Light-Neutrino Exchange and Long-Distance Contributions to $0\nu2\beta$ Decays: An Exploratory Study on $\pi\pi \to ee$

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We present an exploratory lattice QCD calculation of the neutrinoless double beta decay $\pi\pi \to ee$. Under the mechanism of light-neutrino exchange, the decay amplitude involves significant long-distance contributions. The calculation reported here, with pion masses $m_\pi = 420$ and 140 MeV, demonstrates that the decay amplitude can be computed from first principles using lattice methods. At unphysical and physical pion masses, we obtain that amplitudes are 24% and 9% smaller than the predication from leading order chiral perturbation theory. Our findings provide the lattice QCD inputs and constraints for effective field theory. A follow-on calculation with fully controlled systematic errors will be possible with adequate computational resources.

Introduction.—It is a fundamental question whether the neutrinos are Dirac or Majorana-type fermions. Neutrinoless double beta ($0\nu2\beta$) decay, if detected, would prove that neutrinos are Majorana fermions. Besides, it provides direct evidence that the fundamental law of lepton number conservation is violated in nature. According to the light-neutrino exchange mechanism, the observation of $0\nu2\beta$ decay would also give us information about the absolute neutrino mass, which oscillation experiments cannot predict.

Around the world many experiments are underway to hunt for $0\nu2\beta$ decays [1–12]. Recently, four experiments reported the decay's half-lives of $T^{0\nu}_{1/2} > 10^{25}$ yr [9–12] and a fifth experiment reached the level of 1.07×10^{26} yr for ^{126}Xe [5]. With a new generation of ton-scale experiments, the level of sensitivity may be pushed 1 or 2 orders of magnitude higher, yielding the possibility to identify a few decay events per year [13–17].

The standard picture of $0\nu2\beta$ involves the long-range light neutrino exchange—a minimal extension of the standard model. On the other hand, current knowledge of second-order weak-interaction nuclear matrix elements needs to be improved, as various nuclear models lead to discrepancies on the order of 100% [17]. A promising

approach [18,19] to improving the reliability of the $0\nu2\beta$ predication is to constrain the few-body inputs to *ab initio* many-body calculations using lattice QCD [20–23].

In this work we perform the first lattice QCD calculation of the nonlocal matrix elements for the process of $\pi\pi \to ee$, where the light neutrinos are included as active degrees of freedom. We find that the decay amplitude receives dominant long-distance contributions from the $e\bar{\nu}\pi$ intermediate state. Although small, the excited-state contribution is identified with a clear signal in our calculation. At both unphysical and physical pion masses, we find that the lattice results are consistently smaller than the predication from leading order chiral perturbation theory [18].

Light-neutrino exchange in $0\nu2\beta$ decay.—We begin with the effective Lagrangian $\mathcal{L}_{\text{eff}}$ for the single β decay

$$\mathcal{L}_{\text{eff}} = 2\sqrt{2}G_F V_{ud}(\bar{u}_L\gamma_\mu d_L)(\bar{e}_L\gamma_\mu \nu_{eL}), \qquad (1)$$

which represents the standard Fermi charged-current weak interaction involving the left-handed fermionic fields $\bar{u}_L$, d_L, $\bar{e}_L$, and ν_{eL}. Here G_F is the Fermi constant and V_{ud} is the CKM matrix element. One can introduce the neutrino mixing matrix to connect the neutrino flavor eigenstates to the mass eigenstates. For the electron flavor, we have

$$\bar{e}_L\gamma_\mu\nu_{eL} = \sum_{k=1,2,3} \bar{e}_L\gamma_\mu U_{ek}\nu_{kL} \qquad (2)$$

with U_{ek} the mixing matrix element.

The effective Hamiltonian for 2β decay can be constructed as

$$\mathcal{H}_{\text{eff}} = \frac{1}{2!} \int d^4 x T[\mathcal{L}_{\text{eff}}(x)\mathcal{L}_{\text{eff}}(0)]$$

$$= 4G_F^2 V_{ud}^2 \int d^4 x H_{\mu\nu}(x) L_{\mu\nu}(x), \qquad (3)$$

where the hadronic factor $H_{\mu\nu}(x) = T[J_{\mu L}(x)J_{\nu L}(0)]$ with $J_{\mu L}(x) = \bar{u}_L \gamma_\mu d_L(x)$. Under the mechanism that $0\nu2\beta$ decays are mediated by the exchange of light Majorana neutrinos, the leptonic factor can be written as [24]

$$L_{\mu\nu}(x) = -m_{\beta\beta} S_0(x, 0) \bar{e}_L(x) \gamma_\mu \gamma_\nu e_L^c(0) \qquad (4)$$

with $S_0(x, 0) = \int [d^4 q/(2\pi)^4](e^{iqx}/q^2)$ a massless scalar propagator and $m_{\beta\beta} = \sum_k m_k U_{ek}^2$ the effective neutrino mass. The charge conjugate of a fermionic field ψ is given as $\psi^c = C\bar{\psi}^T = \gamma_4 \gamma_2 \bar{\psi}^T$.

For a general $0\nu2\beta$ decay $I(p_I) \to F(p_F)e(p_1)e(p_2)$, its decay amplitude can be written as

$$\mathcal{A} = \langle F, e_1, e_2 | \mathcal{H}_{\text{eff}} | I \rangle$$

$$= -4G_F^2 V_{ud}^2 m_{\beta\beta} \int d^4 x \langle F | H_{\mu\nu}(x) | I \rangle$$

$$\times \int \frac{d^4 q}{(2\pi)^4} \frac{e^{iqx}}{q^2} \langle e_1, e_2 | \bar{e}_L(x) \gamma_\mu \gamma_\nu e_L^c(0) | 0 \rangle. \qquad (5)$$

Here we use $e_{1,2}$ to specify the electron state carrying momentum $p_{1,2}$. The leptonic matrix element is given by

$$\langle e_1, e_2 | \bar{e}_L(x) \gamma_\mu \gamma_\nu e_L^c(0) | 0 \rangle = \bar{u}_L(p_1, x) \gamma_\mu \gamma_\nu u_L^c(p_2, 0)$$
$$- \bar{u}_L(p_2, x) \gamma_\mu \gamma_\nu u_L^c(p_1, 0), \qquad (6)$$

which is antisymmetric under the exchange of two electrons $e_1 \leftrightarrow e_2$ due to the Pauli exclusion principle. Here the spinors are defined as

$$\langle e_i | \bar{e}_L(x) = \bar{u}_L(p_i, x) = \bar{u}_L(p_i) e^{-i\vec{p}_i \cdot \vec{x}} e^{E_i t},$$
$$\langle e_i | e_L^c(x) = u_L^c(p_i, x) = u_L^c(p_i) e^{-i\vec{p}_i \cdot \vec{x}} e^{E_i t}, \qquad (7)$$

for $i = 1, 2$. Inserting the complete set of hadronic intermediate states, the decay amplitude can be written as

$$\mathcal{A} = -4G_F^2 V_{ud}^2 m_{\beta\beta}$$

$$\times \sum_n \left(\frac{\langle F|J_{\mu L}|n\rangle\langle n|J_{\nu L}|I\rangle}{2E_{\nu,2} E_n (E_n + E_{\nu,2} + E_2 - E_I)} \bar{u}_L(p_1)\gamma_\mu\gamma_\nu u_L^c(p_2) \right.$$

$$\left. + \frac{\langle F|J_{\mu L}|n\rangle\langle n|J_{\nu L}|I\rangle}{2E_{\nu,1} E_n (E_n + E_{\nu,1} + E_1 - E_I)} \bar{u}_L(p_1)\gamma_\nu\gamma_\mu u_L^c(p_2) \right).$$

$$(8)$$

Given the spatial momenta $\vec{p}$ for the hadronic intermediate states specified by $|n\rangle$, the neutrino's momenta are constrained by the conservation law $\vec{p}_{\nu,i} = \vec{p}_I - \vec{p} - \vec{p}_i$ and the

corresponding energies are denoted as $E_{\nu,i} = |\vec{p}_{\nu,i}|$. One can write the spinor product as a combination of $\bar{u}_L(p_1) u_L^c(p_2)$ and $\bar{u}_L(p_1)[(\gamma_\mu, \gamma_\nu)/2] u_L^c(p_2)$. The coefficient of the second term is proportional to the difference in electron momenta and generically suppressed by a factor of $|\vec{p}_1 - \vec{p}_2|/k_F \ll 1$, where $|\vec{p}_1 - \vec{p}_2| \sim O(1)$ MeV and $k_F \sim O(100)$ MeV is the typical Fermi momentum of nucleons in a nucleus [18,24]. Keeping only the term of $\bar{u}_L(p_1) u_L^c(p_2)$, the decay amplitude is simplified as

$$\mathcal{A} = -T_{\text{lept}} \sum_n \sum_{i=1,2} \frac{\langle F|J_{\mu L}|n\rangle\langle n|J_{\mu L}|I\rangle}{2E_{\nu,i} E_n (E_n + E_{\nu,i} + E_i - E_I)}, \qquad (9)$$

with $T_{\text{lept}} = 4G_F^2 V_{ud}^2 m_{\beta\beta} \bar{u}_L(p_1) u_L^c(p_2)$.

Calculation of $\pi\pi \to ee$ decay.—In this work we calculate the $\pi\pi \to ee$ decay amplitude with two pions at rest and two electrons carrying spatial momenta $\vec{p}_1 = -\vec{p}_2$, $|\vec{p}_{1,2}| = E_{\pi\pi}/2$. While the condition of $|\vec{p}_1 - \vec{p}_2|/k_F \ll 1$ is no more valid, we target on the determination of the amplitude given in Eq. (9), which is more relevant for chiral effective field theory inputs to *ab initio* many-body calculation [18]. This setup has advantages as follows. (1). Because of the nonzero momentum carried by the electron, the energies of any possible intermediate states $e\bar{\nu}n$ always lie above the initial-state energy $E_{\pi\pi} \approx 2m_\pi$. Therefore, no exponentially growing contamination is associated with the intermediate states when one performs an integral over a Euclidean time. The effects of finite volume on a generic second-order weak amplitude [25] are not relevant here as well. (2). We use the discrete lattice momenta $(2\pi/L)\vec{m}$ for the intermediate hadronic particles and the momenta $\vec{p}_{\nu,i} = -\vec{p}_i - (2\pi/L)\vec{m}$ for the intermediate neutrino, where $\vec{p}_i$ is the momentum carried by the electron. As nonzero momenta are assigned for the neutrino propagator, one can keep the lowest mode of the propagator, which reduces the power-law finite-volume effects.

Note that no short-distance divergence appears as x approaches to 0 in $\mathcal{L}_{\text{eff}}(x)\mathcal{L}_{\text{eff}}(0)$. This can be seen by the power counting in the integral

$$\int d^4 x e^{i\Lambda x} \mathcal{L}_{\text{eff}}(x)\mathcal{L}_{\text{eff}}(0)$$

$$\sim 8G_F^2 V_{ud}^2 \frac{m_{\beta\beta}}{\Lambda^2} (\bar{u}_L \gamma_\mu d_L)(\bar{u}_L \gamma_\mu d_L) \bar{e}_L e_L^c. \qquad (10)$$

In lattice QCD, a hard cutoff is introduced by the inverse of lattice spacing $1/a$. Thus the unphysical short-distance contribution appears as an $O(a^2)$ discretization effect.

Using the Coulomb gauge fixed wall sources for the ϕ_π interpolating operator, we construct the correlation function through

$$C(t_x, t_y, t_{\pi\pi})$$

$$= -4G_F^2 V_{ud}^2 m_{\beta\beta}$$

$$\times \left(\sum_{\vec{x},\vec{y}} \langle 0|T[J_{\mu L}(t_x, \vec{x}) J_{\mu L}(t_y, \vec{y}) \phi_\pi^\dagger \phi_\pi^\dagger(t_{\pi\pi})]|0\rangle \right.$$

$$\left. \times S_0(x,y) \langle e_1 e_2|\bar{e}_L(x) e_L^c(y)|0\rangle \right)$$

$$= -T_{\text{lept}} \sum_{\vec{x},\vec{y}} \langle 0|T[J_{\mu L}(t_x, \vec{x}) J_{\mu L}(t_y, \vec{y}) \phi_\pi^\dagger \phi_\pi^\dagger(t_{\pi\pi})]|0\rangle$$

$$\times S_0(x,y)(e^{-i\vec{p}_1\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}_2\cdot(\vec{x}-\vec{y})}) e^{\frac{E_{\pi\pi}}{2}(t_x+t_y)}. \quad (11)$$

On the lattice, the scalar propagator $S_0(x,y)e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$ with $\vec{k} = \vec{p}_{1,2}$ can be implemented as

$$S_0(x,y)e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} = \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq(x-y)}}{q_t^2 + (\vec{q}+\vec{k})^2}$$

$$\Rightarrow \frac{1}{VT} \sum_{\vec{q},q_t} \frac{e^{iq(x-y)}}{\hat{q}_t^2 + \sum_i \widehat{q_i + k_i}^2} \quad (12)$$

with $\hat{q}_i = 2\sin(q_i/2)$ the lattice discretized momenta. V and T are the spatial volume and time extent of the lattice. We can calculate the zero mode ($\vec{q}=0$) of the propagator as $(1/VT)\sum_{q_t}[e^{iq_t(t_x-t_y)}/\hat{q}_t^2 + \sum_i \hat{k}_i^2]$. The nonzero modes ($\vec{q} \neq \vec{0}$) of the propagator can be constructed as $(1/N_r)\sum_{r=1}^{N_r} \phi_r(x)\phi_r^*(y)$ using the stochastic method, with

$$\phi_r(x) = \frac{1}{\sqrt{VT}} \sum_{\vec{q}\neq\vec{0},q_t} \frac{\xi_r(q)e^{iqx}}{\sqrt{\hat{q}_t^2 + \sum_i \widehat{q_i + k_i}^2}}. \quad (13)$$

Here the stochastic sources $\xi_r(q)$ satisfy

$$\lim_{N_r\to\infty} \frac{1}{N_r} \sum_r \xi_r(q)\xi_r^*(q') = \delta_{q,q'}. \quad (14)$$

It is proposed by the NPLQCD Collaboration that the neutrino propagator can also be computed in an exact way by using double Fourier transformation [26].

Following Refs. [27–33] and integrating t_x and t_y over a fixed window $[t_a, t_b]$ with $t_a \gg t_{\pi\pi}$, we obtain

$$\mathcal{M} = \sum_{t_x=t_a}^{t_b} \sum_{t_y=t_a}^{t_b} C(t_x, t_y, t_{\pi\pi}) \bigg/ \left(V\frac{N_{\pi\pi}}{2E_{\pi\pi}} e^{E_{\pi\pi}t_{\pi\pi}} \right)$$

$$= -T_{\text{lept}} \sum_n \frac{1}{V} \sum_{\vec{p}_n} \sum_{i=1,2} \frac{\langle 0|J_{\mu L}|n\rangle\langle n|J_{\mu L}|\pi\pi\rangle}{2E_{\nu,i}E_n(E_n + E_{\nu,i} + E_i - E_{\pi\pi})}$$

$$\times \left(T_{\text{box}} + \frac{e^{-(E_n+E_{\nu,i}+E_i-E_{\pi\pi})T_{\text{box}}} - 1}{E_n + E_{\nu,i} + E_i - E_{\pi\pi}} \right) \quad (15)$$

with $T_{\text{box}} = t_b - t_a + 1$ the time extent of the integration window. $N_{\pi\pi}$ and $E_{\pi\pi}$ are known from the correlation

TABLE I. Ensembles used in this work. We list the pion mass m_π, the lattice spacing inverse a^{-1}, the space-time volume $L^3 \times T$, the number N_{conf} of configurations used, and the number N_r of stochastic sources for the neutrino propagator.

m_π [MeV]	a^{-1} [GeV]	$L^3 \times T$	N_{conf}	N_r
420	1.73	$16^3 \times 32$	200	32
140	1.01	$24^3 \times 64$	60	64

function $\langle \phi_\pi\phi_\pi\phi_\pi^\dagger\phi_\pi^\dagger(0)\rangle \xrightarrow{T\gg 0} V(N_{\pi\pi}^2/2E_{\pi\pi})(e^{-E_{\pi\pi}t} + e^{-E_{\pi\pi}(T-t)})$ + const by using the methods proposed in Ref. [34]. When T_{box} is sufficiently large, the contamination from the exponential term vanishes as $E_n + E_{\nu,i} + E_i > E_{\pi\pi}$. The coefficient of the term proportional to T_{box} provides a result for the decay amplitude $\mathcal{A}(\pi\pi \to ee)$.

Numerical results.—We use two ensembles with $m_\pi = 420$ and 140 MeV generated by the RBC and UKQCD Collaborations [35]. The corresponding parameters are listed in Table I. We produce wall-source light-quark propagators on all time slices and make use of the time translation invariance to average the correlator over all T time translations. (To reduce the computational costs at $m_\pi = 140$ MeV, we adopt the technique of all mode average [36,37] with T sloppy propagators used for correlator average and 1 precise propagator for correction.) We compute propagators for both periodic and antiperiodic boundary conditions in the temporal direction and use their average in the calculation, which effectively doubles the temporal extent of the lattice.

The Feynman diagrams corresponding to the process of $\pi\pi \to ee$ are shown in Fig. 1. To show the time dependence of the $C(t_x, t_y, t_{\pi\pi})$ explicitly, we define the unintegrated amplitude $\mathcal{M}(t)$ as a function of the variable $t = t_x - t_y$:

$$\mathcal{M}(t) = C(t_x, t_y, t_{\pi\pi}) \bigg/ \left(V\frac{N_{\pi\pi}}{2E_{\pi\pi}} e^{E_{\pi\pi}t_{\pi\pi}} \right). \quad (16)$$

The time t_x and t_y are separated by at least 6 time units from the $\pi\pi$ sources ($t_{x,y} - t_{\pi\pi} \geq 6$) so that the $\phi_\pi^\dagger\phi_\pi^\dagger$ interpolating operators can project onto the ground $\pi\pi$ state. At large $|t|$, the time dependence of $\mathcal{M}(t)$ is saturated by the ground intermediate state-$e\bar{\nu}\pi$
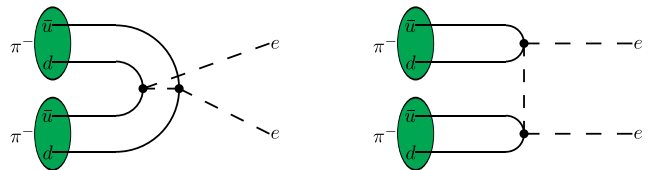

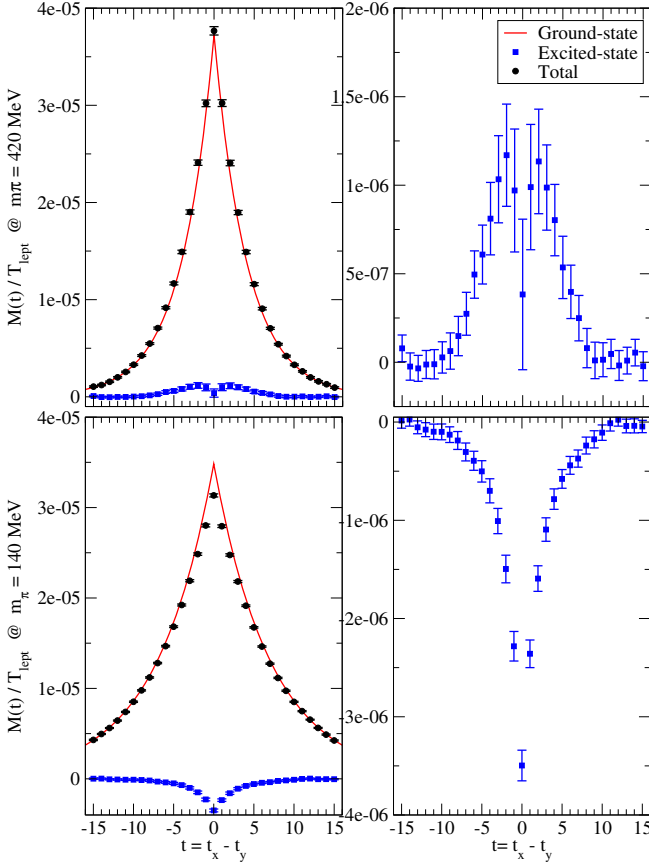FIG. 1. Quark and lepton contractions for the process of $\pi\pi \to ee$.

FIG. 2. Unintegrated amplitude $\mathcal{M}(t)$ defined in Eq. (16) as a function of $t = t_x - t_y$. The black circles show the total contribution of $\mathcal{M}(t)$. The red curve is not a fit to $\mathcal{M}(t)$, but a ground-state contribution predicted by Eq. (17). The blue squares show the remaining excited-state contribution.

$$\mathcal{M}(t) \xrightarrow{|t| \gg 0} - T_{\text{lept}} \frac{1}{V} \frac{2\langle 0|J_{\mu L}|\pi\rangle_V \langle \pi|J_{\mu L}|\pi\pi\rangle_V}{(2m_\pi)(2E_\nu)} e^{-m_\pi|t|}, \quad (17)$$

where the matrix elements of $\langle 0|J_{\mu L}|\pi\rangle_V$ and $\langle \pi|J_{\mu L}|\pi\pi\rangle_V$ are determined from the correlation functions $\langle J_{\mu L}(t)\phi_\pi^\dagger(0)\rangle$ and $\langle \phi_\pi(t_\pi)J_{\mu L}(t_J)\phi_\pi^\dagger\phi_\pi^\dagger(t_{\pi\pi})\rangle$, respectively. The subscript $\langle \cdots \rangle_V$ indicates that the initial and final states are defined in the finite volume. The single-pion states $|\pi\rangle_V$ satisfy the normalization condition $\langle \pi(\vec{p})|\pi(\vec{p}')\rangle_{\text{FV}} = (2E_\pi)V\delta_{\vec{p},\vec{p}'}$, while the two-pion states $|\pi\pi\rangle_V$ can be connected to the states in the finite volume $|\pi\pi\rangle_\infty$ through the Lellouch-Lüscher relation [38,39]

$$|\pi\pi\rangle_\infty = \left(2\pi \frac{E_{\pi\pi}}{k^3}\right)^{\frac{1}{2}} \left(q\frac{d\phi}{dq} + k\frac{d\delta}{dk}\right)^{\frac{1}{2}} |\pi\pi\rangle_V \quad (18)$$

with the momenta $k = \sqrt{(E_{\pi\pi}^2/4) - m_\pi^2}$ and $q = kL/(2\pi)$.

The time dependence of $\mathcal{M}(t)$ is shown in Fig. 2. At large $|t|$ the data of $\mathcal{M}(t)$ are consistent with the contribution from the ground intermediate state. By subtracting the ground-state contribution, the remaining excited-state
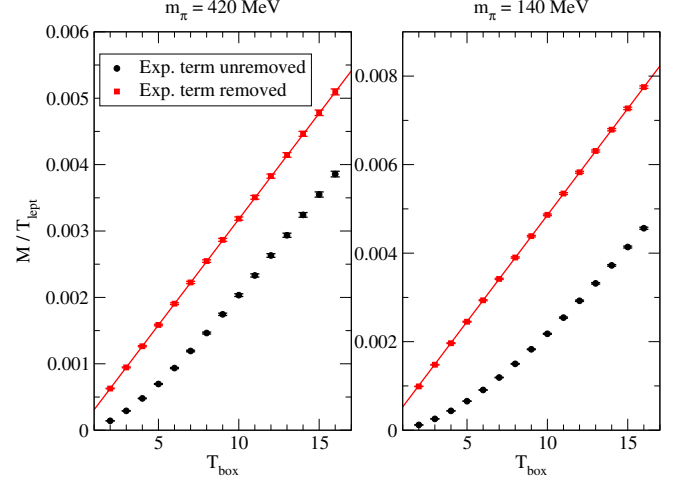

FIG. 3. Integrated matrix element $\mathcal{M}$ as a function of T_{box}. The black circles show the integrated matrix element $\mathcal{M}$ defined in Eq. (15). The red squares show the results of $\mathcal{M}$ with the exponential term for the ground intermediate state $(e^{-m_\pi T_{\text{box}}} - 1)/m_\pi$ subtracted.

contribution is shown by blue square points in the left panel of Fig. 2 and enlarged in the right panel. Although relatively small, the contribution from the excited intermediate states can be identified with a clear signal.

The integrated matrix element defined in Eq. (15) is shown in Fig. 3. We realize that the size of integration window $T_{\text{box}} \approx 16$ is not sufficiently large to discard the exponential term associated with the ground intermediate state. (This can be confirmed in Fig. 2 that at $|t| \approx 15$ the values of $\mathcal{M}(t)$ are statistically larger than 0.) After removing this exponential term, we can fit the lattice data to a linear function of T_{box} and determine the values of $\mathcal{A}_{\text{lat}}(\pi\pi \to ee)$. To convert $\mathcal{A}_{\text{lat}}(\pi\pi \to ee)$ to the physical amplitude $\mathcal{A}(\pi\pi \to ee)$, a renormalization factor square $Z_{V/A}^2$ shall be multiplied, which relates the local lattice vector or axial-vector current (which we use) to the conserved or partially conserved ones. Besides, the Lellouch-Lüscher factor shall be multiplied to relate a finite-volume amplitude to the infinite-volume one. In our calculation, the two pions are in the ground state, i.e., at threshold. The large-L expansion of the Lellouch-Lüscher factor is given by

$$\frac{2\pi}{k^3}\left(q\frac{d\phi}{dq} + k\frac{d\delta}{dk}\right) = V\left[1 + d_1\frac{a_{\pi\pi}}{L} + d_2\left(\frac{a_{\pi\pi}}{L}\right)^2 \right.$$
$$\left. + d_3\left(\frac{a_{\pi\pi}}{L}\right)^3 - 2\pi\frac{a_{\pi\pi}^2 r_{\pi\pi}}{L^3} + O(L^{-4})\right],$$
$$(19)$$

with $a_{\pi\pi}$ the scattering length and $r_{\pi\pi}$ the effective range from the k expansion of $\pi\pi$ scattering phase shift $k\cot\delta(k) = a_{\pi\pi}^{-1} + r_{\pi\pi}(k^2/2) + O(k^4)$. The coefficients d_i are given by

$$d_1 = -2\frac{Z_{00}(1;0)}{\pi} = 5.674\,595,$$

$$d_2 = \frac{Z_{00}(1;0)^2 + 3Z_{00}(2;0)}{\pi^2} = 13.075\,478,$$

$$d_3 = \frac{4\pi^4 - 4Z_{00}(3;0)}{\pi^3} = 11.482\,471. \tag{20}$$

The values of the zeta function $Z_{00}(s,0)$ have been provided by Ref. [40]. We evaluate $a_{\pi\pi}$ using Lüscher's finite-size formula [40] and use it as an input to determine the finite-volume correction up to $O(L^{-2})$.

Another type of power-law finite-volume effect arises from the long-range property of the neutrino propagator. The finite-volume effects relevant for the $e\bar{\nu}\pi$-intermediate state can be evaluated as

$$\Delta_{\rm FV} = \left(\frac{1}{V}\sum_{\vec{p}} - \int \frac{d^3\vec{p}}{(2\pi)^3}\right) \frac{\langle 0|J_{\mu L}|\pi(\vec{p})\rangle\langle\pi(\vec{p})|J_{\mu L}|\pi\pi\rangle}{E_\nu E_\pi(E_\pi + E_\nu + E_e - E_{\pi\pi})}, \tag{21}$$

with $\vec{p} = (2\pi/L)\vec{n}$ the discrete momentum for the pion. The neutrino's energy is given by $E_\nu = |\vec{p} + \vec{p}_e|$, with $\vec{p}_e$ the momentum carried by the electron. We define a function $f(\vec{p}) \equiv \{[\langle 0|J_{\mu L}|\pi(\vec{p})\rangle\langle\pi(\vec{p})|J_{\mu L}|\pi\pi\rangle]/[E_\pi(E_\pi + E_\nu + E_e - E_{\pi\pi})]\}$ and split it as $f(\vec{p}) = f(-\vec{p}_e) + [f(\vec{p}) - f(-\vec{p}_e)]$. The term inside brackets does not contribute a power-law finite-volume effect. We thus simplify $\Delta_{\rm FV}$ as

$$\Delta_{\rm FV} = f(-\vec{p}_e)\left(\frac{1}{V}\sum_{\vec{p}} - \int \frac{d^3\vec{p}}{(2\pi)^3}\right)\frac{1}{|\vec{p}_e + \vec{p}|}$$

$$= f(-\vec{p}_e)\left[-\frac{\kappa(\vec{n}_e)}{2\pi L^2}\right]. \tag{22}$$

The function $\kappa(\vec{n}_e)$ with $\vec{n}_e = \vec{p}_e L/(2\pi)$ can be computed numerically and we find $\kappa(\vec{n}_e) = 0.686(3)$ for $m_\pi = 420$ MeV and $0.517(3)$ for $m_\pi = 140$ MeV. Thus Eq. (22) indicates that the finite-volume correction appears as an $O(L^{-2})$ effect. We expect that the size of $f(-\vec{p}_e)$ is significantly smaller than $f(\vec{0})$, as the total contribution to the decay amplitude from the intermediate hadronic states that carry nonzero lattice momenta only amounts for 3%–4% when compared to the zero-momentum contribution. We therefore neglect this finite-volume effect in this work, and leave it for future studies.

In Table II, we show the ground-state, excited-state, and total contributions to the decay amplitude as $\mathcal{A}^{(g)}$, $\mathcal{A}^{(e)}$, and $\mathcal{A}^{(g)} + \mathcal{A}^{(e)}$, respectively. The results are presented in units of $F_\pi^2 T_{\rm lept}$, where the decay constant F_π is determined from the matrix element $\langle 0|\bar{d}\gamma_\mu\gamma_5 u|\pi(p)\rangle = \sqrt{2}p_\mu Z_A F_\pi$, with Z_A the renormalization constant. Systematic effects associated with three choices of $t_a - t_{\pi\pi} = 6, 7, 8$ are relatively

TABLE II. Results for ground-state ($\mathcal{A}^{(g)}$), excited-state ($\mathcal{A}^{(e)}$), and total ($\mathcal{A}^{(g)} + \mathcal{A}^{(e)}$) contributions to the $\pi\pi \to ee$ decay amplitude. All the results are listed in units of $F_\pi^2 T_{\rm lept}$.

m_π [MeV]	$t_a - t_{\pi\pi}$	$\mathcal{A}^{(g)}$	$\mathcal{A}^{(e)}$	$\mathcal{A}^{(g)} + \mathcal{A}^{(e)}$
420	6		0.055(13)	1.517(13)
	7	1.462(10)	0.060(13)	1.522(13)
	8		0.052(14)	1.514(14)
140	6		−0.0664(70)	1.8200(63)
	7	1.8864(50)	−0.0660(73)	1.8204(62)
	8		−0.0665(70)	1.8199(60)

smaller than the statistical errors, suggesting that a separation of 6 is a safe choice to neglect the excited $\pi\pi$ states.

Conclusion.—We have carried out a lattice QCD calculation of the decay amplitude of $\pi\pi \to ee$ and obtained the result with subpercent statistical errors:

$$\frac{\mathcal{A}(\pi\pi \to ee)}{F_\pi^2 T_{\rm lept}}\bigg|_{m_\pi=420~{\rm MeV}} = 1.517(13),$$

$$\frac{\mathcal{A}(\pi\pi \to ee)}{F_\pi^2 T_{\rm lept}}\bigg|_{m_\pi=140~{\rm MeV}} = 1.820(6). \tag{23}$$

The decay amplitude of $\mathcal{A}(\pi\pi \to ee)$ is mainly contributed by the ground intermediate state via the process of $\pi\pi \to \pi e\bar{\nu} \to ee$. Although the size of the excited-state contribution is only 3%–4%, it is statistically significant (see Fig. 2) as the uncertainty of the amplitude has been reduced to below 1%.

Without the signal-to-noise problem, the case of $\pi\pi \to ee$ serves as an ideal laboratory to develop the necessary methods and tools for a calculation of $0\nu2\beta$ decay with controlled uncertainties. Our exploratory study demonstrates the possibility of a first-principles calculation of the long-distance contribution to $0\nu2\beta$ decay via light-neutrino exchange. At $m_\pi = 420$ and 140 MeV, we find that the decay amplitude $\mathcal{A}(\pi\pi \to ee)$ are 24% and 9% smaller than the leading-order predication $\mathcal{A}^{\rm LO}(\pi\pi \to ee) = 2F_\pi^2 T_{\rm lept}$ in chiral perturbation theory [18]. Various systematic effects such as lattice artifacts and finite-volume effects require an accurate examination in future work but are not expected to qualitatively alter the conclusions of this work. The 9% deviation found here is still quite consistent with power counting in effective field theory. On the other hand, Ref. [41] has found that a leading-order, short-range contribution needs to be introduced in the $nn \to ppee$ decay, which breaks down Weinberg's power-counting scheme. It is interesting to examine the impact of this short-range contribution in our future study. The techniques presented here can be directly applied to the study of other $0\nu2\beta$ decays, such as $n\pi \to pee$ and $nn \to ppee$. From these decays, lattice QCD can provide more low-energy QCD inputs for the effective field theory [18].

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