

Action Principle for Newtonian Gravity

Dennis Hansen,^{1,*} Jelle Hartong,^{2,†} and Niels A. Obers^{3,‡}

¹*Institut für Theoretische Physik, Eidgenössische Technische Hochschule Zürich Wolfgang-Pauli-Strasse 27, 8093 Zürich, Switzerland*

²*School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom*

³*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*



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We derive an action whose equations of motion contain the Poisson equation of Newtonian gravity. The construction requires a new notion of Newton-Cartan geometry based on an underlying symmetry algebra that differs from the usual Bargmann algebra. This geometry naturally arises in a covariant $1/c$ expansion of general relativity, with c being the speed of light. By truncating this expansion at subleading order, we obtain the field content and transformation rules of the fields that appear in the action of Newtonian gravity. The equations of motion generalize Newtonian gravity by allowing for the effect of gravitational time dilation due to strong gravitational fields.

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The idea that gravity is geometry was pioneered by Einstein in his celebrated theory of general relativity (GR). In GR, due to Einstein's equivalence principle, the underlying geometry is (pseudo-)Riemannian, which ensures that one has local Lorentz symmetry and hence the laws of physics locally reduce to those of special relativity. However, spacetime covariance is a property of any physical theory, which led Cartan [1,2] (see also, e.g., [3,4]) to geometrize Newtonian gravity using what is known as Newton-Cartan (NC) geometry. The latter results from applying an equivalence principle that requires freely falling observers to see Galilean laws of physics, giving rise to a geometry with local Galilean invariance.

However, while the Poisson equation of Newtonian gravity can be geometrized using NC geometry, an outstanding question has been to find an action principle for Newtonian gravity, paralleling the Einstein-Hilbert action in GR. In this Letter we present such an action and show that it requires a novel type of geometry. This geometry does encapsulate NC geometry in its original form when time is absolute (as is the case in Newtonian gravity), but is based on an underlying symmetry structure and corresponding set of geometric fields, which goes beyond the Bargmann algebra—the centrally extended Galilean algebra.

NC geometry and its recently discovered version, torsional Newton-Cartan (TNC) geometry [5,6] (referred to as type I TNC geometry below), has been very useful for studying

aspects of field theories with Galilean symmetries. Furthermore, gravity theories for type I TNC geometry (with broken particle number gauge symmetry) have been recently studied as well and shown to correspond to Hořava-Lifshitz gravity (see, e.g., [7,8]). It has proven difficult to write down actions for type I TNC geometry that preserve $U(1)$ particle number, though exceptions exist in $2 + 1$ dimensions [9,10], but these require an additional field.

By taking a critical look at Newtonian gravity, we will show that an action involving type I TNC geometry is incompatible with the way in which the mass source appears in the Poisson equation. This is because in type I TNC geometry mass sources torsion, which is not compatible with the notion of absolute time (and hence zero torsion) of Newtonian gravity. The key to identifying the correct geometry lies in carefully considering the properties of a large speed of light limit of GR, as was recently revisited in [11] following earlier work [12,13].

We present in this Letter a novel type of NC geometry, dubbed type II TNC geometry, which for zero torsion includes the standard (type I) NC geometry used to geometrize Newtonian gravity, and which allows us to formulate an action, in any spacetime dimension $D = d + 1$. To this end, it is crucial to allow for more general time (lapse) functions than the absolute time of Newtonian gravity. We will show that, while type I TNC geometry follows from gauging the Bargmann algebra [7,14] (see also [15,16]), type II TNC geometry follows from a novel nonrelativistic symmetry, which turns out to be a nontrivial contraction of the direct sum of the Poincaré and Euclidean algebras in $D = d + 1$ dimensions.

The action given in this Letter describes the dynamics of a well-defined truncation of the nonrelativistic limit of GR and has direct physical relevance in a post-Newtonian

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regime, including the effects of strong gravitational fields, e.g., via gravitational time dilation. More generally, it can be regarded as providing an off shell definition of the nonrelativistic gravity corner of the *Gch* Bronstein cube of physical theories and, as such, presents a principle towards constructing a nonrelativistic quantum gravity theory. The latter could open up a third road towards (relativistic) quantum gravity, in contradistinction to the usually traveled paths approaching it via relativistic quantum field theory or general relativity.

Newton-Cartan geometry.—Torsional Newton-Cartan geometry consists of a clock form τ_μ , a rank- d symmetric tensor $h_{\mu\nu}$ with signature $(0, 1, \dots, 1)$, and a U(1) connection m_μ . These describe a manifold with a Galilean tangent space structure, geometrizing the Galilean equivalence principle. In Galilean invariant theories, the total mass is conserved with the mass current coupling to a U(1) gauge connection m_μ . The TNC fields transform as tensors under diffeomorphisms (ξ^μ) and exhibit furthermore a set of local symmetries corresponding to Galilean (or Milne) boosts (λ_μ) and a U(1) gauge transformation (σ) associated with mass conservation,

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\xi\tau_\mu, & \delta h_{\mu\nu} &= \mathcal{L}_\xi h_{\mu\nu} + \lambda_\mu\tau_\nu + \lambda_\nu\tau_\mu, \\ \delta m_\mu &= \mathcal{L}_\xi m_\mu + \lambda_\mu + \partial_\mu\sigma, \end{aligned} \quad (1)$$

where \mathcal{L}_ξ denotes the Lie derivative along ξ^μ . The geometric tensors v^μ and $h^{\mu\nu}$ are defined by inverting $-\tau_\mu\tau_\nu + h_{\mu\nu}$ to $-v^\mu v^\nu + h^{\mu\nu}$, with the property that $\tau_\mu h^{\mu\nu} = 0$ and $v^\mu\tau_\mu = -1$. The Galilean boost parameters satisfy $v^\mu\lambda_\mu = 0$. The analog of the absolute value of the determinant of the pseudo-Riemannian metric, which for TNC we denote by e^2 , is given by minus the determinant of the matrix $-\tau_\mu\tau_\nu + h_{\mu\nu}$. Three useful tensors that are invariant under local Galilean boosts (and rotations) are $\hat{v}^\mu \equiv v^\mu - h^{\mu\nu}m_\nu$, $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - 2\tau_\mu m_\nu$, and $\tilde{\Phi} \equiv -v^\mu m_\mu + \frac{1}{2}h^{\mu\nu}m_\mu m_\nu$. We also record the completeness relation $-\hat{v}^\mu\tau_\mu + h^{\mu\lambda}\bar{h}_{\lambda\nu} = \delta^\mu_\nu$.

We will choose the following affine connection to perform covariant differentiation [17–20]:

$$\bar{\Gamma}_{\mu\nu}^\lambda \equiv -\hat{v}^\lambda\partial_\mu\tau_\nu + \frac{1}{2}h^{\lambda\sigma}(\partial_\mu\bar{h}_{\nu\sigma} + \partial_\nu\bar{h}_{\mu\sigma} - \partial_\sigma\bar{h}_{\mu\nu}). \quad (2)$$

This is a metric compatible connection, i.e., $\bar{\nabla}_\mu\tau_\nu = 0 = \bar{\nabla}_\mu h^{\mu\rho}$. Note that this connection is not invariant under the local U(1) transformation with parameter σ . In TNC geometry we cannot make the local Galilean boost and local U(1) symmetries manifest at the same time. We also note that this connection has torsion because $\bar{\Gamma}_{[\mu\nu]}^\lambda = -\hat{v}^\lambda\partial_{[\mu}\tau_{\nu]}$. When the clock 1-form τ_μ obeys $h^{\mu\rho}h^{\nu\sigma}(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu) = 0$, we call the torsion twistless and the resulting geometry is called twistless torsional Newton-Cartan (TTNC) geometry [5–7]. In this Letter we

will assume throughout that τ_μ is twistless, implying that τ_μ obeys the Frobenius integrability condition $\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$ so that τ_μ is hypersurface orthogonal. Thus, in this case, the spacetime allows a foliation in terms of equal-time slices.

A useful property of the connection (2) is $\bar{\Gamma}_{\rho\mu}^\rho = e^{-1}\partial_\mu e - a_\mu$, where we defined the torsion vector $a_\mu \equiv \mathcal{L}_{\hat{v}}\tau_\mu$ and e has been defined above. This implies $(\bar{\nabla}_\mu + a_\mu)X^\mu = e^{-1}\partial_\mu(eX^\mu)$. We define the associated Riemann tensor as usual via

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu]X_\sigma = \bar{R}_{\mu\nu\rho}^\sigma X_\rho - 2\bar{\Gamma}_{[\mu\nu]}^\rho\bar{\nabla}_\rho X_\sigma. \quad (3)$$

Further, we define the Ricci tensor as $\bar{R}_{\mu\nu} = \bar{R}_{\mu\rho\nu}^\rho$. Because of the presence of torsion, one can show using the Bianchi identity for $\bar{R}_{[\mu\nu\sigma]}^\rho$ that the antisymmetric part of the Ricci tensor is nonzero and equal to

$$2\bar{R}_{[\mu\nu]} = (\tau_\mu a_\nu - \tau_\nu a_\mu)\bar{\nabla}_\rho\hat{v}^\rho + \hat{v}^\rho(\tau_\mu\bar{\nabla}_\nu a_\rho - \tau_\nu\bar{\nabla}_\mu a_\rho). \quad (4)$$

The above reviewed standard TNC geometry is referred to as type I TNC geometry below.

Finally, we note that a convenient way to think of type I TNC geometry is via the process of null uplift [21], which will be instrumental below in showing that this geometry cannot correctly describe Newtonian gravity. Any TNC geometry can be written as a Lorentzian geometry with a null isometry in one dimension higher. Parametrizing the null isometry with u we can write the Lorentzian metric \hat{g}_{MN} as

$$\hat{g}_{MN}dx^M dx^N = 2\tau_\mu dx^\mu(du - m_\nu dx^\nu) + h_{\mu\nu}dx^\mu dx^\nu, \quad (5)$$

where $x^M = (u, x^\mu)$. The null Killing vector is ∂_u . The inverse metric is $\hat{g}^{uu} = 2\tilde{\Phi}$, $\hat{g}^{\mu u} = -\hat{v}^\mu$, and $\hat{g}^{\mu\nu} = h^{\mu\nu}$. At the level of symmetries, the null reduction means that the Bargmann algebra is a subalgebra of Poincaré in one dimension higher. Alternatively, it can be obtained by an Inönü-Wigner contraction of the product of the Poincaré algebra (in the same dimension) times a U(1).

A critical look at Newton-Cartan gravity.—Type I Newton-Cartan geometry was initially invented to describe Newtonian gravity in a coordinate independent manner. The equations of motion that covariantize the Poisson equation of Newtonian gravity are

$$\bar{R}_{\mu\nu} = 8\pi G \frac{d-2}{d-1}\rho\tau_\mu\tau_\nu, \quad \partial_\mu\tau_\nu - \partial_\nu\tau_\mu = 0, \quad (6)$$

where ρ is the mass density. On flat spacetime in Cartesian coordinates $\tau = dt$ and $h_{\mu\nu}dx^\mu dx^\nu = dx^i dx^i$ with $m = \Phi dt$, this simply reduces to

$$\partial_i\partial_i\Phi = 8\pi G \frac{d-2}{d-1}\rho. \quad (7)$$

Both sides of (6) are formulated in terms of NC objects and are invariant under all type I NC gauge symmetries for $d\tau = 0$. Yet, the coupling of mass to the geometry is not what one would expect from a theory with local Bargmann U(1) invariance. The gauge field m_μ couples to the conserved mass current J^μ , so any type I invariant action leads to an equation of the form $R^\mu = J^\mu$, where the left-hand side is a geometrical object formed from the type I TNC fields obeying the Bianchi identity $\partial_\mu(eR^\mu) = 0$.

Using null uplift (5), the equation $R^\mu = J^\mu$ reads $\hat{G}_u^\mu = 8\pi G \hat{T}_u^\mu$, where \hat{G}_N^M and \hat{T}_N^M are the higher dimensional Einstein and energy-momentum tensors. From the Bianchi identity for \hat{G}_N^M , it follows that \hat{G}_u^μ is identically conserved. Contracting with τ_μ we see that mass sources $\tau \wedge d\tau \neq 0$ since $\hat{G}_{uu} = 8\pi G \hat{T}_{uu} \equiv 8\pi G \rho$ with $\hat{G}_{uu} = \frac{1}{4} [h^{\mu\rho} h^{\nu\sigma} (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) (\partial_\rho \tau_\sigma - \partial_\sigma \tau_\rho)]^2$. This conflicts with Newtonian gravity since in that case the notion of mass is compatible with $d\tau = 0$; i.e., ρ in Newton's law is not a Bargmann mass.

Newtonian gravity is obtained from a nonrelativistic limit of GR, but we have just shown that this limit cannot be type I TNC geometry. This begs the question what kind of geometry one should employ. The answer comes from studying the large speed of light limit of GR, i.e., the $1/c$ expansion of [11,12]. We will show that this leads to a different notion of Newton-Cartan geometry, which we call type II Newton-Cartan geometry, and that this allows for an off shell formulation of Newtonian gravity.

1/c expansion and type II TNC geometry.—In a $1/c$ expansion the pseudo-Riemannian metric and its inverse are expanded as [11]

$$g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + \bar{h}_{\mu\nu} + \frac{1}{c^2} (2\tau_\mu \hat{B}_\nu - \bar{h}_{\mu\rho} \bar{h}_{\nu\sigma} \hat{\beta}^{\rho\sigma}) + \mathcal{O}(c^{-4}), \quad (8)$$

$$g^{\mu\nu} = h^{\mu\nu} - \frac{1}{c^2} \hat{v}^\mu \hat{v}^\nu + \frac{1}{c^2} \hat{\beta}^{\mu\nu} + \mathcal{O}(c^{-4}), \quad (9)$$

where we note that the 1-form \hat{B}_μ will play no role in what follows. It is convenient to define $\Phi_{\mu\nu}$ via the relation $\hat{\beta}^{\mu\nu} = h^{\mu\rho} h^{\nu\sigma} \Phi_{\rho\sigma}$.

Using the corresponding $1/c$ expansion for the vielbeins [22], it follows that the fields τ_μ and $h_{\mu\nu}$ appearing above transform as in type I TNC geometry [see (1)]. In addition, the fields m_μ and $\Phi_{\mu\nu}$ transform as

$$\begin{aligned} \delta m_\mu &= \mathcal{L}_\xi m_\mu + \lambda_\mu + (\partial_\mu - a_\mu) \lambda + \tau_\mu h^{\rho\nu} a_\rho \zeta_\nu \\ \delta \Phi_{\mu\nu} &= \mathcal{L}_\xi \Phi_{\mu\nu} - 2\lambda \bar{K}_{\mu\nu} + \bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu, \end{aligned} \quad (10)$$

where we defined λ and ζ_ν through the equation $\zeta^\mu = -\hat{v}^\mu \lambda + h^{\mu\nu} \zeta_\nu$, $\bar{K}_{\mu\nu} \equiv -\frac{1}{2} \mathcal{L}_{\hat{v}} \bar{h}_{\mu\nu}$ is the extrinsic curvature tensor, and we recall that $a_\mu \equiv \mathcal{L}_{\hat{v}} \tau_\mu$ is the torsion vector.

These important extra symmetries follow from expanding relativistic diffeomorphism $\Xi^\mu = \xi^\mu + (1/c^2)\zeta^\mu + \dots$, so that ξ^μ parametrizes nonrelativistic diffeomorphisms and ζ^μ the extra symmetries above.

We will refer to the λ transformation in (10) as a torsional U(1) transformation due to the presence of the torsion vector a_μ . One notices that for $d\tau = 0$ the transformation of m_μ above reduces to the one in (1), since the ζ_ν part vanishes in that case while the torsional U(1) takes the same form as the U(1) transformation in (1). However, the gauge field m_μ in type II TNC geometry is quite different from its type I cousin. In particular, we will show in [22] that in type II TNC geometry m_μ couples to the energy current as opposed to type I where it couples to the mass current.

Deferring details to [22] we remark that the transformations of the type II TNC geometry introduced above can be obtained by gauging a novel nonrelativistic algebra of dimension $(d+1)(d+2)$, spanned by the generators $\{H, P_a, G_a, J_{ab}\}$ of the (massless) Galilean algebra augmented with the set $\{N, T_a, B_a, S_{ab}\}$, with nonzero commutators

$$\begin{aligned} [H, G_a] &= P_a, & [P_a, G_b] &= N \delta_{ab}, & [N, G_a] &= T_a, \\ [H, B_a] &= T_a, & [S_{ab}, P_c] &= 2\delta_{c[a} T_{b]}, & [G_a, G_b] &= -S_{ab}, \\ [S_{ab}, G_c] &= 2\delta_{c[a} B_{b]}, & [J_{ab}, J_{cd}] &= 4\delta_{[a[d} J_{c]b]}, \\ [J_{ab}, X_c] &= 2\delta_{c[a} X_{b]}, & [J_{ab}, S_{cd}] &= 4\delta_{[a[d} S_{c]b]}, \end{aligned} \quad (11)$$

where $X_a \in \{P_a, T_a, G_a, B_a\}$. The first line differs from the Bargmann algebra because N is not central. Interestingly, this algebra can be obtained from a contraction of the direct sum of the Poincaré and Euclidean algebras in $d+1$ dimensions and underlies Newtonian gravity in the same way that the Poincaré algebra underlies GR.

Off shell Newtonian gravity.—We now construct a Lagrangian depending on $\tau_\mu, h_{\mu\nu}, m_\mu, \Phi_{\mu\nu}$ that is invariant under the above gauge transformations. The unique two-derivative result is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{16\pi G} e [\hat{v}^\mu \hat{v}^\nu \bar{R}_{\mu\nu} - \tilde{\Phi} h^{\mu\nu} \bar{R}_{\mu\nu} \\ &\quad - \Phi_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} (\bar{R}_{\rho\sigma} - a_\rho a_\sigma - \bar{\nabla}_\rho a_\sigma) \\ &\quad + \frac{1}{2} \Phi_{\mu\nu} h^{\mu\nu} [h^{\rho\sigma} \bar{R}_{\rho\sigma} - 2e^{-1} \partial_\rho (e h^{\rho\sigma} a_\sigma)]], \end{aligned} \quad (12)$$

where e is the integration measure and we have omitted a possible cosmological constant term $e\Lambda$. The Lagrangian is obtained by starting with the (necessary) kinetic term $\hat{v}^\mu \hat{v}^\nu \bar{R}_{\mu\nu}$ and subsequently adding terms such that the entire expression is invariant under the torsional U(1) transformation as well the ζ_μ transformation (10). This invariance follows from the Bianchi identities

$$0 = e^{-1} \partial_\rho \left(e \left[h^{\rho\nu} \hat{\nu}^\mu \bar{R}_{\mu\nu} - \frac{1}{2} \hat{\nu}^\rho h^{\mu\nu} \bar{R}_{\mu\nu} \right] \right) + h^{\mu\rho} h^{\nu\sigma} \bar{K}_{\rho\sigma} \bar{R}_{\mu\nu} - \frac{1}{2} h^{\rho\sigma} \bar{K}_{\rho\sigma} h^{\mu\nu} \bar{R}_{\mu\nu}, \quad (13)$$

$$0 = h^{\mu\nu} h^{\rho\sigma} \bar{\nabla}_\mu \bar{R}_{\nu\rho} - \frac{1}{2} h^{\mu\sigma} h^{\nu\rho} \bar{\nabla}_\mu \bar{R}_{\nu\rho}, \quad (14)$$

which can be derived from $\bar{\nabla}_{[\lambda} \bar{R}_{\mu]\sigma}^k = 0$.

Since we work with off shell TTNC geometries, we need to add the Lagrange multiplier term $\mathcal{L}_{\text{LM}} = e \zeta^{\mu\nu} (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu)$ to the Lagrangian, where $\zeta^{\mu\nu} = -\zeta^{\nu\mu}$ obeying $\tau_\mu \zeta^{\mu\nu} = 0$, so that it only imposes $\tau \wedge d\tau = 0$ but not $d\tau = 0$ [23]. If we were to drop the condition $\tau_\mu \zeta^{\mu\nu} = 0$ so that \mathcal{L}_{LM} enforces $d\tau = 0$, the field $\zeta^{\mu\nu}$ would not decouple from the equations of motion. This is what happens in the 3D Chern-Simons actions for extended Bargmann algebras [9,10], where $\zeta^{\mu\nu} = e^{\mu\nu\rho} \zeta_\rho$ with ζ_ρ associated with the central extension of the 3D Bargmann algebra.

We are going to compute the equations of motion by varying $\tilde{\Phi}$, $\hat{\nu}^\mu$, $\Phi_{\mu\nu}$, and $h^{\mu\nu}$. Let us define

$$\delta\mathcal{L} = -\frac{e}{8\pi G} \left(E_{\tilde{\Phi}} \delta\tilde{\Phi} - E_\mu \delta\hat{\nu}^\mu + \frac{1}{2} E_\mu^h \delta h^{\mu\nu} + \frac{1}{2} E^{\mu\nu} \delta\Phi_{\mu\nu} \right), \quad (15)$$

where $E_{\tilde{\Phi}} = -\frac{1}{2} h^{\mu\nu} \bar{R}_{\mu\nu}$ and

$$E^{\mu\nu} = -h^{\mu\rho} h^{\nu\sigma} (\bar{R}_{\rho\sigma} - a_\rho a_\sigma - \bar{\nabla}_\rho a_\sigma) + \frac{1}{2} h^{\mu\nu} [h^{\rho\sigma} \bar{R}_{\rho\sigma} - 2e^{-1} \partial_\rho (e h^{\rho\sigma} a_\sigma)]. \quad (16)$$

The variations with respect to $P_\mu^\rho \delta\hat{\nu}^\mu$ with P_μ^ρ the spatial projector $P_\mu^\rho \equiv \delta_\mu^\rho + \hat{\nu}^\rho \tau_\mu$ gives

$$h^{\rho\mu} E_\mu = -h^{\rho\mu} \hat{\nu}^\nu \bar{R}_{\mu\nu}. \quad (17)$$

The remaining variations are $\tau_\mu \delta\hat{\nu}^\mu$ and $P_\mu^\alpha P_\nu^\beta \delta h^{\mu\nu}$. Defining $E_h^{\alpha\beta} \equiv h^{\mu\alpha} h^{\nu\beta} E_{\mu\nu}^h$, we find

$$\begin{aligned} -2\hat{\nu}^\mu E_\mu &= -2\tilde{\Phi} E_{\tilde{\Phi}} - \Phi_{\mu\nu} E^{\mu\nu} - h^{\mu\nu} \Phi_{\mu\nu} e^{-1} \partial_\rho (e h^{\rho\sigma} a_\sigma) + h^{\mu\rho} h^{\nu\sigma} \Phi_{\mu\nu} (\bar{\nabla}_\rho a_\sigma + a_\rho a_\sigma) + (h^{\rho\sigma} \bar{K}_{\rho\sigma})^2 \\ &\quad - h^{\rho\sigma} h^{\kappa\lambda} \bar{K}_{\rho\kappa} \bar{K}_{\sigma\lambda} + \bar{\nabla}_\mu [h^{\mu\rho} h^{\nu\sigma} (\bar{\nabla}_\rho \Phi_{\nu\sigma} - \bar{\nabla}_\nu \Phi_{\rho\sigma})], \quad (18) \\ E_h^{\alpha\beta} &= \left(h^{\mu\alpha} h^{\nu\beta} \Phi_{\mu\nu} - \frac{1}{2} h^{\alpha\beta} h^{\mu\nu} \Phi_{\mu\nu} \right) [e^{-1} \partial_\rho (e h^{\rho\sigma} a_\sigma) + E_{\tilde{\Phi}}] - \frac{1}{2} h^{\alpha\beta} \Phi_{\mu\nu} E^{\mu\nu} + h^{\mu\alpha} \Phi_{\mu\rho} E^{\rho\beta} + h^{\mu\beta} \Phi_{\mu\rho} E^{\rho\alpha} \\ &\quad - \frac{1}{2} h^{\rho\sigma} \Phi_{\rho\sigma} E^{\alpha\beta} + \tilde{\Phi} E^{\alpha\beta} - \frac{1}{2} h^{\alpha\beta} [(h^{\mu\nu} \bar{K}_{\mu\nu})^2 - h^{\mu\rho} h^{\nu\sigma} \bar{K}_{\mu\nu} \bar{K}_{\rho\sigma}] + \bar{\nabla}_\rho [\hat{\nu}^\rho h^{\mu\alpha} h^{\nu\beta} \bar{K}_{\mu\nu} - \hat{\nu}^\rho h^{\alpha\beta} h^{\mu\nu} \bar{K}_{\mu\nu}] \\ &\quad + h^{\mu\alpha} h^{\nu\beta} \bar{\nabla}_\mu \partial_\nu \tilde{\Phi} + h^{\mu\alpha} h^{\nu\beta} (a_\mu \partial_\nu \tilde{\Phi} + a_\nu \partial_\mu \tilde{\Phi}) - h^{\alpha\beta} h^{\mu\nu} \bar{\nabla}_\mu \partial_\nu \tilde{\Phi} - 2h^{\alpha\beta} h^{\mu\nu} a_\mu \partial_\nu \tilde{\Phi} \\ &\quad - \frac{1}{2} h^{\alpha\beta} h^{\mu\nu} h^{\rho\sigma} (\bar{\nabla}_\mu + a_\mu) (\bar{\nabla}_\rho + a_\rho) \Phi_{\nu\sigma} + h^{\mu\alpha} h^{\nu\beta} h^{\rho\sigma} (\bar{\nabla}_\rho + a_\rho) \left(\bar{\nabla}_{(\mu} \Phi_{\nu)\sigma} - \frac{1}{2} \bar{\nabla}_\sigma \Phi_{\mu\nu} \right) \\ &\quad + \frac{1}{2} h^{\alpha\beta} h^{\mu\nu} h^{\rho\sigma} (\bar{\nabla}_\mu + a_\mu) \bar{\nabla}_\nu \Phi_{\rho\sigma} - \frac{1}{2} h^{\mu\alpha} h^{\nu\beta} h^{\rho\sigma} \bar{\nabla}_\mu \bar{\nabla}_\nu \Phi_{\rho\sigma}. \quad (19) \end{aligned}$$

We only need to consider the variation $P_\mu^\alpha P_\nu^\beta \delta h^{\mu\nu}$ because we are only interested in the spatial projection of $\Phi_{\mu\nu}$. By taking the trace of $E_h^{\alpha\beta}$ and using $\hat{\nu}^\mu E_\mu$, we find

$$h^{\mu\nu} E_{\mu\nu}^h = -(d-2) \hat{\nu}^\mu E_\mu + \Phi_{\mu\nu} E^{\mu\nu} - (d-1) \left\{ \hat{\nu}^\mu \hat{\nu}^\nu \bar{R}_{\mu\nu} - (\bar{\nabla}_\mu + a_\mu) \left[h^{\mu\nu} a_\nu \left(\tilde{\Phi} + \frac{1}{2} h^{\rho\sigma} \Phi_{\rho\sigma} \right) - h^{\mu\nu} h^{\rho\sigma} a_\rho \Phi_{\nu\sigma} \right] \right\}, \quad (20)$$

where we used the identity

$$\hat{\nu}^\mu \hat{\nu}^\nu \bar{R}_{\mu\nu} = (h^{\mu\nu} \bar{K}_{\mu\nu})^2 - h^{\mu\rho} h^{\nu\sigma} \bar{K}_{\mu\nu} \bar{K}_{\rho\sigma} + 3h^{\mu\nu} a_\mu \partial_\nu \tilde{\Phi} + \bar{\nabla}_\mu (\hat{\nu}^\mu h^{\nu\rho} \bar{K}_{\nu\rho} + h^{\mu\nu} \partial_\nu \tilde{\Phi}) + 2\tilde{\Phi} e^{-1} \partial_\mu (e h^{\mu\nu} a_\nu). \quad (21)$$

Note that for $d\tau = 0$ the field $\Phi_{\mu\nu}$ decouples.

It can be shown that these equations agree with [11], where they were obtained by expanding the Einstein equations in $1/c^2$. However, [11] did not determine the equations of motion for $h^{\mu\alpha} h^{\beta\nu} \Phi_{\mu\nu}$, which we obtain by varying $\hat{\nu}^\mu$ and $h^{\mu\nu}$. These equations are essential in order to obtain a closed system of equations for the general case

$d\tau \neq 0$. Importantly, we note that our action allows for geometries with strong gravitational fields and, in particular, those with τ not closed allow for nonrelativistic gravitational time dilation.

Given the gravity action with type II TNC gauge invariance, we need to understand how matter couples to such a geometry. This will be discussed in [22], but as

remarked before this coupling will be markedly different than the known couplings of matter to type I TNC geometry [18,24–28]. One of the reasons is that, while in type I TNC geometry m_μ couples to the mass current, in type II it couples to the energy current. This will be further discussed in [22] by carefully studying the $1/c$ limit of the worldline action of a relativistic particle as well as the known couplings of Poincaré invariant field theories to pseudo-Riemannian geometry.

Here we will consider only the very special case of a static particle in order to obtain the Poisson equation from an action principle. A static point mass with mass density ρ has a Lagrangian that is simply $\mathcal{L}_m = \alpha e \rho$ with $\alpha = -[(d-2)/2]$. Taking the trace of $E^{\mu\nu}$ gives $\bar{h}_{\mu\nu} E^{\mu\nu} + (d-2)E_\Phi = -(d-1)e^{-1}\partial_\rho(eh^{\rho\sigma}a_\sigma)$. Varying $\mathcal{L} + \mathcal{L}_m$ tells us that the left-hand side vanishes and hence that $\partial_\mu(eh^{\mu\nu}a_\nu) = 0$. Since $\tau \wedge d\tau = 0$ we have that $h^{\mu\rho}h^{\nu\sigma}(\partial_\rho a_\sigma - \partial_\sigma a_\rho) = 0$, so that $h^{\mu\nu}a_\nu = h^{\mu\nu}\partial_\nu F$ for some function F . Hence $\partial_\mu(eh^{\mu\nu}a_\nu) = 0$ states that F is a harmonic function on the d -dimensional Riemannian geometry of the hypersurface to which τ is orthogonal. Regularity requires F to be constant and hence that $d\tau = 0$, as desired in Newtonian gravity which has absolute time. What survives from (20) is then the equation $(d-1)\hat{v}^\mu\hat{v}^\nu\hat{R}_{\mu\nu} = -(d-2)\hat{v}^\mu E_\mu - h^{\mu\nu}E_{\mu\nu}^h$. Then, taking into account the matter contribution to E_μ and $E_{\mu\nu}^h$ in this equation, for which we use the variation $\delta\mathcal{L}_m = \alpha e \rho(\tau_\mu\delta\hat{v}^\mu - \frac{1}{2}h_{\mu\nu}\delta h^{\mu\nu})$, it follows that the equations of motion of $\mathcal{L} + \mathcal{L}_m$ with $d\tau = 0$ are nothing else but Newton’s law (6).

Discussion.—Among the numerous avenues that one may pursue following our action and corresponding novel geometry, we mention a few. It would be interesting to (i) examine if there exists a geometric construction that gives type II TNC geometry from some Lorentzian starting point, just like type I follows from null reduction of a Lorentzian metric, (ii) perform a Hamiltonian analysis along with determining the asymptotic symmetries and examining the solution space of the theory, and (iii) work out how particles, strings, and branes probe type II TNC geometry and see if the equations of motion of the non-relativistic gravity action can be related to consistency conditions of some type of string theory (see [29–31] for nonrelativistic strings in the context of type I TNC geometry). Finally, there are undoubtedly also exciting applications in the realm of the AdS/CFT correspondence and generalizations thereof.

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*dehansen@phys.ethz.ch

†Jelle.Hartong@ed.ac.uk

‡obers@nbi.ku.dk

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