Measuring Nonclassicality of Bosonic Field Quantum States via Operator Ordering Sensitivity

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We introduce a new distance-based measure for the nonclassicality of the states of a bosonic field, which outperforms the existing such measures in several ways. We define for that purpose the operator ordering sensitivity of the state which evaluates the sensitivity to operator ordering of the Renyi entropy of its quasiprobabilities and which measures the oscillations in its Wigner function. Through a sharp control on the operator ordering sensitivity of classical states we obtain a precise geometric image of their location in the density matrix space allowing us to introduce a distance-based measure of nonclassicality. We analyze the link between this nonclassicality measure and a recently introduced quantum macroscopicity measure, showing how the two notions are distinct.

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Introduction.—Questions arising in quantum information theory and quantum chaos drive continued interest in the exploration of the quantum-classical boundary. There is in this context a need to effectively determine the strength of the diverse nonclassical features of quantum states: their Titulaer-Glauber nonclassicality [1–23], their degree of coherence and macroscopic nature [24–35], their degree of entanglement, their entanglement potential [14,20], their semiclassical breaking times in quantum chaos [36,37], and the links between these notions. In this Letter, we investigate the nonclassicality of bosonic quantum field states. The established definition of a Titulaer-Glauber classical state in this context is that it is a statistical mixture of coherent states, or equivalently, that its Glauber-Sudarshan P function defines a probability on phase space [1]. Otherwise, it is *nonclassical*. In this Letter, the term "nonclassical" will always have this precise sense. The two main issues in this respect are the identification of nonclassicality witnesses that establish if a given state is nonclassical and the definition of quantitative measures of nonclassicality, that say how nonclassical a state is.

Indeed, for many states the P function is neither theoretically, nor experimentally readily accessible. Consequently, to test for nonclassicality, various sufficient and more easily verified criteria have been designed. Some generalize the well-known quantum optics criteria such as the negativity of the Mandel parameter and of the degree of squeezing [7,12,18]. Others involve the negativity of the Wigner function [6,9,13], the entanglement potential of the state [14], or the minimal number of coherent states allowing us to write it as a superposition ([20,22], and references therein). While they capture various aspects of nonclassicality, they do not furnish a nonclassicality measure. An alternative approach uses a distance between a given state and the set $\mathcal{C}$ of all classical states as nonclassicality measure. This idea was pursued using the trace norm [3,4,21], the Hilbert-Schmidt norm [10], and the Bures distance [11]. It has been argued, however, that the resulting nonclassicality measure depends on the arbitrariness in the choice of norm [19]. Also, computing or estimating these distances has been possible only in a few cases [14,21,22].

We propose a distance-based measure of nonclassicality avoiding those drawbacks. We construct a specifically adapted Hilbert norm on the density operators, whose square we refer to as the "operator ordering sensitivity" (OS) of the state ρ [see Eqs. (6) and (9)]. It measures the sensitivity of the Renyi entropy of its quasiprobability distributions to operator ordering, an eminently nonclassical notion. We show OS provides a simple and efficient sufficient condition for nonclassicality [see (6)] that is also necessary for pure states. Furthermore, as the square of a norm, the OS induces a distance $\mathcal{N}(\rho)$ from ρ to the set $\mathcal{C}$ of all classical states that we propose as a new measure of nonclassicality [see (10)].

We establish that the OS of ρ yields a good approximation of the nonclassicality distance $\mathcal{N}(\rho)$, that it captures the intuitive physical ideas underlying nonclassicality well, and that it can often be more easily determined than existing criteria.

Another feature of the quantum-classical boundary is "quantum macroscopicity" which, loosely speaking, evaluates the degree to which a quantum state is the superposition of macroscopically distinct states. Various measures of quantum macroscopicity have been proposed [24–27,31–34]. We will compare a proposal based on the quantum Fisher information to the nonclassicality measure $\mathcal{N}(\rho)$ and explain their relation.

Ordering sensitivity: A nonclassicality witness.—For ease of notation, we shall concentrate on one-dimensional systems, characterized by an annihilation-creation operator pair a, $a^\dagger$. We introduce the s-ordered quasiprobabilities $W_s(\alpha)$ of a state with density matrix ρ [38,39]. Let

$$\chi_s(\xi) = \exp\left(s\frac{|\xi|^2}{2}\right)\chi_0(\xi), \qquad \chi_0(\xi) = \mathrm{Tr}\rho D(\xi),$$

where $D(\xi) = \exp(\xi a^\dagger - \xi^* a)$. Then

$$W_s(\alpha) = \frac{1}{\pi^2}\int \chi_s(\xi)\exp(\xi^*\alpha - \xi\alpha^*)d^2\xi; \tag{1}$$

$W_0(\alpha)$ is the Wigner function of ρ, $W_1(\alpha)$ its Glauber-Sudarshan P function, $W_{-1}(\alpha)$ its Husimi function and $\chi_s(\xi)$ is the characteristic function of $W_s(\alpha)$ and

$$\partial_s W_s(\alpha) = -\frac{1}{8}\Delta_\alpha W_s(\alpha). \tag{2}$$

Here, $\alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$ and $\Delta_\alpha = \partial^2_{\alpha_1} + \partial^2_{\alpha_2}$. There exists $x(\rho) \geq 0$ so that, for all $s < x(\rho)$, W_s is analytical and square integrable, meaning that $\|W_s\|_2^2 \coloneqq \int |W_s|^2(\alpha)d^2\alpha < +\infty$; $W_0(\rho)$ is always square integrable [38–40]. Since $\mathrm{Tr}\rho = 1$, one has, for all $s < x(\rho)$, $\int W_s(\alpha)d^2\alpha = 1$, which, together with the fact that $W_s(\alpha)$ is real valued but not necessarily nonnegative, explains the term "quasiprobability distribution."

Following [1,2,41], we say a quantum state with density matrix ρ is classical when its P function is a probability on the phase space $\mathbb{C}$, which implies $x(\rho) \geq 1$. Below, we first establish a sufficient condition for a state ρ to be nonclassical based on its *ordering sensitivity* $S_o(\rho)$, which measures the variation in the quasiprobability distributions $W_s(\alpha)$, for s close to 0: see (6).

For that purpose, we introduce the *s-ordered entropy*:

$$H(s,\rho) = -\ln(\pi\|W_s\|_2^2). \tag{3}$$

This is well defined for all s so that W_s is square integrable and hence at least for all $-1 \leq s \leq 0$. This terminology is motivated by the observation that, when $W_s \geq 0$, it defines a bona fide probability. $H(s,\rho)$ is then its (second order) Renyi entropy, one of many measures of its uncertainty or unpredictability. A strongly localized probability distribution corresponds to a low degree of uncertainty and a strongly negative Renyi entropy. Conversely, a strongly delocalized probability distribution has a large positive Renyi entropy. For $s = 0$, $H(0,\rho) = -\ln\mathrm{Tr}\rho^2 \geq 0$ is directly expressed in terms of the purity $\mathcal{P} = \mathrm{Tr}\rho^2$ of ρ. It reaches its minimal value 0 for pure states. Note that, from (2) one finds

$$H'(s,\rho) = \frac{1}{4}\frac{\langle W_s, \Delta W_s\rangle}{\|W_s\|_2^2} = -\frac{1}{4}\frac{\|\nabla W_s\|_2^2}{\|W_s\|_2^2} \leq 0, \tag{4}$$

so H decreases with s. This reflects the fact that $W_s(\alpha)$ solves the well-studied [42–44] backward or antidiffusion equation (2) leading to an increase in entropy backward in the "time" s and a decrease forward in time [45]. One easily sees $H''(\rho,s) \leq 0$ so that H is concave.

Our main tool for the characterization of the nonclassicality of quantum states is the following bound on H'. *Theorem.*—If ρ is a classical state, then

$$0 \leq -(1-s)H'(s,\rho) \leq 1, \qquad -1 \leq s < 1. \tag{5}$$

The proof [46] relies on (2). The upper bound 1 is sharp, since one easily checks that, for coherent states, $H'(s,|\alpha\rangle\langle\alpha|) = (s-1)^{-1}$. The lower bound follows from (4). By evaluating (5) at $s = 0$, we infer the following sufficient condition for nonclassicality of ρ:

$$S_o(\rho) \coloneqq -H'(0,\rho) > 1 \Rightarrow \rho \text{ is nonclassical.} \tag{6}$$

We call $S_o(\rho)$ the *ordering sensitivity (OS)* of ρ. It measures the change in the s-ordered entropy of ρ, and hence the change in $W_s(\alpha)$, as s varies close to $s = 0$. This terminology is justified because different values of s correspond to different operator orderings in the quantization procedure [38,39]. The condition $S_o(\rho) > 1$ can hence be paraphrased by saying that the state ρ is strongly ordering sensitive and (6) says this implies the state is not classical, in agreement with "operator ordering" as a typical quantum feature.

A second argument in favor of $S_o(\rho)$ as a nonclassicality probe comes from the observation that ∇W_0 probes the oscillations and short range structures of W_0, associated in particular with interference fringes and with negativity of W_0. Hence (4) implies $S_o(\rho)$ measures such features. Since interference fringes are a hallmark of quantum mechanics, physical intuition suggests large values of $S_o(\rho)$ are associated to strong quantum features of the state. *A contrario*, if ρ is a classical state then, setting $s = 0$ in (5) and using (4), one finds $S_o(\rho) \leq 1$ or $\|\nabla W_0\|_2^2 \leq (4/\pi)\mathrm{Tr}\rho^2$. Hence, for classical states, these oscillations in the Wigner distribution are *purity limited*. The less pure a classical state, the smaller they are.

Equation (4) therefore links two quantum phenomena: the sensitivity of $W_s(\alpha)$ to the operator ordering parameter s and the oscillations of $W_s(\alpha)$ at fixed s. That such oscillations increase with s has been argued generally in [38,39] and illustrated on examples in [49,50]. It finds a quantitative expression in our setting in the fact that $-H''(s,\rho) \geq 0$. The quantity $\|\nabla W_0\|^2/\|W_0\|^2$ has been used previously in quantum chaos studies [36,37] and both $\|\nabla W_0\|^2/\|W_0\|^2$ and $\|\nabla W_0\|^2$ have been proposed as measures of "quantum macroscopicity" [25–27], but their relevance for that latter purpose has been contested [34] (see below for details.) Our results reinterpret $\|\nabla W_0\|^2/\|W_0\|^2$ as the OS of the state and show it is a nonclassicality witness. We use it below to construct a nonclassicality measure.

Third, we consider the behavior of the OS when the system interacts with a thermal bath with mean photon number $\langle n\rangle$. We use a simple input-output model [14,15] that can alternatively be interpreted as the action of a beam splitter [51]. The system, initially in the state $\rho_{\rm in}$, ends up in the state $\rho_{\rm out}$ after interaction with the bath, characterized by an efficiency $0 \le \lambda \le 1$, where

$$\chi_{{\rm out},1}(\xi) = \chi_{{\rm in},1}(\sqrt{\lambda}\xi)\exp[-(1-\lambda)\langle n\rangle|\xi|^2].$$

As a result, with $\bar{s} = 1 + \lambda^{-1}[(s-1) - 2(1-\lambda)\langle n\rangle] \le 1$,

$$W_{s,{\rm out}}(\alpha) = \lambda^{-1} W_{\bar{s},{\rm in}}\left(\frac{\alpha}{\sqrt{\lambda}}\right).$$

It follows that $S_o(\rho_{\rm out}) = -\lambda^{-1}H'(\bar{s},\rho_{\rm in})$, with $\bar{s} = 1-\lambda^{-1}(1+2(1-\lambda)\langle n\rangle) < 0$. Since $-H'$ is a nondecreasing function of s, this yields $S_o(\rho_{\rm out}) \le \lambda^{-1}S_o(\rho_{\rm in})$. For λ close to 1 and $\langle n\rangle$ large enough, this shows that $S_o(\rho_{\rm out})$ is lower than or equal to $S_o(\rho_{\rm in})$. More precisely, in the weak coupling limit $\lambda \to 1$ and $(1-\lambda)\langle n\rangle \to \bar{e}$, this yields

$$\lim_{\lambda\to 1} S_o(\rho_{\rm out}) = -H'(-2\bar{e},\rho_{\rm in}) \le S_o(\rho_{\rm in}).$$

Since noisy environments destroy the quantal nature of states, this is again compatible with the interpretation of $S_o(\rho)$ as an indicator of the level of nonclassicality of ρ, a point further developed below, see Eqs. (10) and (11). In fact, the above equation shows that, the noisier the environment (large $\bar{e}$), the more it decreases $S_o(\rho)$ and hence the nonclassicality of the initial state.

A final argument in favor of the pertinence of $S_o(\rho)$ as a nonclassicality probe comes from the analysis of $S_o(\rho)$ for pure states $\rho = |\psi\rangle\langle\psi|$. In that case, (9) below implies

$$\begin{aligned} S_o(\rho) &= \langle (Q-\langle Q\rangle)^2\rangle + \langle (P-\langle P\rangle)^2\rangle \\ &= 2\langle (a^\dagger - \langle a^\dagger\rangle)(a-\langle a\rangle)\rangle + 1. \end{aligned} \tag{7}$$

Here, $Q = (1/\sqrt{2})(a^\dagger + a)$, $P = (i/\sqrt{2})(a^\dagger - a)$. Hence, for pure states $S_o(\rho)$ captures the intuitive idea that they are strongly nonclassical when they have a large uncertainty. Indeed, in classical mechanics pure states are points in phase space displaying no uncertainty, whereas in quantum mechanics, pure states must have uncertainty, of which $S_o(\rho)$ is a natural measure. The uncertainty principle then implies $S_o(\rho) \ge 1$; it is equal to one only if $|\psi\rangle$ is a coherent state. Equations (5) and (7) therefore provide an alternative proof of the known fact that the only pure classical states are the coherent states [2]. The condition $S_o(\rho) > 1$ is therefore both necessary and sufficient for the nonclassicality of pure states.

We now show how to use the OS to construct a nonclassicality measure, extending these ideas to all states.

A new nonclassicality measure.—We first interpret S_o geometrically. We define, for two operators A, B

$$\langle A, B\rangle = \frac{1}{2}{\rm Tr}([A^\dagger, Q][Q,B] + [A^\dagger, P][P,B]). \tag{8}$$

This expression is linear in B, antilinear in A, and positive when $B = A$. We set $|||A||| = \langle A, A\rangle^{1/2}$. If $|||A||| = 0$, one shows A vanishes [46]. Hence, the above expression defines an inner product. We write $\mathcal{L}^{(1)}_{\rm HS}$ for the corresponding Hilbert space of operators [46]. One has [36,46]

$$S_o(\rho) = -\frac{1}{2}\frac{{\rm Tr}([Q,\rho]^2 + [P,\rho]^2)}{{\rm Tr}\rho^2} = |||\tilde{\rho}|||^2, \tag{9}$$

where $\tilde{\rho} = \rho/\sqrt{{\rm Tr}(\rho^2)}$, showing OS is a norm. The map $\rho \to \tilde{\rho}$ is the normalization of ρ for the Hilbert-Schmidt norm.

We now reformulate (6): $\rho \in \mathcal{C} \Rightarrow |||\tilde{\rho}||| \le 1$. In other words, $\tilde{\mathcal{C}}$, which is the image of $\mathcal{C}$ under the map $\rho \to \tilde{\rho}$, is contained inside the unit ball of $\mathcal{L}^{(1)}_{\rm HS}$. Conversely, when $\tilde{\rho}$ is outside this unit ball, ρ is nonclassical. We define the distance from ρ to $\mathcal{C}$ by $d(\rho,\mathcal{C}) = \inf_{\sigma\in\mathcal{C}}|||\tilde{\rho}-\tilde{\sigma}|||$ and propose it as a quantitative nonclassicality measure for ρ by defining the *nonclassicality* $\mathcal{N}(\rho)$ of ρ via

$$\mathcal{N}(\rho) = d(\rho,\mathcal{C}). \tag{10}$$

Clearly, $\mathcal{N}(\rho) > 0$ implies ρ nonclassical and ρ classical implies $\mathcal{N}(\rho) = 0$ [46].

One could object that, since we have no good understanding of the precise shape of $\mathcal{C}$, this distance cannot be readily computed, as for the distances previously introduced in the literature. However, since $\tilde{\mathcal{C}}$ lies inside the unit ball, and since the OS of classical states can be arbitrarily small, the triangle inequality for norms implies [46]

$$|||\tilde{\rho}||| - 1 \le \mathcal{N}(\rho) \le |||\tilde{\rho}|||. \tag{11}$$

Hence, if $|||\tilde{\rho}||| \gg 1$, then $|||\tilde{\rho}|||$ provides a very good estimate of $\mathcal{N}(\rho)$. In addition, (9) expresses $S_o(\rho)$ directly in terms of the density matrix ρ itself, without referring to its quasiprobabilities $W_s(\alpha)$. This, as we will see, is a distinct advantage in its computation.

Computing the ordering sensitivity: pure states.—One finds, using (7), that $S_o(|\alpha\rangle\langle\alpha|) = 1$, as anticipated above. For squeezed states $|\alpha, z\rangle, z = re^{i\varphi}$, one finds $S_o(|\alpha,z\rangle\langle\alpha,z|) = \cosh(2r)$: increased squeezing leads to increased nonclassicality. Also, $S_o(|n\rangle\langle n|) = 2n+1$: the number states are increasingly far from the set $\mathcal{C}$ of classical states as n grows, corroborating their increasing nonclassicality. For N-component cat states [51,52], the nonclassicality grows as $|\alpha|^2$ [46]. In contrast, for such states, the Mandel parameter and the degree of squeezing, as well as the method of moments, provide inefficient nonclassicality witnesses when α is large [46]. The entanglement potential of the N-component cat states saturates at $\ln N$ [14,52] for large $|\alpha|$ failing to capture the nonclassicality growth with growing $|\alpha|$. Similarly, the degree of nonclassicality introduced in [17,22] equals N, independently of α. Finally, our approach here has essential advantages over the one using the trace distance $\delta_{\mathcal{C}}(\rho) = \frac{1}{2}\inf_{\sigma\in\mathcal{C}}\mathrm{Tr}(|\rho-\sigma|)$ as a measure of nonclassicality. Indeed, $\delta_{\mathcal{C}}(|n\rangle\langle n|)$ tends to its maximal value 1 as n grows, whereas $\delta_{\mathcal{C}}(|\psi_\pm\rangle\langle\psi_\pm|)$ saturates at 1/2 for large $|\alpha|$ for the even/odd cat states $|\psi_\pm\rangle \propto (|\alpha\rangle \pm |-\alpha\rangle)$ [21]. It is therefore insensitive to their increased phase space spread. In addition, to the best of our knowledge, sharp computable estimates such as (11) are not available for $\delta_{\mathcal{C}}(|\psi\rangle\langle\psi|)$ with general $|\psi\rangle$. Indeed, (11) provides both upper and lower bounds on $\mathcal{N}(|\psi\rangle\langle\psi|)$ showing it is equivalent to the right hand side of (7), known as the total noise [53]. The total noise is easily computable in terms of $\langle a\rangle$ and $\langle a^\dagger a\rangle$ and is experimentally accessible [5]. While it is large when $\delta_{\mathcal{C}}(|\psi\rangle\langle\psi|)$ is close to its maximal value 1 [5], the converse is not true as the above examples show.

Computing the ordering sensitivity: Mixed states.—Let $\rho = \sum_i p_i|i\rangle\langle i|$, where $\langle i|j\rangle = \delta_{ij}$. Then $\langle\tilde{\rho},\tilde{\rho}\rangle = \tilde{p}^T K\tilde{p}$, with, for $i \neq j$ [46],

$$K_{ii} = \Delta Q_i^2 + \Delta P_i^2, \quad K_{ij} = -(|\langle i|Q|j\rangle|^2 + |\langle i|P|j\rangle|^2). \tag{12}$$

The computation of $S_o(\rho)$ is therefore reduced to the computation of field quadratures in the eigenstates $|i\rangle$ of ρ followed by the analytical or numerical computation of a matrix element of K. In comparison, the computation of the trace, Hilbert-Schmidt, or Bures distances has not been achieved for mixed states. Also, the determination of the Mandel parameter and *a fortiori* the use of the moment method, require the computation of higher moments in a, $a^\dagger$ [46]. From (12) it follows $S_o(\rho)$ is less than the weighted average $\sum_i \tilde{p}_i^2(\Delta Q_i^2 + \Delta P_i^2)$ of the ordering sensitivities of the eigenstates $|i\rangle$. This bound can be reached. For example, when the $|i\rangle$ are the number states $|n\rangle$, one has $K_{nn} = 2n+1$ and $K_{nn+1} = -(n+1)$. For $\rho_{\mathrm{even}} = \sum_n p_{2n}|2n\rangle\langle 2n|$ this gives

$$S_o(\rho_{\mathrm{even}}) = \sum_n \tilde{p}_{2n}^2(\Delta Q_{2n}^2 + \Delta P_{2n}^2) = 1 + 4\sum_n \tilde{p}_{2n}^2 n.$$

When $p_0 = 0$, $p_2 = \cdots = p_{2M} = 1/M$, this yields $S_o(\rho_{\mathrm{even},M}) = 1 + 2(M+1)$. These states are therefore increasingly nonclassical as M grows and show strong oscillations in their Wigner function. For $\rho_M = M^{-1}\sum_{n=1}^M |n\rangle\langle n|$ [22,25], on the contrary, $S_o(\rho_M) = 1 + 2\,M^{-1}$. The ρ_M are only weakly nonclassical: they remain at a distance at most $1 + M^{-1}$ from $\mathcal{C}$ and their Wigner function shows only small fluctuations (Fig. 1). On the other hand, the Mandel parameter grows for both $\rho_{\mathrm{even},M}$ and ρ_M as M, failing to detect their nonclassicality for large M. Also, although these two states are very different, the degree of nonclassicality introduced in [17,22] is $2M$ for both and does not distinguish them. Similar computations allow us to determine the ordering sensitivity of a mixture of a thermal and a Fock state $|m\rangle$ [25], which is strongly nonclassical for large m, as well as for truncated thermal states [6] and for (single) photon added thermal [7,16] states, which are found to be weakly nonclassical.

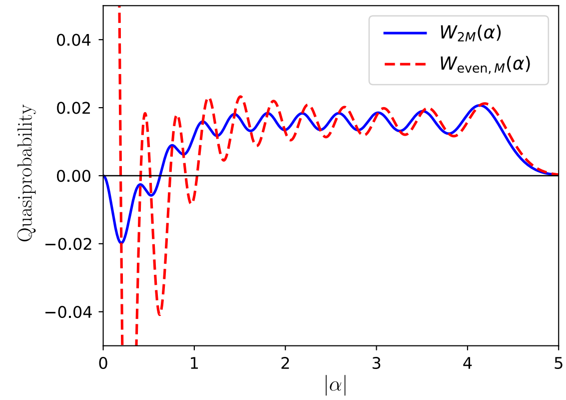

FIG. 1. Plots of the Wigner functions of ρ_{2M} (solid line) and of $\rho_{\mathrm{even},M}$ (dashed line) as a function of $|\alpha|$ for $M = 10$. Both states have comparable photon number: $\mathrm{Tr}(\rho_{2M}a^\dagger a) = M + \frac{1}{2}$, $\mathrm{Tr}(\rho_{\mathrm{even},M}a^\dagger a) = M + 1$. The oscillations in the Wigner function of one are visibly much more pronounced than in the other, as detailed in the text.

Nonclassicality versus quantum macroscopicity.—In contrast to the "nonclassicality" of a state, there is no generally agreed upon definition of its quantum macroscopicity, a property for which a variety of measures have been proposed recently [24–26,28,31–34]. One such measure [33,34] uses the quantum Fisher information $\mathcal{F}(\rho, Q_\theta)$ [54] of the quadratures $Q_\theta = Q\cos\theta + P\sin\theta$:

$$\mathcal{M}_{\mathrm{QFI}}(\rho) = \frac{1}{4}\max_\theta \mathcal{F}(\rho, Q_\theta).$$

We show in the Supplemental Material [46] that

$$\mathcal{M}_{\mathrm{QFI}}(\rho) > \frac{1}{2} \Rightarrow \rho\ \text{nonclassical}, \tag{13}$$

proving that $\mathcal{M}_{\mathrm{QFI}}$ is a nonclassicality witness, as is S_o. It is natural to ask how $\mathcal{M}_{\mathrm{QFI}}$ and S_o are related. One may notice that on many states they behave similarly. Indeed, on thermal states and on ρ_M (defined above), $S_o = 2\mathcal{M}_{\mathrm{QFI}}$, so that they coincide except for normalization [46]. The two can also be very different. For example, $\mathcal{M}_{\mathrm{QFI}}$ is a less efficient nonclassicality witness than S_o for truncated vacuum states, while it is more efficient than S_o for squeezed thermal states [46,55]. However, and more importantly, a large $\mathcal{M}_{\mathrm{QFI}}$ does not imply a large $\mathcal{N}$, as the example $\rho_k = [1-(M_*/k)]|0\rangle\langle 0| + (M_*/k)|k\rangle\langle k|$ shows: $\mathcal{M}_{\mathrm{QFI}}(\rho_k) = \frac{1}{2} + M_*$, $\mathcal{N}(\rho_k) \to 0$ when $k \gg 1$. So, when M_* is large, $\mathcal{M}_{\mathrm{QFI}}$ is large, while $\mathcal{N}$ remains small [46]. Consequently, if $\mathcal{M}_{\mathrm{QFI}}$ does indeed correctly capture the idea of quantum macroscopicity, as proposed in [33,34], then large quantum macroscopicity does not imply large Glauber-Titulaer nonclassicality. This would seem to indicate that, even for a single mode, $\mathcal{M}_{\mathrm{QFI}}$ captures "macroscopic" and/or "quantum" features of such states that are different from the nonclassicality associated with a nonpositive Sudarshan-Glauber P function and revealed by their OS. What these features are, remains unclear.

Conclusions.—We have constructed a new nonclassicality measure $\mathcal{N}(\rho)$ for the states of a single-mode boson field. $\mathcal{N}(\rho)$ is a distance to the set $\mathcal{C}$ of classical states, defined in terms of the ordering sensitivity (OS) of the state, a new entropic notion that evaluates the sensitivity of the state to operator ordering. We have proven that all classical states have an OS less than or equal to one and that, when the OS of a density matrix is large, it provides a good approximation of $\mathcal{N}(\rho)$. The OS is easily computable in terms of field quadratures, captures several intuitive features of nonclassicality naturally, and detects in many cases nonclassicality more efficiently than previously known indicators. We have finally compared the nonclassicality $\mathcal{N}(\rho)$ to a recent proposal for the measure of quantum macroscopicity based on the quantum Fisher information.

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