Mixed Axial-Torsional Anomaly in Weyl Semimetals

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We show that Weyl semimetals exhibit a mixed axial-torsional anomaly in the presence of axial torsion, a concept exclusive of these materials with no known natural fundamental interpretation in terms of the geometry of spacetime. This anomaly implies a nonconservation of the axial current—the difference in the current of left- and right-handed chiral fermions—when the torsion of the spacetime in which the Weyl fermions move couples with opposite sign to different chiralities. The anomaly is activated by driving transverse sound waves through a Weyl semimetal with a spatially varying tilted dispersion, which can be engineered by applying strain. This leads to a sizable alternating current in the presence of a magnetic field that provides a clear-cut experimental signature of our predictions.

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Introduction.—Weyl semimetals [1–10] are gapless three-dimensional topological materials whose low energy excitations are Weyl fermions. In field theory, Weyl fermions exhibit the chiral anomaly [11,12]: Their current in the presence of nonorthogonal electric $\vec{E}$ and magnetic $\vec{B}$ fields is not conserved at the quantum level. The continuity equation for the four-current $J^{\mu}_{L/R}$ of a single Weyl fermion of a given chirality (left or right handed) reads [13] $\partial_{\mu}J^{\mu}_{L/R} = \pm e\vec{E}\cdot\vec{B}/12\pi^2\hbar^2$. In the condensed-matter realization of Weyl semimetals, Weyl fermions necessarily come in pairs of opposite chirality [14]. Because of the chiral anomaly, their associated currents are not separately conserved; due to the freedom of having independent gauge fields coupling to each chirality, a cancellation of the chiral anomaly between the two chiralities does not necessarily happen, and conservation of the total vector current $J^{\mu} = J^{\mu}_L + J^{\mu}_R$ is not guaranteed [15]. Vector-current conservation is recovered by picking a specific, left-right asymmetric regularization of the underlying quantum field theory, which results in the axial current $J^{\mu}_5 = J^{\mu}_L - J^{\mu}_R$ being nonconserved [11,12,15–18]: $\partial_{\mu}J^{\mu}_5 = e\vec{E}\cdot\vec{B}/2\pi^2\hbar^2$. This is referred to as the axial anomaly (in the condensed-matter literature, it is often also called the chiral anomaly). Importantly, at the field theory level the chiral anomaly only forbids simultaneous conservation of the vector and axial currents, but it is natural to impose conservation of the vector current. The axial anomaly is predicted to result in a negative magnetoresistance in Weyl semimetals [19–22], which is experimentally observed [23–26].

The aforementioned freedom of independent gauge fields for each chirality means that the axial anomaly gets a contribution beyond the electromagnetic one. This occurs in the presence of axial gauge fields $A^5_{\mu} = (A^L_{\mu} - A^R_{\mu})/2$, which arise in helium-3 [27,28] and induced by strain or inhomogeneous magnetization [29,30] in Weyl semimetals. The axial fields couple to the two chiralities with opposite sign, and in analogy to electromagnetic fields give rise to the continuity equation for the axial current [15,31] $\partial_{\mu}J^{\mu}_5 = e\vec{E}^5\cdot\vec{B}^5/6\pi^2\hbar^2$, where the axial electric and magnetic fields $\vec{E}^5$ and $\vec{B}^5$ are obtained from A^5_{μ} analogous to their electromagnetic counterparts. This leads to alternative signatures for the axial anomaly in strained Weyl semimetals [32,33] and various other phenomena [34–42].

A further, less studied, contribution to the axial anomaly results from torsion of spacetime. An intuitive notion of torsion comes from its effect on vector fields: Vectors are twisted when parallel transported around a curve in a differential manifold with torsion [43]. While there is no experimental evidence for torsion in the spacetime of our Universe, extensions of general relativity that include torsion, such as the Einstein-Cartan theory [44], exist, as well as studies of their cosmological implications in the presence of chiral matter [45–50]. In condensed matter, torsion is, however, allowed and has been discussed in the context of Weyl semimetals [51–58], topological insulators [51,59,60], graphene [61], and helium-3 [62]. Since torsion affects spacetime, it influences the energy-momentum tensor, which for a single Weyl fermion cannot be jointly conserved with the electric current. This obstruction to a simultaneous conservation of energy-momentum and current is usually referred to as a mixed anomaly. However, as before, in the presence of pairs of opposite chirality Weyl fermions, one can impose both energy-momentum and current conservation at the cost of nonconservation of the axial current, which acquires torsional corrections beyond the (axial) electromagnetic contributions [63–67]—this is the mixed axial-torsional anomaly. Spacetime curvature additionally results in gravitational contributions to the axial anomaly [68].

A natural question arises: Is there, analogous to axial electromagnetic fields, a notion of axial-torsional fields, and if so, do they give rise to additional axial anomaly terms? At the face of it, the answer would seem to be no, since torsion is a property of spacetime, and as such, it does not know about chiralities, at least not at a fundamental level. However, we show in this work that in a material such as a Weyl semimetal, axial torsion is realized under the application of strain. We derive the resulting mixed axial-torsional anomaly with axial torsion and propose a realistic experimental setup that activates it. This work constitutes the first proposal for the realization and measurement of torsional contributions to the axial anomaly.

Mixed axial-torsional anomaly.—In a system consisting of a pair of left- and right-handed Weyl fermions, the torsional contribution to the axial anomaly reads

$$\partial_\mu J^\mu_5 = \frac{e}{16\pi^2 l^2}\epsilon^{\mu\nu\rho\lambda}\left(T^a_{\mu\nu}T^b_{\rho\lambda} + \frac{l^2}{l_5^2}T^{5,a}_{\mu\nu}T^{5,b}_{\rho\lambda}\right)\eta_{ab}, \quad (1)$$

where η_{ab} is the Minkowski metric, a, $\mu = t, x, y, z$, and $T^a_{\mu\nu}$ and $T^{5,a}_{\mu\nu}$ are the torsion and axial-torsion tensors, respectively, defined below. The first term in Eq. (1) is known as the Nieh-Yan term [69] and was derived [70] in Refs. [63–67]; the second axial-torsion term is new and is our main result. The derivation of the axial-torsion term proceeds similarly to that of the axial anomaly in the presence of axial gauge fields [15]: We start from the known expression [67] of the mixed axial-torsional anomaly of Weyl fermions and allow for axial torsion. This directly results in an apparent nonconservation of both axial and vector currents; to restore vector-current conservation, we depart from a left-right symmetric regularization by introducing Bardeen counterterms [71] resulting in Eq. (1). For details, see Ref. [72]. An important difference between electromagnetic and the torsional contributions to the axial anomaly is that, while the former are universal, the later are nonuniversal [73] and depend explicitly on the regularization through the cutoff length scales l and l_5. Moreover, different regularizations, still respecting current and energy-momentum conservation, characterized by additional Bardeen counterterms, change the coefficient of the axial-torsion term, such that even the ratio l/l_5 is nonuniversal.

To define the (axial) torsion tensor, we introduce a set of four orthonormal basis vectors $\underline{e}_a^\mu$, one for each spacetime component a, at each point of the manifold [43,74]. Being an orthonormal basis, the vectors fulfill $g_{\mu\nu}\underline{e}_a^\mu \underline{e}_b^\nu = \eta_{ab}$, where $g_{\mu\nu}$ is the (covariant) metric tensor. $\underline{e}_a^\mu$ is usually referred to as the frame field, and we define its inverse, the coframe field e^a_μ, such that $\underline{e}_a^\mu e^a_\nu = \delta^\mu_\nu$. In terms of these fields, the contravariant and covariant metrics are $g^{\mu\nu} = \underline{e}_a^\mu \underline{e}_b^\nu \eta^{ab}$ and $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. The torsion tensor is simply defined as the field strength of the coframe field $T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu$; in analogy with the electromagnetic field, we then define a set of four (one for each spacetime component a) torsional electric and magnetic fields $\mathcal{E}^a_i = \partial_t e^a_i - \partial_i e^a_t$ and $\mathcal{B}^a_i = \epsilon^{ijk}\partial_j e^a_k$, where $i = x, y, z$. Similarly, we define the axial-torsion tensor to be the field strength of the axial coframe field $e^{5,a}_\mu = (e^{L,a}_\mu - e^{R,a}_\mu)/2$, where we allow for the possibility that left- and right-handed fermions have different coupling to the background geometry described by the two distinct frame fields $e^{L/R,a}_\mu$. The axial-torsional electric and magnetic fields are then given by $\mathcal{E}^{5,a}_i = \partial_t e^{5,a}_i - \partial_i e^{5,a}_t$ and $\mathcal{B}^{5,a}_i = \epsilon^{ijk}\partial_j e^{5,a}_k$, and the anomaly Eq. (1) becomes $2\pi^2 l^2 \partial_\mu J^\mu_5 = e\vec{\mathcal{E}}_a \cdot \vec{\mathcal{B}}^a + e l^2/l_5^2 \vec{\mathcal{E}}^5_a \cdot \vec{\mathcal{B}}^{5,a}$. Activating the mixed axial-torsional anomaly therefore requires the presence of nonorthogonal (axial) torsional electric and torsional magnetic fields.

Weyl semimetals with spatially varying dispersion.—To put the anomaly Eq. (1) in the context of a specific system, we consider a minimal linear model of a Weyl semimetal consisting of two opposite chirality Weyl nodes separated in momentum space by a reciprocal vector $2K_i = (0, 0, 2K)$. The Hamiltonian for each chirality

$$\mathcal{H}_{L/R} = \frac{i\hbar v}{2}[\bar{\Psi}(\underline{e}_t^{L/R,i} \pm \sigma^j \underline{e}_j^{L/R,i})\partial_i\Psi - (\partial_i\bar{\Psi})(\underline{e}_t^{L/R,i} \pm \sigma^j \underline{e}_j^{L/R,i)})\Psi], \quad (2)$$

where σ^j are the Pauli matrices. The term $v\underline{e}_j^{L/R,i}$ is a generalized anisotropic Fermi velocity for each chirality, whereas $v\underline{e}_t^{L/R,i}$ tilts the Weyl cones [75–77]. The frame field notation naturally accounts for inhomogeneities by allowing the tilt and Fermi velocity to depend on space [78], which can be seen as a distortion of the geometry of the medium in which the Weyl fermions move [79–81]. It is precisely a frame field, as in Eq. (2) but without chirality dependence, which gives the coupling of Weyl fermions to the background geometry in the standard field theoretical formalism describing Weyl fermions in curved space [43,51,60]. In our model, in contrast, each chirality is allowed to couple differently to the geometry and there is no spin connection. The latter feature means that the inhomogeneous tilt and Fermi velocity are only equivalent to a distortion of space when the spacetime curvature vanishes; although not necessary for our results, all configurations we consider have vanishing curvature.

We turn to a possible specific realization of axial torsion. Take the boundary of the Weyl semimetal (2), with a tilt along the z direction and an isotropic and homogeneous Fermi velocity v. We model a boundary at $x = 0$ by the space-dependent Weyl-node-separation vector $K_z = K\Theta(x)$ and tilt $\underline{e}_t^{L/R,z} = \mp r\Theta(x)$, with r a constant. The gradient in K_z gives an axial magnetic field $B^5_y = \hbar K\delta(x)$ at the surface [82], whereas the tilt gradient gives the desired

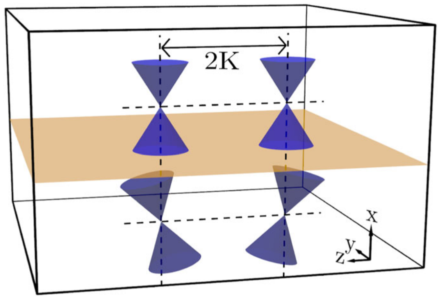

FIG. 1. Weyl semimetal heterostructure with a tilted interface. Two Weyl semimetal slabs are stacked along the x axis, both with two Weyl nodes separated by identical distance $2K$ along the z axis in reciprocal space. There is a finite inversion-symmetric tilt along the z direction in the lower slab, while the tilt vanishes in the upper one.

axial-torsional magnetic field $\mathcal{B}_y^{5,t} = r\delta(x)$. In order to isolate the torsional field, we take an interface between the above-defined tilted Weyl semimetal, and the same without the tilt, as represented in Fig. 1. The tilt gradient still gives rise to $\mathcal{B}_y^{t,5}$ at the interface, but, crucially, the absence of a gradient in the Weyl node separation means that the axial magnetic field vanishes, and there are no Fermi arcs at the interface. Such a stacked Weyl semimetal configuration will be used below for the activation of the axial-torsional anomaly and will be important to isolate the torsional contribution from the axial gauge field contribution. For practical purposes, it is enough that the torsional contribution dominates, so it is sufficient that the gradient of tilt is considerably bigger than the gradient of Weyl node separation, relaxing the strict condition of equal Weyl node separation across the interface.

Strain in Weyl semimetals.—In the model Hamiltonian (2), the microscopic origin of the inhomogeneous tilt and Fermi velocities was unspecified; we now argue that both arise from the application of strain. In the continuum limit, atomic displacements in a solid are captured by the displacement vector $u^i(x, y, z)$. An inhomogeneous displacement-vector field generates strain, and nonzero strain indicates that the spatial geometry of the elastic medium has been distorted. In fact, the change in the spatial components of the metric is given in terms of the displacement vector [83] as $g_{ij} = \delta_{ij} + 2u_{ij}$, with u_{ij} the strain tensor $u_{ij} = 1/2(\partial_i u_j + \partial_j u_i)$.

While such a strain-based elasticity theory is quite useful, it is not general enough to model all effects generated by the coupling of spin-orbit coupling to geometric deformations; the more fundamental frame field $\underline{e}_a^\mu$ is required [60]. From a lattice point of view, the frame is a set of four vectors residing on each lattice site at any given time, encoding the local bond stretching through their spatial lengths and the local orbital orientation through their relative angles. Importantly, while the metric does not capture local orbital deformations, the frame field does. This modification of elasticity theory is related to micropolar or "Cosserat" elasticity [84]. To first order in the displacement vector, the frame and coframe fields are given by [60] $\underline{e}_a^i = \delta_a^i - \delta_{ak}\partial^i u^k$ and $e_i^a = \delta_i^a + \delta_k^a \partial_i u^k$.

The presence in a Weyl semimetal of the vector scale K_i that gives the node separation implies that the above expression for the frame field is not sufficient to encode all strain effects. Symmetry arguments [85] and tight-binding derivations [29,35] entail two strain terms in the Hamiltonian, constructed by contracting the strain tensor with K_i: a pseudoscalar $C\hbar v K_j u^{ij} k_i$, with k_i the momentum, and a pseudovector (or axial vector) $A_i^5 = \beta\hbar K_j u_i^j$, with C and β model-dependent constants. Since C has units of length, we write $C = \gamma a$, where a is the typical lattice spacing and γ is a dimensionless model-dependent parameter; β is a dimensionless constant that in tight-binding calculations [29] is equal to the Grüneisen parameter, which is a measure of a crystal's sensitivity to strain. The "pseudo" prefix implies that these terms couple with opposite sign to the two chiralities. The pseudoscalar term contributes to $\underline{e}_0^i$ and tilts the Weyl cones, whereas the pseudovector term acts as an axial gauge field called the elastic gauge field [86]. Hence, the strained system is described by the Hamiltonian (2), fixing the Weyl node separation to $2K_i = (0, 0, 2K)$, with the modified frame field

$$\underline{e}_a^{L/R,i} = \delta_a^i - \delta_{ak}\partial^i u^k \mp \gamma a K \delta_a^t \delta_k^i u_z^k, \tag{3}$$

coframe field

$$e_i^{L/R,a} = \delta_i^a + \delta_k^a \partial_i u^k \pm \gamma a K \delta_t^a \delta_i^k u_k^z, \tag{4}$$

and minimal axial coupling to the elastic gauge field

$$\hbar\partial_i \to \hbar\partial_i \pm iA_i^5, \qquad A_i^5 = \hbar\beta K u_i^z. \tag{5}$$

Realizing the mixed axial-torsional anomaly.—Having established a possible microscopic origin for the chiral frame fields, we return to the heterostructure of Fig. 1, with constant Weyl node separation but varying tilt. One way to achieve this is to stack (here in the x direction) two Weyl semimetals with vanishing tilt but different Weyl node separation (here in the z direction). To make the Weyl node separation similar in the two samples, we apply an uniaxial strain $u^z = \alpha z$ to one side, where $\alpha = \Delta L/L$ measures the elongation of the crystal, modifying the Weyl node separation to $K \to K + A_z^5/\hbar = K(1 + \beta\alpha)$. At the same time, the strained sample is tilted in the z direction, resulting in a tilt gradient across the interface: $\underline{e}_t^{L/R,z} = \mp\gamma a K u_z^z = \mp\, \gamma a K \alpha \Theta(x)$. Alternatively, the Weyl node separation can be tuned with a magnetic field through the Zeeman term [87]. Although the above procedure can be generalized to Weyl semimetals with multiple Weyl nodes, it may be technically challenging. Encouragingly, proposals for minimal time-reversal breaking Weyl semimetals with a single pair of Weyl nodes in magnetic Heusler alloys have been put forward [88]; these would be ideal for realizing the tilted interface just described.

The tilt gradient through the interface generates an axial-torsional magnetic field $\mathcal{B}_y^{5,t} = \gamma a K \alpha \delta(x)$, while the axial magnetic field generated by spatially varying node separation vanishes. To activate the anomaly, we additionally need a torsional axial electric field. This can be achieved by a displacement-vector component $u^y(z,t)$, which can be realized by driving transverse sound waves through the crystal, resulting in $u^y(z,t) = u_0 \sin(k_s z - \omega t)$, with $k_s = \omega/c_s$ the wave number, ω the frequency, and c_s the sound velocity. Such a displacement vector gives rise to $\mathcal{E}_y^{5,t} = \gamma a K u_0 \omega^2 \sin(k_s z - \omega t)/c_s$. It also gives rise to an axial electric field E_y^5, but since the axial magnetic field vanishes, the axial gauge field contribution to the anomaly vanishes. The mixed axial-torsional anomaly is therefore the only anomaly contribution in this setup.

We solve the torsional anomaly equation for the axial charge density assuming $\partial_i J_5^i = 0$. In the presence of intervalley scattering with scattering time τ_v [89], the anomaly equation takes the form $\partial_t n_5 = \vec{\mathcal{E}}_a^5 \cdot \vec{\mathcal{B}}^{5,a}/2\pi^2 l_5^2 - n_5/\tau_v$, where $n_5 = J_5^0/e$ is the axial number density. Inserting the explicit form of the torsional fields and solving for the density in the limit where the phonon frequency is much larger than the intervalley scattering rate $\omega\tau_v \gg 1$, we get, at long times $t \gg \tau_v$,

$$n_5 = -\frac{\gamma^2 K^2 \alpha u_0 \omega}{2\pi^2 c_s} \cos(k_s z - \omega t)\delta(x). \tag{6}$$

In arriving at Eq. (6), we have taken the cutoff length scale $l_5 = a$ equal to the lattice spacing, the physical cutoff length scale of the crystal. Notice that the (co)frame field vanishes outside the material, and therefore, its total flux through the sample must vanish $\int dx dz \mathcal{B}_y^{5,t} = 0$ [21]. Consequently, there must be a contribution to the anomaly localized at the lower surface (Fig. 1), where the tilt gradient has a sign opposite to that at the interface, such that the total (spatially integrated) axial number density is conserved [90].

Experimental detection.—To experimentally detect the spacetime oscillating axial charge (6), we make use of the chiral magnetic effect [91,92]. Because of this effect, a magnetic field in the z direction [93] induces a current $J_{\mathrm{CME}}^z = e^2 \mu_5 B/2\pi^2\hbar^2$ parallel to the applied magnetic field, where $\mu_5 = (\mu_L - \mu_R)/2$ is the axial chemical potential. In the weak field limit $\hbar e B \ll \mu_5^2/v^2$, n_5 is related to μ_5 according to $3\pi^2\hbar^3 v^3 n_5 = \mu_5^3 + \mu_5(\pi^2\kappa_B^2 T^2 + \mu^2)$, where T and μ are the temperature and chemical potential [91]. In the realistic limit $\mu, \kappa_B T \gg \mu_5$ we can drop the μ_5^3 term. Crucially, the chiral magnetic current J_{CME}^z being proportional to the axial chemical potential is only nonzero if the axial anomaly is activated.

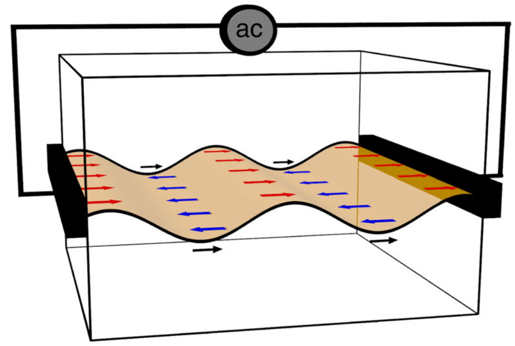

FIG. 2. The mixed axial-torsional anomaly is activated by driving transverse phonons through the Weyl semimetal heterostructure presented in Fig. 1. The subsequent application of a magnetic field in the z direction (horizontal axis in the picture) yields a charge density wave (CDW) at the tilted interface. The sinusoidal pattern represents the magnitude of the charge at each point in the interface. The red and blue arrows represent the direction of the current where the CDW amplitude has its maxima and minima, respectively, while the black arrows point towards the direction of propagation of the CDW. Two leads are placed at each side of the interface, such that the propagating CDW generates an alternating current through the circuit.

The two-dimensional chiral magnetic current density at the interface between the two Weyl semimetals, using Eq. (6), now reads

$$J = \int dx J_{\mathrm{CME}}^z = -\frac{3\hbar^3 v^3 \gamma^2 K^2 \alpha u_0 \omega B}{2c_s(\pi^2\kappa_B^2 T^2 + \mu^2)} \cos(k_s z - \omega t). \tag{7}$$

Because of current conservation $\partial_\mu J^\mu = 0$, this generates a propagating charge density wave with charge density $\rho = J^0 = J/c_s$. The mixed axial-torsional anomaly can then be tested by measuring the ac current J flowing through the current leads placed at opposite sides of the interface (Fig. 2). For typical values $v = 10^6$ m/s, $K = 10^9$ m^{-1}, $\gamma = 1$, $\mu = 10$ meV, for strain of 3%, phonon amplitude $u_0 = a/10$, with lattice constant $a = 5$ Å, sound speed $c_s = 2 \times 10^3$ m/s, and driving frequency $\omega = 1$ THz, we estimate the amplitude of the high frequency ac current at room temperature to be $J = 0.9B$ A/m, with B in Tesla. For a magnetic field of $B = 10$ mT, fulfilling the weak field condition, we obtain $J = 9$ mA/m. Assuming a 5 μm wide sample, the total current amplitude is then $J = 45$ nA, well within current experimental range. The linear dependence of the current amplitude on the magnetic field serves to distinguish it from noise.

Discussion.—In this work, we demonstrated that a hitherto overlooked anomaly, the mixed axial-torsional anomaly with axial torsion, is naturally realized in a condensed-matter setting. In particular, we demonstrated that this anomaly should be detectable within the current experimental capabilities by driving transverse sound waves through tilted Weyl semimetal interfaces and measuring the induced alternating currents in the presence of an external magnetic field. For driving phonon frequencies in the THz regime and a small value of the magnetic field of 10 mT, we predicted a current amplitude of around 40 nA.

Our treatment of the axial-torsional anomaly with axial torsion opens the path for a more in-depth study of anomaly-induced torsional responses in Weyl systems. On the experimental and phenomenological side, other implementations of inhomogeneities, such as magnetization in magnetic Weyl semimetals, which could give rise to nonvanishing torsion, are worth exploring. Another appealing direction is to extend the methodology presented here to study the realization of a mixed axial-gravitational anomaly to come up with a realization of axial curvature and study how it modifies the gravitational contributions to the anomaly. All these effects could, in principle, be engineered in Weyl semimetals.

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