional Operator Ideals* (Longman Scientific & Technical, New York, 1989).

[20] N. Tomczak-Jaegermann, Stud. Math. **50**, 163 (1974).

[21] C. A. Fuchs and J. van de Graaf, IEEE Trans. Inf. Theory **45**, 1216 (2006).

[22] M. Hillery, V. Bužek, and M. Ziman, Phys. Rev. A **65**, 022301 (2002).

[23] G. Vidal and J. I. Cirac, arXiv:quant-ph/0012067.

[24] A. Y. Kitaev, Russ. Math. Surv. **52**, 1191 (1997).

[25] E. G. Effros and Z.-J. Ruan, *Operator Spaces* (Oxford University, New York, 2000).

[26] W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1991).

[27] F. G. S. L. Brandao and A. W. Harrow, arXiv:1509.05065.

[28] M. Sedlák, M. Ziman, O. Přibyla, V. Bužek, and M. Hillery, Phys. Rev. A **76**, 022326 (2007); G. Sentís, E. Bagan, J. Calsamiglia, and R. Muñoz-Tapia, Phys. Rev. A **82**, 042312 (2010); T. Zhou, J. X. Cui, X. Wu, and G. L. Lon, Quantum Inf. Comput. **12**, 1017 (2012); T. Zhou, Quantum Inf. Process. **11**, 1669 (2012); G. Sentís, E. Bagan, J. Calsamiglia, and R. Muñoz-Tapia, Phys. Rev. A **88**, 052304 (2013).

[29] Z. Ji, G. Wang, R. Duan, Y. Feng, and M. Ying, IEEE Trans. Inf. Theory **54**, 5172 (2008); J. Suzuki, Phys. Rev. A **94**, 042306 (2016).

[30] V. Giovannetti, L. Maccone, T. Morimae, and T. G. Rudolph, Phys. Rev. Lett. **111**, 230501 (2013); C. A. Pérez-Delgado and J. F. Fitzsimons, Phys. Rev. Lett. **114**, 220502 (2015).

[31] L. Yu, C. A. Pérez-Delgado, and J. F. Fitzsimons, Phys. Rev. A **90**, 050303 (2014).

[32] A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, and P. Perinotti, Phys. Rev. A **81**, 032324 (2010).