Strong Quantum Nonlocality without Entanglement

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Quantum nonlocality is usually associated with entangled states by their violations of Bell-type inequalities. However, even unentangled systems, whose parts may have been prepared separately, can show nonlocal properties. In particular, a set of product states is said to exhibit "quantum nonlocality without entanglement" if the states are locally indistinguishable; i.e., it is not possible to optimally distinguish the states by any sequence of local operations and classical communication. Here, we present a stronger manifestation of this kind of nonlocality in multiparty systems through the notion of local irreducibility. A set of multiparty orthogonal quantum states is defined to be locally irreducible if it is not possible to locally eliminate one or more states from the set while preserving orthogonality of the postmeasurement states. Such a set, by definition, is locally indistinguishable, but we show that the converse does not always hold. We provide the first examples of orthogonal product bases on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d = 3, 4$ that are locally irreducible in all bipartitions, where the construction for $d = 3$ achieves the minimum dimension necessary for such product states to exist. The existence of such product bases implies that local implementation of a multiparty separable measurement may require entangled resources across all bipartitions.

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Composite quantum systems, parts of which are physically separated, can possess nonlocal properties. The most well-known manifestation of quantum nonlocality—Bell nonlocality [1], arises from entangled states [2]. Entangled states are nonlocal for they violate Bell-type inequalities [3–11]—the family of inequalities that must be satisfied by probabilities arising from any local realistic theory. Apart from the foundational implications, Bell nonlocality tests have applications in quantum technologies as they quantify nonclassicality in a device-independent manner [12–16].

Nonlocal properties, however, are not restricted only to entangled systems. In the seminal paper [17] *Quantum nonlocality without entanglement*, Bennett *et al.* showed that product states can exhibit nonlocal properties in a way fundamentally different from Bell nonlocality. In particular, they considered the following problem: Suppose that a quantum system, consisting of two parts held by separated observers, was prepared in one of several known orthogonal product states. The task is to identify, as well as possible, in which state the system is in, using local operations and classical communication (LOCC). The question they asked was whether for any known set of orthogonal product states exact discrimination is always possible using LOCC.

Now intuition suggests that the answer to the above question ought to be yes because product states admit local preparation (following some known set of rules), and therefore, it should be possible to learn about the state of the system with local measurements alone. But surprisingly, the authors presented an orthogonal product basis (OPB) on $\mathbb{C}^3 \otimes \mathbb{C}^3$ for which exact discrimination is not possible using LOCC [17]. Subsequently, more such examples were found in both bipartite [18–23] and multiparty systems [18,20,21,24–30] and their properties explored [18,19,31–34].

The possibility of this kind of result was, in fact, first pointed out by Peres and Wootters a few years earlier [35] (also see Refs. [36,37]). For a specific set of three non-orthogonal product states, they conjectured that LOCC measurements are suboptimal for state discrimination.

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The conjecture was only recently shown to be true [38]. We now say that any set of product states that cannot be optimally (exactly, if and only if the states are orthogonal) distinguished by LOCC exhibit nonlocality without entanglement. Here, nonlocality is in the sense that a measurement on the whole system reveals more information about the state of the system than any sequence of LOCC on their parts, even though they may have been prepared in different labs. Let us also note that the recent PBR theorem [39] (also see Refs. [40,41]) reveals yet another nonlocal feature of nonorthogonal product states, where state elimination with certainty becomes possible only by entangled measurements on the joint system.

The results of Peres-Wootters [35] and Bennett *et al.* [17] initiated a plethora of studies on more general local state discrimination problems—the task of optimal discrimination of multiparty states, not necessarily product, by means of LOCC [17,18,21–23,25–30,32–34,37,38,42–72]. It was found that in some cases, e.g., a set of two pure states, LOCC can indeed accomplish the task as efficiently as global measurements [51,52], whereas in some other cases, e.g., orthogonal entangled bases [53,55–58,61,63,65,68,70] they cannot, and we call such states locally indistinguishable. Locally indistinguishable states have found useful applications in quantum cryptography primitives such as data hiding [73–76] and quantum secret sharing [77].

In this Letter, we report new nonlocal properties of multiparty orthogonal product states—within the framework of local state discrimination, but considering instead a more basic problem—quantum state elimination using orthogonality-preserving local measurements (a measurement is orthogonality preserving if the postmeasurement states remain orthogonal). The motivation stemmed from the observation that some sets of orthogonal states on a composite Hilbert space are locally reducible; i.e., it is possible to locally eliminate one or more states from the set while preserving orthogonality of the postmeasurement states. For such sets, the task of local state discrimination is therefore reduced to that of a subset of states.

While a locally distinguishable set is locally reducible (trivially), the opposite is not true in general. In the following examples, the locally indistinguishable sets are locally reducible to a union of two or more disjoint subsets, each of which can be addressed individually.

(a) Consider the entangled orthogonal basis on $\mathbb{C}^2 \otimes \mathbb{C}^4$:

$$\begin{array}{ll} |00\rangle \pm |11\rangle & |02\rangle \pm |13\rangle \\ |01\rangle \pm |10\rangle & |03\rangle \pm |12\rangle. \end{array} \tag{1}$$

Here, Bob performs a local measurement to distinguish the subspaces spanned by $\{|0\rangle, |1\rangle\}$ and $\{|2\rangle, |3\rangle\}$. Depending upon the outcome, Alice and Bob end up with a state belonging to one of the two subsets (left or right). Note that, neither subset is locally distinguishable [53] (in fact, neither is locally reducible—see Proposition 2).

(b) Consider the orthogonal basis on $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\begin{array}{llll} |00\rangle \pm |11\rangle & |02\rangle & |20\rangle & |22\rangle \\ |01\rangle \pm |10\rangle & |12\rangle & |21\rangle. & \end{array} \tag{2}$$

Here, if the unknown state is one of the product states, it can always be correctly identified, and if it is not, all product states can be locally eliminated. In the latter case, Alice and Bob will end up with one of the four Bell states (local protocol given in Appendix A [78]). Note that, unlike the previous example, here not all the subsets are locally indistinguishable.

The above examples give rise to the following question: Are all locally indistinguishable sets locally reducible? The answer is no. As will be shown, some of the well-known locally indistinguishable sets are not locally reducible. First, we have the following definition.

Definition 1.—(Locally irreducible set) A set of orthogonal quantum states on $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$ with $n \geq 2$ and $\dim \mathcal{H}_i \geq 2$, $i = 1, \ldots, n$, is locally irreducible if it is not possible to eliminate one or more states from the set by orthogonality-preserving local measurements.

A locally indistinguishable set in general is not locally irreducible except when it contains three orthogonal pure states.

Proposition 1: Any set of three locally indistinguishable orthogonal pure states on $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$ with $n \geq 2$ and $\dim \mathcal{H}_i \geq 2$, $i = 1, \ldots, n$, is locally irreducible.

Since any two orthogonal pure states can be exactly distinguished by LOCC [51], a locally reducible set containing three orthogonal pure states must be locally distinguishable. But this contradicts the fact that the set is known to be locally indistinguishable. This proves the proposition.

We will now describe a sufficient condition for local irreducibility. The formalism was originally developed [54] (also see Refs. [43,44]) for local indistinguishability. We begin by defining a nontrivial measurement [54].

Definition 2.—A measurement is nontrivial if not all the POVM elements are proportional to the identity operator. Otherwise, the measurement is trivial.

The crux of the argument [54] was that, in any local protocol one of the parties must go first, and whoever goes first must be able to perform some nontrivial orthogonality-preserving measurement (NOPM). This fits naturally into our scenario for the following reasons. The measurement should be orthogonality preserving because we require that any measurement outcome must leave the postmeasurement states mutually orthogonal, possibly eliminating some states but not all (unless it correctly identifies the input right away). It is also essential that the measurement is nontrivial because a trivial measurement despite satisfying (trivially) the orthogonality-preserving conditions, gives us no information about the state. The sufficient condition follows by noting that, if none of the parties can perform a local NOPM, the states must be locally irreducible.

Following Ref. [54] we now discuss how to apply this condition when a set contains only orthogonal pure states. The basic idea is to check whether an orthogonality-preserving POVM on any of the subsystems is trivial or not. If it is trivial for all subsystems, the states are locally irreducible.

Let $S = \{|\psi_i\rangle\}_{i=1}^{k}$ be a set of orthogonal pure states on $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$, where $n \geq 2$, and $\dim \mathcal{H}_i \geq 2$, $i = 1, \ldots, n$. Consider a POVM $\{\pi_\alpha^i\}$, $\alpha = 1, 2, \ldots$ that may be performed on the ith subsystem. The POVM elements π_α^i are positive operators summing up to identity and correspond to the measurement outcomes. Further, each element admits the Krauss form: $\pi_\alpha^i = M_\alpha^{i\dagger} M_\alpha^i$, where M_α^is are the Krauss operators. The probability that an input state $|\psi_x\rangle \in S$ yields the outcome α is $p_\alpha = \langle\psi_x|\mathbb{I}_1 \otimes \cdots \otimes \pi_\alpha^i \otimes \cdots \otimes \mathbb{I}_n|\psi_x\rangle$ with the corresponding postmeasurement state given by $(1/\sqrt{p_\alpha})(\mathbb{I}_1 \otimes \cdots \otimes M_\alpha^i \otimes \cdots \otimes \mathbb{I}_n)|\psi_x\rangle$. Since we require the POVM to be orthogonality preserving, for all pairs of states $\{|\psi_x\rangle, |\psi_y\rangle\}$, $x \neq y$ and all outcomes α, the conditions

$$\langle\psi_x|\mathbb{I}_1 \otimes \cdots \otimes \pi_\alpha^i \otimes \cdots \otimes \mathbb{I}_n|\psi_y\rangle = 0 \tag{3}$$

need to be satisfied. To use the above conditions effectively, we represent each POVM element π_α^i, $\alpha = 1, 2, \ldots$ by a $d_i \times d_i$ matrix (in the computational basis) and solve for the matrix elements by choosing suitable pairs of vectors (also expressed in the computational basis of $\mathcal{H}$). This can be done exactly in many problems of interest. Now if we find that the conditions (3) are satisfied only if π_α^i is proportional to the identity for all α, then the measurement is trivial. This means the ith party cannot begin a LOCC protocol, and if this is true for all i, then none of the parties can go first. Therefore, S is locally irreducible. We will use this condition extensively in our proofs.

The OPB on $\mathbb{C}^3 \otimes \mathbb{C}^3$ [17] is locally irreducible. This follows from the proof showing that the states are locally indistinguishable [54]. We now show that the Bell basis and the three-qubit Greenberger-Horne-Zeilinger (GHZ) basis are locally irreducible (both are locally indistinguishable [53,55]) using the method just described.

Proposition 2: The two-qubit Bell basis (unnormalized): $|00\rangle \pm |11\rangle$, $|01\rangle \pm |10\rangle$ is locally irreducible.

The proof is by contradiction. Suppose that the Bell basis is locally reducible. Then, either Alice or Bob must be able to begin the protocol by performing some local NOPM. Without loss of generality assume that Bob goes first. Bob's general measurement can be represented by a set of 2×2 POVM elements $\pi_\alpha = \left(\begin{smallmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{smallmatrix}\right)$ written in the $\{|0\rangle, |1\rangle\}$ basis. Since this measurement is orthogonality preserving, for any pair of Bell states the conditions (3) must hold. By choosing suitable pairs, it is easy to show that π_α must be proportional to the identity (details in Appendix B [78]). As the argument holds for all outcomes, all of Bob's POVM elements are proportional to the identity. This means Bob cannot go first, and from the symmetry of the Bell states, neither can Alice. This completes the proof.

Proposition 3: The three-qubit GHZ basis (unnormalized): $|000\rangle \pm |111\rangle$, $|011\rangle \pm |100\rangle$, $|001\rangle \pm |110\rangle$, $|010\rangle \pm |101\rangle$, is locally irreducible.

The proof is along the same lines as in the previous one and is given in Appendix C [78] (can be extended for a N-qubit GHZ basis).

We now come to the main part of the Letter. Here we consider the following question: Do there exist multiparty orthogonal sets that are locally irreducible in every bipartition? The motivation for asking this question is that many of the properties of multiparty states in general are not preserved if we change the spatial configuration. For example, the three-party (A, B, and C) unextendible product basis (UPB) on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ [18] is locally indistinguishable (hence, nonlocal when all parts are separated) but can be perfectly distinguished across all bipartitions $A|BC$, $B|CA$, and $C|AB$ [18] using LOCC (and therefore, not nonlocal in the bipartitions). In fact, one can also find sets of entangled states that are locally distinguishable in one bipartition but not in others (see Appendix D [78]).

So which sets of orthogonal states are expected to remain locally irreducible in all bipartitions? Intuition suggests that a genuinely entangled orthogonal basis (the basis vectors are entangled in every bipartition) is a promising candidate because in any bipartition, the states are not only locally indistinguishable but also none can be correctly identified with a nonzero probability using LOCC [55]. However, we find that the GHZ basis, which is genuinely entangled and locally irreducible [Proposition 3], is locally reducible in all bipartitions.

Proposition 4: The three-qubit *GHZ* basis given in Proposition 3 is locally reducible in all bipartitions.

The proof is simple. Note that, one can always perform a joint measurement on any two qubits to distinguish the subspaces spanned by $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$. Thus in any bipartition, the whole set can be locally reduced to two disjoint subsets, each of which is locally equivalent to the Bell basis (the proof can be extended *mutatis mutandis* for a N-qubit GHZ basis with the identical conclusion).

Proposition 4 gives rise to an interesting question: Can multiparty orthogonal product states be locally irreducible in all bipartitions? If such sets exist, then they would clearly demonstrate quantum nonlocality stronger than what we presently understand.

First we observe that such product states cannot be found in systems where one of subsystems has dimension two: If the system contains a qubit, then the set is locally distinguishable in the bipartition **qubit**|**rest** because orthogonal product states on $\mathbb{C}^2 \otimes \mathbb{C}^d$, $d \geq 2$ are known to be locally distinguishable [20]. So they can only exist, if at all, on $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$, $n \geq 3$, where $\dim \mathcal{H}_i \geq 3$ for every i. Thus the minimum dimension corresponds to a three-qutrit system.

We checked all the known examples (to the best of our knowledge) of locally indistinguishable multiparty orthogonal product states, but did not find any with the desired property. Some were ruled out by the dimensionality constraint, and the rest turned out to be either locally distinguishable [20,24,26,28,29,43,44,48] or locally reducible [30] in one or more bipartitions.

The main result of this Letter lies in showing that multiparty orthogonal product states that are locally irreducible in all bipartitions, exist. We call such sets strongly nonlocal.

Definition 3.—Consider a composite quantum system $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$ with $n \geq 3$ and $\dim \mathcal{H}_i \geq 3$, $i = 1, \ldots, n$. A set of orthogonal product states $|\psi_i\rangle = |\alpha_i\rangle_1 \otimes |\beta_i\rangle_2 \otimes \cdots \otimes |\gamma_i\rangle_n$ on $\mathcal{H}$ is strongly nonlocal if it is locally irreducible in every bipartition.

We now give an example of an OPB on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and prove it strongly nonlocal. Note that, this construction achieves the minimum dimension required (as discussed earlier). We will use the notation $|1\rangle, |2\rangle, |3\rangle$ for the bases of Alice, Bob, and Charlie's Hilbert spaces. Consider the following OPB on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\begin{array}{lll} |1\rangle|2\rangle|1\pm 2\rangle & |2\rangle|1\pm 2\rangle|1\rangle & |1\pm 2\rangle|1\rangle|2\rangle \\ |1\rangle|3\rangle|1\pm 3\rangle & |3\rangle|1\pm 3\rangle|1\rangle & |1\pm 3\rangle|1\rangle|3\rangle \\ |2\rangle|3\rangle|1\pm 2\rangle & |3\rangle|1\pm 2\rangle|2\rangle & |1\pm 2\rangle|2\rangle|3\rangle \\ |3\rangle|2\rangle|1\pm 3\rangle & |2\rangle|1\pm 3\rangle|3\rangle & |1\pm 3\rangle|3\rangle|2\rangle \\ |1\rangle|1\rangle|1\rangle & |2\rangle|2\rangle|2\rangle & |3\rangle|3\rangle|3\rangle, \end{array} \tag{4}$$

where $|1\pm 2\rangle$ stands for $(1/\sqrt{2})(|1\rangle \pm |2\rangle)$, etc. Note that, the set (4) is invariant under cyclic permutation of the parties A, B, and C. We first show that the states are locally irreducible.

Lemma 1: The set of states given by (4) on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is locally irreducible.

To prove the lemma, we first consider the following states

$$\begin{array}{lll} |1\rangle|2\rangle|1\pm 2\rangle & |2\rangle|1\pm 2\rangle|1\rangle & |1\pm 2\rangle|1\rangle|2\rangle \\ |1\rangle|3\rangle|1\pm 3\rangle & |3\rangle|1\pm 3\rangle|1\rangle & |1\pm 3\rangle|1\rangle|3\rangle, \end{array} \tag{5}$$

chosen from the whole set. For the above states it was shown [26] that any 3×3 orthogonality-preserving POVM acting on any subsystem must be proportional to the identity. Clearly, this must also hold for the whole set (4) of which the states (5) form a subset because all the states belong to the same state space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Therefore, none of the parties can begin a LOCC protocol by performing some local NOPM. Hence, the proof (for completeness, we have included the details in Appendix E [78]).

Theorem 1: The orthogonal product basis (4) is strongly nonlocal.

We need to show that the states (4) form a locally irreducible set in any bipartition. To begin with, consider the bipartition $A|BC$ ($\mathcal{H}_A \otimes \mathcal{H}_{BC}$). In this bipartition the states (4) take the form

$$\begin{array}{lll} |1\rangle|21\pm 22\rangle & |2\rangle|11\pm 21\rangle & |1\pm 2\rangle|12\rangle \\ |1\rangle|31\pm 33\rangle & |3\rangle|11\pm 31\rangle & |1\pm 3\rangle|13\rangle \\ |2\rangle|31\pm 32\rangle & |3\rangle|12\pm 22\rangle & |1\pm 2\rangle|23\rangle \\ |3\rangle|21\pm 23\rangle & |2\rangle|13\pm 33\rangle & |1\pm 3\rangle|32\rangle \\ |1\rangle|11\rangle & |2\rangle|22\rangle & |3\rangle|33\rangle. \end{array} \tag{6}$$

Physically this means the subsystems B and C are treated together as a nine-dimensional subsystem BC. For clarity, denote the elements of the basis $\{|ij\rangle\}_{i,j=1}^3$ on $\mathcal{H}_{BC}$ as: $\forall\, i = 1, 2, 3$, $|1i\rangle \to |\mathbf{i}\rangle$, $|2i\rangle \to |\mathbf{i+3}\rangle$, and $|3i\rangle \to |\mathbf{i+6}\rangle$ and rewrite the states (6) as

$$\begin{array}{lll} |1\rangle|\mathbf{4\pm 5}\rangle & |2\rangle|\mathbf{1\pm 4}\rangle & |1\pm 2\rangle|\mathbf{2}\rangle \\ |1\rangle|\mathbf{7\pm 9}\rangle & |3\rangle|\mathbf{1\pm 7}\rangle & |1\pm 3\rangle|\mathbf{3}\rangle \\ |2\rangle|\mathbf{7\pm 8}\rangle & |3\rangle|\mathbf{2\pm 5}\rangle & |1\pm 2\rangle|\mathbf{6}\rangle \\ |3\rangle|\mathbf{4\pm 6}\rangle & |2\rangle|\mathbf{3\pm 9}\rangle & |1\pm 3\rangle|\mathbf{8}\rangle \\ |1\rangle|\mathbf{1}\rangle & |2\rangle|\mathbf{5}\rangle & |3\rangle|\mathbf{9}\rangle. \end{array} \tag{7}$$

We now show that any orthogonality-preserving local POVM performed either on A or BC must be trivial. Therefore, neither Alice (A) nor Bob and Charlie together (BC) can go first.

First, consider Alice. Recall that, Lemma 1 holds because none of the parties can perform a local NOPM when all parts are separated. Since in the bipartition $A|BC$ Alice's subsystem is still separated from the rest, we conclude that Alice cannot go first.

We now consider whether it is possible to initiate a local protocol by performing some NOPM on BC. Let the POVM $\{\Pi_\alpha\}$ describe a general orthogonality-preserving measurement on BC. Each POVM element Π_α can be written as a 9×9 matrix in the $\{|\mathbf{1}\rangle, \ldots, |\mathbf{9}\rangle\}$ basis of $\mathcal{H}_{BC}$:

$$\Pi_\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} \end{pmatrix}. \tag{8}$$

The measurement must leave the postmeasurement states mutually orthogonal. By choosing suitable pairs of vectors $\{|\psi_i\rangle, |\psi_j\rangle\}$, $i \neq j$, we find that all the off-diagonal matrix elements a_{ij}, $i \neq j$ must be zero if the orthogonality-preserving conditions $\langle\psi_i|\mathbb{I}\otimes\Pi_\alpha|\psi_j\rangle = 0$ are to be satisfied. Table I in Appendix F [78] shows the complete analysis. Similarly, we find that the diagonal elements are all equal. For example, by setting the inner product $\langle 1|\langle \mathbf{4}+\mathbf{5}|\mathbb{I}\otimes\Pi_\alpha|1\rangle|\mathbf{4}-\mathbf{5}\rangle = 0$, we get $a_{44} = a_{55}$. Table II (Appendix F [78]) summarizes this analysis. As the diagonal elements of Π_α are all equal and the off-diagonal elements are all zero, Π_α must be proportional to the identity. The argument applies to all measurement outcomes, and thus all POVM elements $\{\Pi_\alpha\}$ must be proportional to the identity. This means the POVM must be trivial, and therefore, BC cannot go first. Thus the states (7) form a locally irreducible set in the bipartition $A|BC$. Now from the symmetry of the states (4) (invariant under cyclic permutation of the parties), it follows that the states (4) are also locally irreducible in the bipartitions $C|AB$, and $B|CA$. This completes the proof of the theorem.

In Appendix G [78] we have given an example of a strongly nonlocal OPB on $\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$ along with the complete proof.

We now discuss the question of local discrimination of strongly nonlocal product states using entanglement as a resource. Note that, in our examples, the three-party separable measurements cannot be locally implemented even if any two share unlimited entanglement. So exact local implementation would require a resource state which must be entangled in all bipartitions (this also holds for product states that are locally indistinguishable in all bipartitions [30] but not strongly nonlocal). As to how much entanglement must one consume, we do not have any clear answer. A teleportation protocol can perfectly distinguish the states (4) using $\mathbb{C}^3\otimes\mathbb{C}^3$ maximally entangled states shared between any two pairs, but whether one can do just as well using cheaper resources (see Ref. [31]) is an intriguing question.

The results in this Letter also leave open other interesting questions. One may consider generalizing our constructions on $\bigotimes_{i=1}^n \mathbb{C}^d$ for $n \geq 4$, and $d \geq 3$. Another problem worth considering is whether incomplete orthogonal product bases can be strongly nonlocal, e.g., can we have a strongly nonlocal UPB? Finally, one may ask, whether one can find entangled bases that are locally irreducible in all bipartitions. In view of Proposition 4, it seems that to satisfy "local irreducibility in all bipartitions," the structure of the states is likely to be more important than their entanglement. Here, even examples in the simplest case of $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ can help us to understand this property better.

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