Nonclassicality as a Quantifiable Resource for Quantum Metrology

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(Received 16 May 2018; revised manuscript received 27 July 2018; published 1 February 2019)

We establish the nonclassicality of continuous-variable states as a resource for quantum metrology. Based on the quantum Fisher information of multimode quadratures, we introduce the metrological power as a measure of nonclassicality with a concrete operational meaning of displacement sensitivity beyond the classical limit. This measure belongs to the resource theory of nonclassicality, which is nonincreasing under linear optical elements. Our Letter reveals that a single copy, highly nonclassical quantum state is intrinsically advantageous when compared to multiple copies of a quantum state with moderate nonclassicality. This suggests that metrological power is related to the degree of quantum macroscopicity. Finally, we demonstrate that metrological resources useful for nonclassical displacement sensing tasks can be always converted into a useful resource state for phase sensitivity beyond the classical limit.

DOI: 10.1103/PhysRevLett.122.040503

Recognizing the differences between classical and quantum physics has changed our viewpoint of nature, while developments in quantum information theory have shown that these differences lead to quantum advantages in informational tasks [1–7]. Nonclassicality can be defined by the negativity of the Glauber-Sudarshan P representation in the context of light fields [8–10]. An N-mode continuous-variable state $\hat{\rho}$ can be represented as

$$\hat{\rho} = \frac{1}{\pi^N} \int d^{2N}\alpha P_{\hat{\rho}}(\boldsymbol{\alpha}) |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|,$$

where $P_{\hat{\rho}}(\boldsymbol{\alpha})$ is the P function and the set of coherent states forms an overcomplete basis $|\boldsymbol{\alpha}\rangle = \otimes_{n=1}^{N} |\alpha_n\rangle$ in the corresponding Hilbert space. As coherent states are considered to be the most classical states among all pure states [8,9,11,12], a nonclassical quantum state, which cannot be represented as a statistical mixture of coherent states, should contain negativity in its P function [10].

A diverse range of studies have been performed to characterize nonclassicality [13–19], as well as its relationship to entanglement [20–22] and quantum communications [23,24]. For the quantification of nonclassicality, various approaches have been suggested, including distance-based measures [25,26], nonclassicality depth [27], entanglement potential [21,22], characteristic function methods [28], and operational approaches [29,30]. Recently, the nonclassicality based on the negativity of the P function was investigated using the resource theory of coherence [31]. The orthogonalization process suggested in Ref. [31] successfully unifies the old notion of nonclassicality [8–10] and the new concept of coherence [32] in the coherent-state basis. Emerging from this characterization is a resource theory of nonclassicality based on linear optics, where the set of classical operations are naturally chosen as linear optical operations. The challenge is then to find a quantifier of nonclassicality based on the resource theory that possesses a clear operational significance, paralleling the developments in the entanglement [33] and coherence [32,34] theories. It has been found that, in metrological tasks, nonclassicality rather than entanglement is a necessary resource to achieve quantum advantages [35–37], while the operational meaning of nonclassicality was very recently studied based on the quadrature fluctuations in a similar vein [38,39].

In this Letter, we demonstrate that the nonclassicality of a continuous-variable state is a quantifiable resource for parameter estimation tasks. We show that the mean quadrature variance captures every pure-state nonclassicality, and its convex roof construction is a strict measure of nonclassicality. Extending this concept, we introduce the metrological power to quantify nonclassical resources that lead to quantum enhancement in displacement metrology, given in the form of the quantum Fisher information (QFI). We prove that this quantifier witnesses the negativity of the P function and does not increase by linear optical operations, so that it belongs to the family of monotones within the resource theory of nonclassicality. In addition, it is shown that a collection of many small-size nonclassical states cannot achieve a large degree of nonclassicality; this is consistent with the notion of quantum macroscopicity [40–42]. Interestingly, nonclassical resources for displacement sensing can always be converted into a useful resource for phase sensing tasks using linear optical operations. Our Letter provides a concrete operational meaning for the nonclassicality of a continuous-variable state as a potential resource for quantum metrology that can be quantified by a computable measure.

Resource theory of nonclassicality.—We define a resource theory of nonclassicality based on Ref. [31]. Consider a linear optical unitary for the N-mode bosonic system belonging to the $O(2N)$ rotation group of the quadratures $\hat{\boldsymbol{R}} \coloneqq (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_N, \hat{p}_N)^T$, in addition to the displacement operation $\hat{D}_n(\alpha_n) = \exp[\alpha_n \hat{a}_n^\dagger - \alpha_n^* \hat{a}_n]$. Such a unitary transforms a multimode bosonic operator $\hat{a}_{\boldsymbol{\mu}}^\dagger \coloneqq \sum_{n=1}^N \mu_n \hat{a}_n^\dagger$ into $\hat{a}_{\boldsymbol{\mu}'}^\dagger + \bigoplus_{n=1}^N \alpha_n \mathbb{1}_n$, where $\boldsymbol{\mu} \coloneqq (\mathrm{Re}[\mu_1], \mathrm{Im}[\mu_1], \mathrm{Re}[\mu_2], \mathrm{Im}[\mu_2], \ldots, \mathrm{Re}[\mu_N], \mathrm{Im}[\mu_N])^T$ is a real $2N$-dimensional unit vector and $\alpha_n \mathbb{1}_n$ corresponds to the displacement on the nth mode. Consequently, a mutimode quadrature operator can be defined as $\hat{X}_{\boldsymbol{\mu}} \coloneqq (\hat{a}_{\boldsymbol{\mu}} + \hat{a}_{\boldsymbol{\mu}}^\dagger)/\sqrt{2} = \hat{\boldsymbol{R}}^T \boldsymbol{\mu}$. Using linear optical unitary operations, we define a linear optical map

$$\Phi_L(\hat{\rho}_A) \coloneqq \mathrm{Tr}_E[\hat{U}_L(\hat{\rho}_A \otimes \hat{\sigma}_E)\hat{U}_L^\dagger],$$

where $\hat{\sigma}_E$ is a classical state (see Fig. 1), as a free operation since it maps every classical state into another classical state. A selective linear operation can be defined by a set of Kraus operators $\{\hat{K}_i\}$ when there exists $\hat{U}_L$, classical ancilla $\hat{\sigma}_{EE'}$, and a set of orthogonal vectors $\{|i\rangle_{E'}\}$ such that $\mathrm{Tr}_E[\hat{U}_L(\hat{\rho}_A \otimes \hat{\sigma}_{EE'})\hat{U}_L^\dagger] = \sum_i p_i \hat{\rho}_A^i \otimes |i\rangle_{E'}\langle i|$, where $p_i \hat{\rho}_A^i \coloneqq \hat{K}_i \hat{\rho}_A \hat{K}_i^\dagger$ and $p_i \coloneqq \mathrm{Tr}(\hat{K}_i \hat{\rho}_A \hat{K}_i^\dagger)$. One might expect that a complete set of classicality preserving maps could be expressed in the form of dilations of a linear optical unitary with classical ancilla, but this is not the case. We note that a classicality preserving map $\Lambda: \hat{\rho} \to \int (d^2\alpha/\pi) Q_{\hat{\rho}}(\alpha)|\alpha\rangle\langle\alpha|$, where $Q_{\hat{\rho}}(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle$ is the Husimi Q function, is not a linear optical map [43] since it involves a metaplectic unitary corresponding to two-mode squeezing [44]. Nevertheless, a set of linear optical maps serves as an important class of operations that can be relatively easily performed in laboratories, compared to nonlinear operations such as squeezing.

In this framework, nonclassicality for a pure state $|\psi\rangle$ can be quantified by the mean quadrature variance

$$\bar{\mathcal{V}}(|\psi\rangle) \coloneqq \frac{1}{N} \sum_{k=1}^{2N} \mathrm{Var}(\psi, \hat{R}^{(k)}), \tag{1}$$

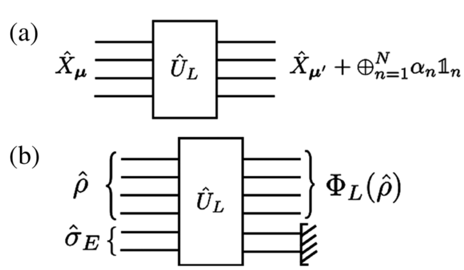

FIG. 1. (a) Linear optical unitary and (b) linear optical map.

where $\mathrm{Var}(\psi, \hat{O}) \coloneqq \langle\psi|\hat{O}^2|\psi\rangle - \langle\psi|\hat{O}|\psi\rangle^2$ and $\hat{R}^{(k)}$ is the kth element of $\hat{\boldsymbol{R}}$. It is important to note that $\bar{\mathcal{V}} \geq 1$, and the equality holds if and only if the state is a coherent state. We extend this measure to quantify the nonclassicality of a mixed state by taking the convex roof

$$\mathcal{Q}(\hat{\rho}) \coloneqq \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \bar{\mathcal{V}}(|\psi_i\rangle) - 1, \tag{2}$$

where $\{p_i, |\psi_i\rangle\}$ is a pure-state decomposition of $\hat{\rho}$. We show that $\mathcal{Q}$ is a faithful measure of nonclassicality [31].

Theorem 1.—$\mathcal{Q}$ is a nonclassicality measure satisfying the following conditions. (1) $\mathcal{Q}(\hat{\rho}) = 0$ if and only if $\hat{\rho}$ is classical. (2) (a) (Weak monotonicity) $\mathcal{Q}(\hat{\rho}) \geq \mathcal{Q}[\Phi_L(\hat{\rho})]$. (b) (Strong monotonicity) $\mathcal{Q}(\hat{\rho}) \geq \sum_i p_i \mathcal{Q}(\hat{\rho}_i)$ where $p_i \coloneqq \mathrm{Tr}(\hat{K}_i^\dagger \hat{K}_i \hat{\rho})$ and $\hat{\rho}_i \coloneqq (\hat{K}_i \hat{\rho} \hat{K}_i^\dagger)/p_i$. (3) (Convexity), i.e., $\mathcal{Q}(\sum_i p_i \hat{\rho}_i) \leq \sum_i p_i \mathcal{Q}(\hat{\rho}_i)$.

We note that the value of nonclassicality is bounded by $\mathcal{Q}(\hat{\rho}) \leq 2(\bar{n}/N)$, where $\bar{n} \coloneqq \mathrm{Tr}[\sum_{n=1}^N \hat{a}_n^\dagger \hat{a}_n \hat{\rho}]$ is the mean photon number. The upper bound saturates in the case of pure states if and only if $\langle\psi|\hat{R}^{(k)}|\psi\rangle = 0$ for every k, a condition which holds for, e.g., Fock states, cat states, or squeezed coherent states. Another interesting point is that $N(\bar{\mathcal{V}} - 1)$ is equivalent to the phase space macroscopicity measure proposed in Ref. [45]. Thus, $\mathcal{Q}$ can be understood as the convex roof extension of the macroscopicity measure per mode.

Nonclassicality and metrological power.—We now establish the relationship between the quadrature variance and the displacement sensitivity. Suppose that we want to estimate the parameter θ when a quantum state $\hat{\rho}$ is displaced into $\hat{\rho}_{\theta,\boldsymbol{\mu}} = e^{-i\theta \hat{X}_{\boldsymbol{\mu}}} \hat{\rho} e^{i\theta \hat{X}_{\boldsymbol{\mu}}}$. In this case, a tight bound for the variance of the estimator $(\Delta\theta)_{\boldsymbol{\mu}}^2$ by performing the optimal measurements on $\hat{\rho}_{\theta,\boldsymbol{\mu}}$ is given by the quantum Cramér-Rao bound [46]

$$(\Delta\theta)_{\boldsymbol{\mu}}^2 \geq \frac{1}{I_F(\hat{\rho}, \hat{X}_{\boldsymbol{\mu}})} = \frac{1}{\boldsymbol{\mu}^T \boldsymbol{F} \boldsymbol{\mu}}. \tag{3}$$

The QFI can be calculated as $I_F(\hat{\rho}, \hat{X}_{\boldsymbol{\mu}}) = 2\sum_{i,j}[(\lambda_i - \lambda_j)^2/\lambda_i + \lambda_j]|\langle i|\hat{X}_{\boldsymbol{\mu}}|j\rangle|^2$ by using the eigenvalue decomposition $\hat{\rho} = \sum_i \lambda_i |i\rangle\langle i|$, and $\boldsymbol{F}$ is the QFI matrix with real symmetric $2N \times 2N$ elements $F_{kl} = 2\sum_{i,j}[(\lambda_i - \lambda_j)^2/\lambda_i + \lambda_j] \langle i|\hat{R}^{(k)}|j\rangle\langle j|\hat{R}^{(l)}|i\rangle$. For a pure state, four times the variance is equal to the QFI, so that a large quadrature variance directly implies high displacement sensitivity. The QFI has been also studied to quantify multipartite entanglement [47–49] and macroscopic quantum coherence [42,50–52].

From the observation that $\boldsymbol{F}/2 = \mathbb{1}$ for every coherent state, $(\Delta\theta)_{\boldsymbol{\mu}}^2$ is lower bounded by $1/2$ for any classical state, so-called the standard quantum limit (SQL) for displacement metrology. The pure-state nonclassicality measure $\mathcal{Q}(|\psi\rangle) = \mathrm{Tr}[\boldsymbol{F}]/(4N) - 1$ has the meaning of the metrological advancement beyond the SQL, on average over all possible values of $\boldsymbol{\mu}$. For a mixed state, however, it is unknown if $\mathcal{Q}$ has a direct operational meaning in terms of quantum metrology, while $\mathrm{Tr}[\boldsymbol{F}]$ cannot fully capture nonclassicality of a mixed state when some eigenvalues of $\boldsymbol{F}$ are smaller than 2.

Nonetheless, we shall consider another quantifier of nonclassicality, the "metrological power," which has the concrete operational meaning of the maximal metrological advantage by performing a linear optical unitary with a vacuum ancilla,

$$\mathcal{M}(\hat{\rho}) := \frac{1}{2} \max_{\hat{\sigma}=\hat{U}_L(\hat{\rho}\otimes|0\rangle\langle 0|)\hat{U}_L^\dagger} I_F(\hat{\sigma}, \hat{x}_1) - 1$$
$$= \max\left\{\frac{\lambda_{\max}(\boldsymbol{F})}{2} - 1, 0\right\}, \quad (4)$$

where $\lambda_{\max}(\boldsymbol{F})$ is the maximum eigenvalue of $\boldsymbol{F}$. This quantifies the optimal sensitivity among all possible parametrizations since $\min_{\boldsymbol{\mu}}(\Delta\theta)^2_{\boldsymbol{\mu}} = [\lambda_{\max}(\boldsymbol{F})]^{-1}$, together with the fact that one can always find a linear optical unitary operator $\hat{U}_L$ such that $\hat{U}_L^\dagger e^{-i\theta\hat{X}_{\boldsymbol{\mu}}}\hat{U}_L = e^{-i\theta\hat{x}_1}$, and displacement operations do not change the quadrature QFI. We show the following useful properties of $\mathcal{M}$.

Theorem 2.—The metrological power $\mathcal{M}$ satisfies the following properties: (1) $\mathcal{M} \geq 0$, and $\mathcal{M} = 0$ for every classical state. For a pure state, $\mathcal{M} = 0$ if and only if the state is a coherent state. (2) $\mathcal{M}$ is invariant under linear optical unitaries $\hat{U}_L$ and a monotone under linear optical maps Φ_L. (3) $\mathcal{M}$ is convex. (4) $\mathcal{M}(\hat{\rho}_A \otimes \hat{\sigma}_B) = \max\{\mathcal{M}(\hat{\rho}_A), \mathcal{M}(\hat{\sigma}_B)\}$.

The first property shows that every pure quantum state except coherent states outperforms all classical states in terms of the metrological power. For a mixed state, this quantifier can witness nonclassicality whenever $\mathcal{M} > 0$, although there can exist nonclassical states having $\mathcal{M} = 0$. This is, however, offset by the computational advantages and operational interpretation of $\mathcal{M}$. The metrological power also satisfies monotonicity and convexity, which are necessary conditions for nonclassicality monotones. The last property fulfills one of the proposed requirements to quantify genuine quantum macroscopicity: the accumulation microscopic quantum coherence should be distinguished from the genuine macroscopic coherence [40].

Similar quantum macroscopicity measures for optical systems have been proposed [52–54] based on the QFI, for instance, the quantity $\max_{\{\phi_n\}} I_F(\hat{\rho}, \hat{X}_{\{\phi_n\}})/N$ using the sum of quadratures $\hat{X}_{\{\phi_n\}} = \sum_{n=1}^N [\cos\phi_n \hat{x}_n + \sin\phi_n \hat{p}_n]$. In this case, however, we point out that a linear optical unitary can increase the measure, since $\hat{X}_{\{\phi_n\}}$ in general does not transform in a covariant way, i.e., $\hat{U}_L^\dagger \hat{X}_{\{\phi_n\}}\hat{U}_L \neq \hat{X}_{\{\phi'_n\}}$. Thus, measures of this type do not belong to nonclassicality monotones, although they capture many useful properties of quantum macroscopicity. It is worth noting that utilizing the quadrature QFI to characterize nonclassicality was recently studied with a slightly different set of free operations [38].

Examples.—We first observe that both the Fock state $|n\rangle$ and NOON state $|n\rangle|0\rangle + |0\rangle|n\rangle$ give $\mathcal{Q} = 2\bar{n}/N$ and $\mathcal{M} = 2\bar{n}$. A cat state $|\alpha\rangle \pm |-\alpha\rangle$ gives $\mathcal{Q} = 2\bar{n}$ and $\mathcal{M} = 2(\bar{n} + |\alpha|^2)$, while a decohered cat state $\hat{\rho}_\Gamma = N_\Gamma^{-1}[|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha| + \Gamma(|\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha|)]$ gives $\mathcal{M}(\hat{\rho}_\Gamma) = \max\{[16|\alpha|^2/N_\Gamma^2]\Gamma(\Gamma + e^{-2|\alpha|^2}), 0\}$, where $N_\Gamma = 2 + 2\Gamma e^{-2|\alpha|^2}$. A decohered even cat state with positive Γ is nonclassical unless $\Gamma = 0$, while a decohered odd cat state with negative Γ can be nonclassical when $\mathcal{M} = 0$. Because of invariance under linear optical unitary operations, nonclassicality between different modes can also be fairly compared throughout our measure. For example, $\mathcal{M}$ for an entangled coherent state $|\alpha\rangle|\alpha\rangle \pm |-\alpha\rangle|-\alpha\rangle$ is equivalent to a single-mode cat state with an amplitude $\sqrt{2}\alpha$ since they are interconvertible via a 50:50 beam splitter. This can be extended to a multimode entangled coherent state $|\alpha_1\rangle|\alpha_2\rangle\cdots|\alpha_N\rangle \pm |-\alpha_1\rangle|-\alpha_2\rangle\cdots|-\alpha_N\rangle$, which is convertible into $(|\gamma\rangle \pm |-\gamma\rangle)|0\rangle\cdots|0\rangle$ via beam splitter operations, where $|\gamma| = \sqrt{\sum_{n=1}^N |\alpha_n|^2}$.

We apply our result to a multimode Gaussian state characterized by its mean value $\boldsymbol{d}$ with $d_k = \mathrm{Tr}[\hat{\rho}\hat{R}^{(k)}]$ and the covariance matrix $\boldsymbol{V}$ with $V_{kl} = \mathrm{Tr}[\hat{\rho}\{\hat{R}^{(k)} - d_k, \hat{R}^{(l)} - d_l\}]$, where $\{\hat{A}, \hat{B}\} := \hat{A}\hat{B} + \hat{B}\hat{A}$. The symplectic transform of $\boldsymbol{V}$ and corresponding symplectic matrix $\boldsymbol{S}$ then always exist. This allows us to decompose every Gaussian state into single-mode squeezing combined with linear optical operations acting on the product of thermal states [55]. In this case, the following closed form formula is obtained: $\mathcal{M} = \max\{\lambda_{\max}[\boldsymbol{S}^{-1}\boldsymbol{S}^T\boldsymbol{V}^{-1}\boldsymbol{S}(\boldsymbol{S}^{-1})^T] - 1, 0\}$. Especially for a single-mode Gaussian state $\hat{D}(\alpha)\hat{S}(\xi)\hat{\tau}\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)$ with $\hat{S}(\xi) = \exp[(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2)/2]$ and $\hat{\tau} = \sum_{n=0}^\infty \bar{n}_{\mathrm{th}}^n/(1+\bar{n}_{\mathrm{th}})^{(n+1)}|n\rangle\langle n|$, a direct relationship between nonclassicality and squeezing [56] can be derived as $\mathcal{M} = e^{2G(\boldsymbol{V})} - 1 = \max\{\exp(2|\xi|)/(2\bar{n}_{\mathrm{th}} + 1) - 1, 0\}$, where $G(\boldsymbol{V}) := \inf[-\sum_{i=1}^N \log s_i^\downarrow(\boldsymbol{S})|\boldsymbol{V} \geq \boldsymbol{S}^T\boldsymbol{S}]$ with $s_i^\downarrow(\boldsymbol{S})$ being singular values of $\boldsymbol{S}$ in decreasing order. This observation also leads to the following corollary.

Corollary.—The metrological power $\mathcal{M}$ is zero if and only if a single-mode Gaussian state is classical.

Similar to the case of entangled coherent states, the metrological power of two-mode and single-mode squeezed states can be equivalently compared as they are interconvertible by using the beam splitter. Figure 2 shows $\mathcal{Q}$ and $\mathcal{M}$ for various types of quantum states.

Quantum phase estimation assisted by linear optical unitaries.—We discuss how a nonclassical resource for the displacement metrology can be utilized in phase estimation tasks beyond the classical limit. Quantum phase estimation aims to measure the relative phase of a chosen mode of a

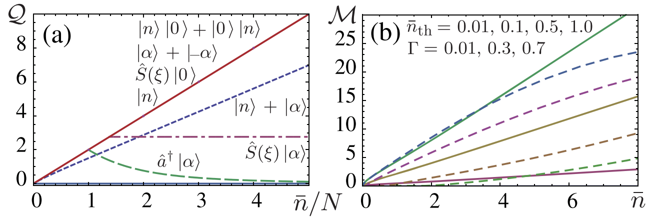

FIG. 2. (a) Nonclassicality measure $\mathcal{Q}$ achieves the maximum value (solid line) for NOON, cat, squeezed, and Fock states. Also, superposition between Fock state and coherent state $|n\rangle + |\alpha\rangle$ with $n = |\alpha|^2$ (dotted line), squeezed coherent states $\hat{S}(\xi)|\alpha\rangle$ for $\xi = 1$ (dot-dashed line), and photon-added coherent states $\hat{a}^\dagger|\alpha\rangle$ (dashed line) are evaluated. (b) Metrological power $\mathcal{M}$ for decohered cat states $\hat{\rho}_\Gamma$ (solid lines) and squeezed thermal states $\hat{S}(\xi)\hat{\tau}\hat{S}^\dagger(\xi)$ (dashed lines) with the parameters $\Gamma = 0.01, 0.3, 0.7$ and $\bar{n}_{\mathrm{th}} = 0.01, 0.1, 0.5, 1.0$ (both starting from above).

interferometer whose dynamics is given by $e^{-i\theta\hat{a}^\dagger\hat{a}}\hat{\rho}e^{i\theta\hat{a}^\dagger\hat{a}}$. The sensitivity of the phase estimation task is bounded by $(\Delta\theta)^2_{\mathrm{phase}} \geq I_F(\hat{\rho}, \hat{a}^\dagger\hat{a})^{-1}$. It was shown [29] that a nonclassical quantum state can be identified whenever the QFI is larger than four times the mean photon number $I_F(\hat{\rho}, \hat{a}^\dagger\hat{a}) > 4\mathrm{Tr}[\hat{\rho}\hat{a}^\dagger\hat{a}]$, where the SQL for the phase metrology can be considered as $I_F(\hat{\rho}, \hat{a}^\dagger\hat{a}) \leq 4\mathrm{Tr}[\hat{\rho}\hat{a}^\dagger\hat{a}]$. Although this condition is useful to witness nonclassicality, we highlight that it is not sufficient to detect every nonclassical pure state. For example, the Fock state $|n\rangle$ is obviously nonclassical for $n > 0$, but $I_F(|n\rangle, \hat{a}^\dagger\hat{a}) = 0$.

In order to overcome this problem, we optimize the phase sensitivity over linear optical unitaries, analogously to the displacement metrology. However, we should additionally take into account that displacement can increase the phase estimation sensitivity even for classical states as $I_F(|\alpha\rangle, \hat{a}^\dagger\hat{a}) \propto |\alpha|^2$ for a coherent state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$. It is therefore necessary to characterize the linear optical unitaries according to the degree of displacement. This can be done by decomposing a linear optical unitary into $\hat{U}_L^\alpha \coloneqq [\bigotimes_{n=1}^N \hat{D}_n(\alpha_n)]\hat{U}_L^0$ with $|\alpha|^2 = \sum_{n=1}^N |\alpha_n|^2$, and $\hat{U}_L^0$ is a linear optical unitary without any displacement. We then define the α-invested metrological power for phase estimation as

$$\mathcal{M}^\alpha_{\mathrm{phase}}(\hat{\rho}) \coloneqq \max_{\hat{\sigma}=\hat{U}_L^\alpha(\hat{\rho}\otimes|0\rangle\langle 0|)\hat{U}_L^{\alpha\dagger}} \left(\frac{I_F(\hat{\sigma}, \hat{a}_1^\dagger\hat{a}_1)}{4} - \mathrm{Tr}[\hat{\sigma}\hat{a}_1^\dagger\hat{a}_1]\right), \tag{5}$$

where $\mathcal{M}^\alpha_{\mathrm{phase}} \geq \mathcal{M}^\beta_{\mathrm{phase}} \geq 0$ for $|\alpha| \geq |\beta|$ and $\mathcal{M}^\alpha_{\mathrm{phase}} = 0$ for every classical state; thus the phase sensitivity beyond the SQL ($\mathcal{M}^\alpha_{\mathrm{phase}} > 0$) directly captures the negativity in the P distribution. Additionally, $\mathcal{M}^\alpha_{\mathrm{phase}}$ enjoys convexity and is invariant under $\hat{U}_L^0$. We demonstrate a remarkable relationship between the displacement and phase metrological powers.

Theorem 3.—Provided $\mathcal{M}(\hat{\rho}) > 0$, there exists a linear optical unitary $\hat{U}_L^\alpha$ to reach the sensitivity beyond the SQL for phase estimation, i.e., $\mathcal{M}^\alpha_{\mathrm{phase}}(\hat{\rho}) > 0$.

In particular, $\mathcal{M} \leq \lim_{|\alpha|\to\infty}[\mathcal{M}^\alpha_{\mathrm{phase}}/|\alpha|^2] \leq \mathcal{M}+1$ ([57], see also the Supplemental Material [58]), which can be intuitively understood by the fact that a large displacement followed by a small rotation can be approximated by two sequential displacement operations in orthogonal directions. Another important figure of merit for phase metrology is the scaling behavior with the mean photon number $\bar{n}$. In phase estimation, the classical limit with coherent states is $\Delta\theta_{\mathrm{cl}} \propto 1/\sqrt{\bar{n}}$, while quantum states can achieve the sensitivity of $\Delta\theta_{\mathrm{HS}} \propto 1/\bar{n}$, referred to as Heisenberg-like scaling (HS) [59]. In order to reach HS, the corresponding QFI should scale quadratically with $\bar{n}$. The following Theorem demonstrates that high nonclassicality in displacement sensing is sufficient to achieve HS.

Theorem 4.—If $\mathcal{M}(\hat{\rho}) \propto \bar{n}_{\hat{\rho}}$, there exists $\hat{\sigma} = \hat{U}_L^\alpha(\hat{\rho} \otimes |0\rangle\langle 0|)\hat{U}_L^{\alpha\dagger}$ such that $I_F(\hat{\sigma}, \hat{a}_1^\dagger\hat{a}_1) \propto \bar{n}^2_{\hat{\sigma}}$. More precisely, HS can be achieved if and only if $\mathcal{M}(\hat{\rho}_0) \propto \bar{n}_{\hat{\rho}_0}$ or $\mathcal{M}^0_{\mathrm{phase}}(\hat{\rho}_0) \propto \bar{n}^k_{\hat{\rho}_0}$ with $k \geq 2$, where $\hat{\rho}_0 = \hat{V}_L\hat{\rho}\hat{V}_L^\dagger$ is the state centered in phase space ($\mathrm{Tr}\hat{\rho}_0\hat{\boldsymbol{R}} = 0$) by acting the linear optical unitary $\hat{V}_L$ on $\hat{\rho}$. Here, $\bar{n}_{\hat{\sigma}}$ is the mean photon number of a quantum state $\hat{\sigma}$.

We note that the Fock state and cat state cannot reach HS via only linear interferometers without additional displacement. However, Theorem 4 guarantees that an appropriate displacement operation will allow the system to reach HS. According to Theorems 3 and 4, a nonclassical resource for displacement sensing always implies the quantum enhancement of phase sensing. However, it is unclear at this point whether (1) negative P function of a mixed state always implies quantum enhancement in phase sensing and whether (2) nonclassical phase sensing implies nonclassical displacement sensing (see Fig. 3). These two statements are incompatible, thus both cannot be simultaneously true, but both can be false.

Remarks.—We have identified nonclassicality of continuous-variable states as a quantifiable resource for quantum metrology. We have shown that any pure state with negativity in the P function provides metrological enhancement over all classical states in displacement estimation tasks, and so does every single-mode Gaussian state. This metrological power is found to be a measure of nonclassicality based on a quantum resource theory that does not increase under linear optical elements. It is demonstrated that every state displaying metrological enhancement in displacement sensing can be converted into nonclassical phase sensitivity by utilizing a linear optical unitary.

The metrological power also satisfies the necessary conditions for a valid measure of quantum macroscopicity. Our study provides a possible avenue to a unified understanding of nonclassicality, quantum macroscopicity, and

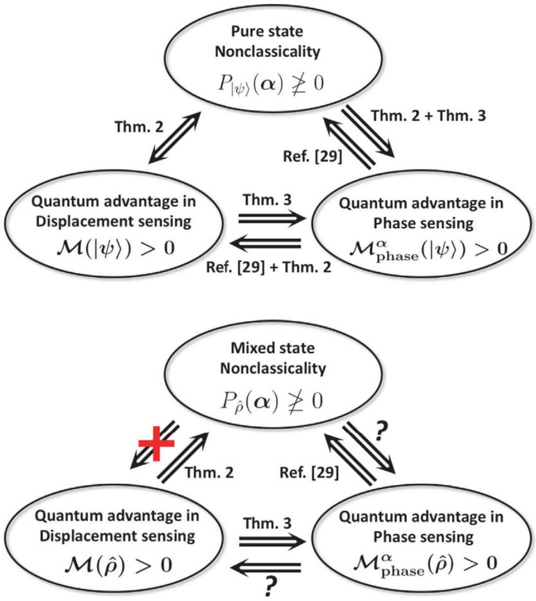

FIG. 3. Relationship between nonclassical resources for the displacement and phase estimation tasks.

the metrological usefulness in the framework of the quantum resource theory. Our measures could possibly be applied not only to multimode bosonic systems, but also to other many-body systems, including spin, atomic, and optomechanical systems. In these systems, a more general notion of coherent states [60,61] and the metrological usefulness with nonclassical states [62,63] can be considered. This may lead to a unified description of nonclassicality for both discrete and continuous systems.

The authors thank Jean-Michel Raimond, Serge Haroche, Benjamin Yadin, and Changhun Oh for helpful discussions. This work was supported by a National Research Foundation of Korea grant funded by the Korea government (Grant No. 2010-0018295). K. C. T and T. J. V. were supported by Korea Research Fellowship Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science and Information and Communications Technology (Grants No. 2016H1D3A1938100 and No. 2016H1D3A1908876).

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[1] A. K. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).
[2] C. H. Bennett and S. J. Wiesner, Communication via One- and Two-Particle Operators on Einstein-Podolsky-Rosen States, Phys. Rev. Lett. **69**, 2881 (1992).
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).
[4] R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett. **86**, 5188 (2001).
[5] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum Metrology, Phys. Rev. Lett. **96**, 010401 (2006).
[6] N. Gisin and R. Thew, Quantum communication, Nat. Photonics **1**, 165 (2007).
[7] S. Pironio, A. Acin, S. Massar, A. B. de La Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).
[8] R. J. Glauber, The quantum theory of optical coherence, Phys. Rev. **130**, 2529 (1963).
[9] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. **10**, 277 (1963).
[10] U. M. Titulaer and R. J. Glauber, Correlation functions for coherent fields, Phys. Rev. **140**, B676 (1965).
[11] L. Mandel, Non-classical states of the electromagnetic field, Phys. Scr. **T12**, 34 (1986).
[12] W. H. Zurek, S. Habib, and J. P. Paz, Coherent States via Decoherence, Phys. Rev. Lett. **70**, 1187 (1993).
[13] W. Vogel, Nonclassical States: An Observable Criterion, Phys. Rev. Lett. **84**, 1849 (2000).
[14] T. Richter and W. Vogel, Nonclassicality of Quantum States: A Hierarchy of Observable Conditions, Phys. Rev. Lett. **89**, 283601 (2002).
[15] A. Kenfack and K. Zyczkowski, Negativity of the Wigner function as an indicator of nonclassicality, J. Opt. B **6**, 396 (2004).
[16] Z. Jiang, M. D. Lang, and C. M. Caves, Mixing nonclassical pure states in a linear-optical network almost always generates modal entanglement, Phys. Rev. A **88**, 044301 (2013).
[17] L. Rigovacca, C. Di Franco, B. J. Metcalf, I. A. Walmsley, and M. S. Kim, Nonclassicality Criteria in Multiport Interferometry, Phys. Rev. Lett. **117**, 213602 (2016).
[18] J. Peřina, Jr., I. I. Arkhipov, V. Michálek, and O. Haderka, Nonclassicality and entanglement criteria for bipartite optical fields characterized by quadratic detectors, Phys. Rev. A **96**, 043845 (2017).
[19] F. Damanet, J. Kübler, J. Martin, and D. Braun, Nonclassical states of light with a smooth P function, Phys. Rev. A **97**, 023832 (2018).
[20] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, Entanglement by a beam splitter: Nonclassicality as a prerequisite for entanglement, Phys. Rev. A **65**, 032323 (2002).
[21] J. K. Asbóth, J. Calsamiglia, and H. Ritsch, Computable Measure of Nonclassicality for Light, Phys. Rev. Lett. **94**, 173602 (2005).
[22] W. Vogel and J. Sperling, Unified quantification of nonclassicality and entanglement, Phys. Rev. A **89**, 052302 (2014).
[23] G. Smith, J. A. Smolin, and J. Yard, Quantum communication with Gaussian channels of zero quantum capacity, Nat. Photonics **5**, 624 (2011).
[24] D. Lercher, G. Giedke, and M. M. Wolf, Standard superactivation for Gaussian channels requires squeezing, New J. Phys. **15**, 123003 (2013).
[25] M. Hillery, Nonclassical distance in quantum optics, Phys. Rev. A **35**, 725 (1987).
[26] P. Marian, T. A. Marian, and H. Scutaru, Quantifying Nonclassicality of One-Mode Gaussian States of the Radiation Field, Phys. Rev. Lett. **88**, 153601 (2002).

[27] C. T. Lee, Measure of the nonclassicality of nonclassical states, Phys. Rev. A **44**, R2775(R) (1991).

[28] S. Ryl, J. Sperling, and W. Vogel, Quantifying nonclassicality by characteristic functions, Phys. Rev. A **95**, 053825 (2017).

[29] Á. Rivas and A. Luis, Precision Quantum Metrology and Nonclassicality in Linear and Nonlinear Detection Schemes, Phys. Rev. Lett. **105**, 010403 (2010).

[30] C. Gehrke, J. Sperling, and W. Vogel, Quantification of nonclassicality, Phys. Rev. A **86**, 052118 (2012).

[31] K. C. Tan, T. Volkoff, H. Kwon, and H. Jeong, Quantifying the Coherence between Coherent States, Phys. Rev. Lett. **119**, 190405 (2017).

[32] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying Coherence, Phys. Rev. Lett. **113**, 140401 (2014).

[33] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[34] A. Streltsov, G. Adesso, and M. B. Plenio, Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).

[35] J. Sahota and N. Quesada, Quantum correlations in optical metrology: Heisenberg-limited phase estimation without mode entanglement, Phys. Rev. A **91**, 013808 (2015).

[36] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Heisenberg scaling in Gaussian quantum metrology, Phys. Rev. A **92**, 022106 (2015).

[37] W. Ge, K. Jacobs, Z. Eldredge, A. V. Gorshkov, and M. Foss-Feig, Distributed Quantum Metrology with Linear Networks and Separable Inputs, Phys. Rev. Lett. **121**, 043604 (2018).

[38] B. Yadin, F. C. Binder, J. Thompson, V. Narasimhachar, M. Gu, and M. S. Kim, Operational Resource Theory of Continuous-Variable Nonclassicality, Phys. Rev. X **8**, 041038 (2018).

[39] The author of the present Letter and the authors of Ref. [38] have independently developed the two closely related methods and found each other's work before concurrent submission to a journal.

[40] A. J. Leggett, Macroscopic Quantum Systems and the Quantum Theory of Measurement, Prog. Theor. Phys. Suppl. **69**, 80 (1980).

[41] H. Jeong, M. Kang, and H. Kwon, Characterizations and quantifications of macroscopic quantumness and its implementations using optical fields, Opt. Commun. **337**, 12 (2015).

[42] F. Fröwis, P. Sekatski, W. Dür, N. Gisin, and N. Sangouard, Macroscopic quantum states: Measures, fragility, and implementations, Rev. Mod. Phys. **90**, 025004 (2018).

[43] M. G. Díaz, K. Fang, X. Wang, M. Rosati, M. Skotiniotis, J. Calsamiglia, and A. Winter, Using and reusing coherence to realize quantum processes, Quantum **2**, 100 (2018).

[44] G. De Palma, The Wehrl entropy has Gaussian optimizers, Lett. Math. Phys. **108**, 97 (2018).

[45] C.-W. Lee and H. Jeong, Quantification of Macroscopic Quantum Superpositions within Phase Space, Phys. Rev. Lett. **106**, 220401 (2011).

[46] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72**, 3439 (1994).

[47] L. Pezzé and A. Smerzi, Entanglement, Nonlinear Dynamics, and the Heisenberg Limit, Phys. Rev. Lett. **102**, 100401 (2009).

[48] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Fisher information and multiparticle entanglement, Phys. Rev. A **85**, 022321 (2012).

[49] G. Tóth, Multipartite entanglement and high-precision metrology, Phys. Rev. A **85**, 022322 (2012).

[50] F. Fröwis and W. Dür, Measures of macroscopicity for quantum spin systems, New J. Phys. **14**, 093039 (2012).

[51] B. Yadin and V. Vedral, General framework for quantum macroscopicity in terms of coherence, Phys. Rev. A **93**, 022122 (2016).

[52] F. Fröwis, Lower bounds on the size of general Schrödinger-cat states from experimental data, J. Phys. A **50**, 114003 (2017).

[53] E. Oudot, P. Sekatski, F. Fröwis, N. Gisin, and N. Sangouard, Two-mode squeezed states as Schrödinger cat-like states, J. Opt. Soc. Am. B **32**, 2190 (2015).

[54] T. J. Volkoff and K. B. Whaley, Measurement- and comparison-based sizes of Schrödinger cat states of light, Phys. Rev. A **89**, 012122 (2014).

[55] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[56] M. Idel, D. Lercher, and M. M. Wolf, An operational measure for squeezing, J. Phys. A **49**, 445304 (2016).

[57] This can be obtained by using the following property of the QFI: $\left|\sqrt{I_F(\hat{\rho},\hat{A})}-\sqrt{I_F(\hat{\rho},\hat{B})}\right| \le \sqrt{I_F(\hat{\rho},\hat{A}+\hat{B})} \le \sqrt{I_F(\hat{\rho},\hat{A})}+\sqrt{I_F(\hat{\rho},\hat{B})}$. See Ref. [58] for a detailed proof.

[58] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.040503 for the proofs and detailed analysis.

[59] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics **5**, 222 (2011).

[60] J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys. **4**, 313 (1971).

[61] P. D. Drummond and M. D. Reid, Coherent states in projected Hilbert spaces, Phys. Rev. A **94**, 063851 (2016).

[62] G. Björk, A. B. Klimov, P. de la Hoz, M. Grassl, G. Leuchs, and L. L. Sánchez-Soto, Extremal quantum states and their Majorana constellations, Phys. Rev. A **92**, 031801(R) (2015).

[63] F. Bouchard, P. de la Hoz, G. Björk, R. W. Boyd, M. Grassl, Z. Hradil, E. Karimi, A. B. Klimov, G. Leuchs, J. Řeháček, and L. L. Sánchez-Soto, Quantum metrology at the limit with extremal Majorana constellations, Optica **4**, 1429 (2017).