Slow Growth of Out-of-Time-Order Correlators and Entanglement Entropy in Integrable Disordered Systems

Max McGinley,[1] Andreas Nunnenkamp,[1] and Johannes Knolle[2,1]
[1]*Cavendish Laboratory, University of Cambridge, Cambridge CB3 0HE, United Kingdom*
[2]*Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom*

(Received 16 July 2018; revised manuscript received 11 October 2018; published 16 January 2019)

We investigate how information spreads in three paradigmatic one-dimensional models with spatial disorder. The models we consider are unitarily related to a system of free fermions and, thus, are manifestly integrable. We demonstrate that out-of-time-order correlators can spread slowly beyond the single-particle localization length, despite the absence of many-body interactions. This phenomenon is shown to be due to the nonlocal relationship between elementary excitations and the physical degrees of freedom. We argue that this nonlocality becomes relevant for time-dependent correlation functions. In addition, a slow logarithmic-in-time growth of the entanglement entropy is observed following a quench from an unentangled initial state. We attribute this growth to the presence of strong zero modes, which gives rise to an exponential hierarchy of time scales upon ensemble averaging. Our work on disordered integrable systems complements the rich phenomenology of information spreading and we discuss broader implications for general systems with nonlocal correlations.

DOI: 10.1103/PhysRevLett.122.020603

Introduction.—The presence of spatial disorder in quantum systems can have profound effects on their static and dynamical properties, leading in particular to the phenomenon of localization [1–6]. Localized systems are often characterized by an absence of diffusion and, therefore, are capable of retaining information about the initial state for arbitrarily long times.

Recently, there has been a surge of interest in studying localization in noninteracting and interacting systems [3–6], termed Anderson localization (AL) and many-body localization (MBL), respectively. Although transport phenomena are the same in AL and MBL, the presence of interactions in MBL systems leads to a slow growth of entanglement entropy (EE) [7–9], indicating a propagation of information across the system, albeit at an exponentially slow rate. In a similar light, it has recently been shown that out-of-time-order correlators (OTOCs)—two-time correlation functions in which operators are not chronologically ordered—are also capable of detecting this slow spread of information in MBL systems [10–15]. Evidently, the presence of many-body interactions in localized systems has a drastic effect on the spreading of information, as witnessed by the EE and OTOCs. Excitingly, the realization of MBL systems in cold atom [16,17] and trapped ion [18] experiments, wherein the EE [19,20] and OTOC [21–26] can be measured, allows for this slow information spreading to be directly observed [27].

In this Letter, focussing on EE and OTOCs, we study how information spreads in three disordered models whose Hamiltonians can be brought into free-fermion form and which, in that sense, are manifestly integrable. In all of our models, we observe slow dynamics in the EE which yields a logarithmic-in-time growth upon disorder averaging (Fig. 1)—we associate this growth with the presence of strong zero modes [28]. Furthermore, as our central result, we find that the OTOC slowly spreads beyond the single-particle localization length over long timescales (Fig. 2)

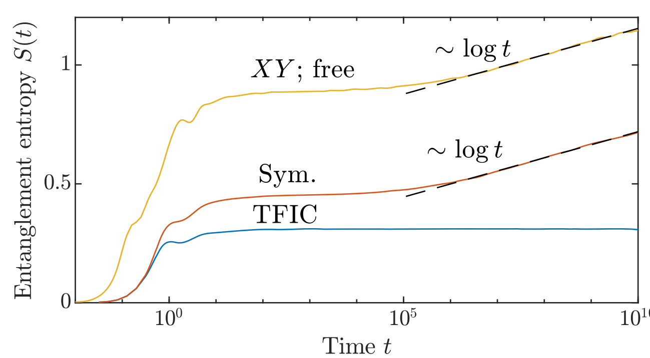

FIG. 1. Growth of the second Renyi entropy after a quench under Hamiltonians (1), (2), and (3), and the transverse-field Ising chain (TFIC). We use clean $T_j = 1$ and disordered $R_j \in 2 + [-5, 5]$. For each disorder realization, the initial state used is a random unentangled product state of fermion occupation or spin quantum numbers. The system size is $N = 56$ in all cases, and the entropy is averaged over $M = 10^4$ disorder realizations. In systems (1), (2), and (3) at late times, the entanglement entropy grows logarithmically despite the lack of many-body interactions usually associated with slow growth in MBL systems. The onset time of slow growth depends on the system size, whilst the final value of $S(t)$ as $t \to \infty$ is a constant of order 1, in contrast to MBL systems (see the discussion).

 © 2019 American Physical Society

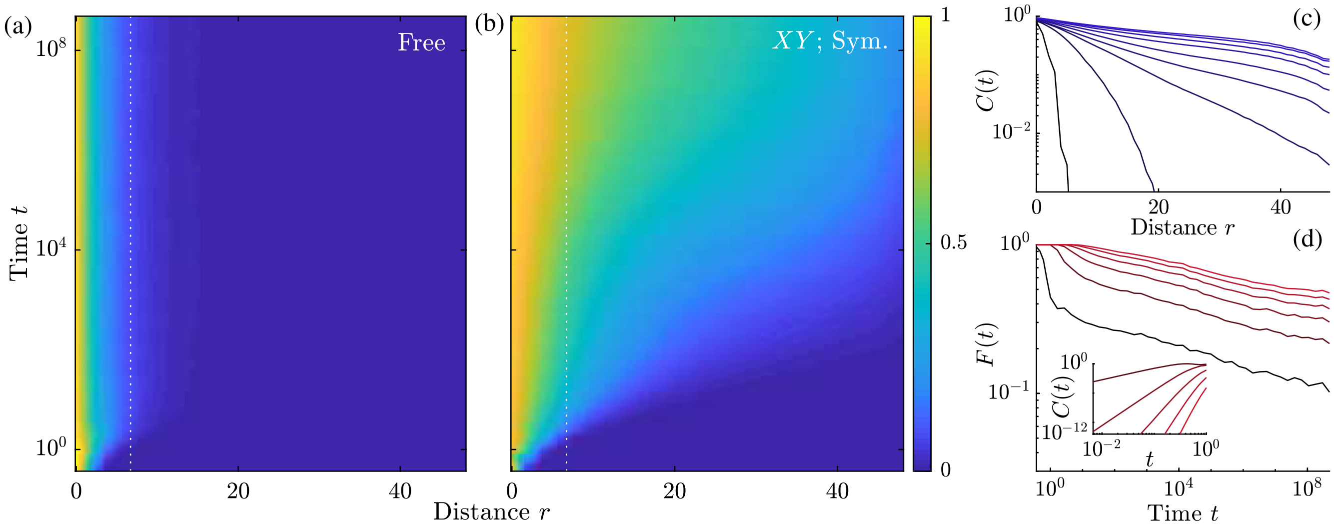

FIG. 2. Dynamics of the out-of-time-order correlator $C(t)$ [Eq. (4) with $\hat{A}$ and $\hat{B}$ defined in the main text] for integrable disordered models (1), (2), and (3) with mean $\langle R \rangle = 1$ and width $W_R = 6$. The data are averaged over $M = 5 \times 10^3$ disorder realizations. The system size is $N = 100$ and we fix $j = 50$. Panel (a): color plot of $C(t)$ for the free fermion model as a function of time t and operator distance r. Panel (b): equivalent color plot for the XY and symmetric Kitaev models, which have identical OTOCs. The white dotted line indicates the single-particle localization length. Panel (c): Data from (b) as a function of r for fixed times t varying from $t = 0.5$ (black) to $t = 5 \times 10^5$ (light blue), showing an exponential decay with time-dependent decay constant: $C \sim e^{-\lambda(t)|r|}$—this differs from the profile seen for typical MBL systems. Panel (d): Data from (b) [Plotted as $F(t) \equiv 1 - \mathrm{Re}C(t)$] as a function of t for various distances r, varying from $r = 1$ (black) to $r = 23$ (red). Inset: short time behavior of $C(t)$.

despite the lack of genuine many-body interactions. We attribute this to the nonlocal relationship between the physical and diagonal (free-fermion) bases, allowing nontrivial *dynamical* correlations to appear which are not reflected in the static properties of eigenstates.

Although these signatures are generally associated with MBL phases, quantitative differences from typical MBL phenomenology are seen in the EE saturation value, which is order 1, and the profile of OTOC growth. Indeed, the exact solvability of our models used here implies that slow OTOC and EE growth in localized systems is not always mediated by many-body interactions; thus, this phenomenology cannot necessarily be used as signatures to distinguish AL and MBL systems. Our results highlight the role of these nonlocal correlations in nonequilibrium dynamics, and have broader implications for the diagnostics of localized phases.

Models.—We study three disordered one-dimensional chains with open boundary conditions. Our first system of interest is the celebrated XY spin chain with spatial disorder. The Hamiltonian is

$$\hat{H}_{XY} = \sum_{j=1}^{N-1} T_j \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + R_j \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y. \quad (1)$$

The above Hamiltonian can be mapped to a 1D system of free fermions $\{\hat{f}_j^{(\dagger)}\}$ via the Jordan-Wigner (JW) transform $\hat{f}_j^\dagger = \hat{\sigma}_j^+ \prod_{k<j} \hat{\sigma}_k^z$ [29]. The transformed system constitutes our second model, describing manifestly free fermions with anomalous terms

$$\hat{H}_{\mathrm{free}} = \sum_{j=1}^{N-1} (T_j + R_j) \hat{f}_j^\dagger \hat{f}_{j+1} + (T_j - R_j) \hat{f}_j^\dagger \hat{f}_{j+1}^\dagger + \mathrm{H.c.} \quad (2)$$

This quadratic Hamiltonian, which is a disordered generalization of the Kitaev chain [30], can be efficiently diagonalized by a Bogoliubov transformation $\hat{a}_n = \sum_j u_{n,j} \hat{c}_j + v_{n,j} \hat{c}_j^\dagger$, such that $\hat{H}_{\mathrm{free}} = \sum_n \epsilon_n \hat{a}_n^\dagger \hat{a}_n$ [29]. The JW transform relates the eigenstates of (1) and (2) while preserving the spectrum.

Our third system can also be obtained through a JW transform, with the crucial difference that the XY model is first rotated by $\pi/2$ into an "XZ" model. This yields a manifestly interacting fermionic Hamiltonian which is integrable, known as the symmetric interacting Kitaev chain [31]

$$\hat{H}_{\mathrm{sym}} = \sum_{j=1}^{N-1} T_j (\hat{c}_j^\dagger \hat{c}_{j+1} + \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger) + \mathrm{H.c.} + R_j (2\hat{c}_j^\dagger \hat{c}_j - 1)(2\hat{c}_{j+1}^\dagger \hat{c}_{j+1} - 1), \quad (3)$$

with fermionic operators $\hat{c}_j^{(\dagger)}$. The system features hopping and p-wave pairing with equal amplitudes T_j, supplemented with nearest-neighbor density-density interactions.

After diagonalizing each system in the basis of quadratic Jordan-Wigner fermions, one can express the single-particle occupation numbers $\{\hat{a}_n^\dagger \hat{a}_n\}$ in the physical basis; this defines a collection of conserved quantities for each system. When either T_j or R_j are disordered, system (2) exhibits Anderson localization [32], which makes each $\hat{a}_n^\dagger \hat{a}_n$ local in terms of $\hat{f}_j$ operators; we show that this locality also holds for the other two systems in the Supplemental Material (SM) [33]. Such an extensive set of local conserved quantities leads to the absence particle transport [41]. This does not necessarily preclude information spreading—e.g., in MBL systems, interactions between the conserved quantities can lead to a slow growth of entanglement entropy [8,9] and out-of-time-order correlators [10–15]. Even so, since all our systems are spectrally equivalent to the Anderson insulator (2), such interactions are absent and we might expect that the EE and OTOC will quickly saturate to nonextensive values.

However, the presence of nonlocal "JW strings" in the transformations relating our systems plays an important role out of equilibrium. In systems (1) and (3), the excitation operators $\hat{a}_n^\dagger$ which relate different eigenstates are highly nonlocal, unlike in a typical Anderson insulator. We will see that the dynamics of these systems can unveil these nonlocal correlations which would, otherwise, cancel for eigenstates in equilibrium. The impact of JW strings on dynamical correlators for clean systems has been observed previously [42–44].

We use the Jordan-Wigner transforms to derive expressions for the EE and OTOC of all models in the SM [33]; these can be efficiently computed for large system sizes and long times. Hereon, we choose $T_j = 1$ and a uniform distribution for R_j, with mean $\langle R \rangle$ and width W_R.

Entanglement entropy.—Our quench protocol for the EE dynamics is as follows: a random unentangled product state of the relevant degrees of freedom (fermion occupation numbers or spins $\hat{\sigma}_j^z$) is time evolved under a disordered Hamiltonian. (The initial state energy densities are equal on average for all models.) The time-evolved density matrix $\hat{\rho}(t)$ is partitioned into left (A) and right (B) halves of the system, and the Renyi entropy $S^{(2)}(t) = -\ln\{[\mathrm{Tr}_B \hat{\rho}(t)]^2\}$ is calculated. We then average over $M = 10^4$ disorder realizations, yielding $\bar{S}^{(2)}(t)$.

Figure 1 shows the disorder-averaged EE dynamics after a quench for each of the systems (1), (2), and (3). We also show results for the transverse-field Ising chain (TFIC) for reference, where the second term in (1) is replaced by $R_j \hat{\sigma}_j^z$. The EE first grows ballistically, before plateauing after a short time as expected for an AL system. However, we see that, after a long time ($\sim 10^5$), the EE in models (1)–(3) starts to slowly grow as $\log t$, in contrast to that of the TFIC.

Such a logarithmic-in-time growth of EE is often associated with MBL phases [8,9], where interactions between conserved quantities lead to dephasing. However, since our systems are one-body reducible, they do not possess such interactions, so a different mechanism must be responsible.

We attribute the unusual EE growth to the presence of nonlocal Majorana edge modes in systems (2) and (3), which also manifest themselves as strong zero modes of (1) [28], and to ensemble averaging. Except at criticality, these systems always possess edge modes [31,45] with an energy that is exponentially small in the system size. For a given realization of disordered R_j, the energy of the zero mode is approximately $|\log E_{\mathrm{maj}}| \sim |\sum_j \log R_j|$ [45,46]. Thus, the distribution of this energy scale is Gaussian in its logarithm, and therefore, the statistical ensemble of systems possesses an exponential hierarchy of timescales. Such a distribution of energy scales can in general lead to quantities which depend logarithmically on time [47]. The TFIC does not possess zero modes for the parameters chosen and, thus, does not exhibit this slow growth; similarly, we have verified that the slow growth is absent in systems with periodic boundary conditions.

The above argument can be intuitively captured with a two-site fermionic toy model, described by four Majorana operators $\hat{\gamma}_{1,2}^{A,B}$. We construct a Hamiltonian which features one "edge mode" $\hat{f}_e = \hat{\gamma}_1^A + i\hat{\gamma}_2^B$ and one "bulk mode" $\hat{f}_b = \hat{\gamma}_2^A + i\hat{\gamma}_1^B$. The Hamiltonian is $\hat{H} = E_{\mathrm{maj}} \hat{f}_e^\dagger \hat{f}_e + E_b \hat{f}_b^\dagger \hat{f}_b$. We show, in the SM [33], that if one averages the EE $S^{(2)}(t)$ for this model over the appropriate distribution of the edge mode energies, i.e., $P(E_{\mathrm{maj}}) \sim 1/E_{\mathrm{maj}}$ for $E_- < E_{\mathrm{maj}} < E_+$, then we obtain $\bar{S}^{(2)}(t) \propto \log t$ for times $E_+^{-1} < t < E_-^{-1}$.

Out-of-time-order correlators.—We now study the dynamics of OTOCs in models (1)–(3). Specifically, we calculate the quantity (first proposed in Ref. [48] and recently revived in [49,50])

$$C(t) = \frac{1}{2} \langle [\hat{A}, \hat{B}(t)]^\dagger [\hat{A}, \hat{B}(t)] \rangle_\beta, \tag{4}$$

where $\hat{A}$ and $\hat{B}$ are local Hermitian operators which commute and each square to 1, $\langle \cdot \rangle_\beta$ denotes a thermal average at temperature β^{-1}. The above quantity contains the term $F(t) = \langle \hat{A}\, \hat{B}(t) \hat{A}\, \hat{B}(t) \rangle_\beta$ which features operators that are not time ordered from right to left. Clearly, $C(t) = 1 - \mathrm{Re}F(t)$. The physical intuition behind this quantity is that, in a chaotic system, the operator support of $\hat{B}(t)$ will spread and eventually overlap with the support of $\hat{A}$, at which point $C(t)$ will become nonzero. Thus, $C(t)$ measures operator spreading under the Hamiltonian of interest. The OTOC provides a way to understand how information spreads in localized systems. Logarithmic OTOC spreading has been proposed as a signature of MBL [10–15].

We compare how the OTOC evolves for each of the systems (1)–(3). We choose $\hat{A}_j$ and $\hat{B}_{j+r}$ to be the same local operator shifted by r and which will fix j while varying r. We choose $\hat{A}_j$ to be $\hat{\sigma}_j^y$, $(2\hat{f}_j^\dagger \hat{f}_j - 1)$, and $(2\hat{c}_j^\dagger \hat{c}_j - 1)$ for the XY, free, and symmetric models, respectively. We calculate the OTOC at infinite temperature by evaluating the operator in (4) on a randomly selected eigenstate for each disorder realization. The OTOCs of models (1) and (3) can be shown to be identical. Importantly, the OTOC expression for these two cases features JW strings between sites j and $j+r$.

The OTOC for our three models is plotted in Fig. 2, which was calculated using the formula derived in the SM [33]. In the free fermion case, the OTOC spreads for a short time and then saturates at time $t\sim\mathcal{O}(1)$, as one would expect for an AL system. However, the presence of strings qualitatively changes the behavior of the OTOC for systems (1) and (3). As one of the central results of this Letter, we find that the OTOC does not saturate at short times but spreads out. By plotting the OTOC at constant times, we see that $C(t)$ as a function of r always decays exponentially with r, but with a decay constant that decreases with time beyond the single-particle localization length, unlike in the free fermion case. For fixed distance, the onset as a function of time, as well as the approach to the long-time value, appears to be power-law, similar to other integrable systems [44,51].

Discussion.—We have identified two features in the dynamics of three integrable disordered models which lie beyond the physics expected for a typical Anderson insulator.

First, we observed a slow logarithmic-in-time growth of the disorder-averaged bipartite entanglement entropy; we argued, using a toy model, that this was due to the presence of strong zero modes in our models. The significance of the strong zero modes (as opposed to, e.g., ground state degeneracy due to spontaneous symmetry breaking) is twofold: it ensures that the entire spectrum is nearly pairwise degenerate; and it constitutes a mode whose wave function is delocalized across the chain, such that it is picked up by the entanglement cut (see the Supplemental Material [33] for details). We expect that this underlying mechanism for slow entanglement dynamics also applies to nonintegrable systems featuring strong zero modes, e.g., parafermionic models [46].

The limits of the energy distribution E_+ and E_- determine the timescales when the logarithmic growth begins and ends. Their values depend on the Hamiltonian parameters as well as the system size. Away from criticality, the energies $E_\pm$ decrease for larger system sizes, leading to a later onset of logarithmic growth; this explains the late onset of slow growth in Fig. 1. However, we expect this phenomenon to appear at earlier times in critical systems for arbitrarily large N or in cases where edge modes appear at finitely separated topological domain walls. Moreover, the infinite time value of the EE is expected to be a constant of order 1. Indeed, our results appear to be consistent with previous studies of entanglement dynamics in the disordered XX chain, i.e., the critical version of the XY model with $R_j=T_j$ [52].

Second, we observed a slow growth of the OTOC in models (1) and (3). The profile of OTOC spreading we see is not typical for MBL or ergodic systems, where an "information front" emerges separating regions of $C(t)\approx 0$ and $C(t)\approx 1$ [10–15]. This is in line with previous proofs of zero Lieb-Robinson velocities in related models [53]. However, the OTOC can reach appreciable values at spatial separations well beyond the single-particle localization length ξ (dotted line in Fig. 2), unlike one would expect for a typical Anderson localized system.

Indeed, in the language of [41], the single-particle orbital occupations $\hat{a}_n^\dagger\hat{a}_n$ in the free system (2) constitute a set of "l-bits": each forms a two-level system which can be defined locally in terms of the physical operators. However, the analogous quantities in the other models are not strictly l-bits, since the excitation operators $\hat{a}_n^\dagger$ are not local in the physical basis due to the JW strings. This subtlety does not affect the properties of static correlation functions, where $\langle b|\hat{a}_n^\dagger|b\rangle$ is necessarily zero if $|b\rangle$ is an eigenstate, but matrix elements between different eigenstates $\langle b|\hat{a}_n^\dagger|c\rangle$ are sensitive to this nonlocality, and such terms do appear in dynamical correlation functions.

Accordingly, let us express the OTOC $F(t)$ for an eigenstate $|\Psi\rangle=|b\rangle$ in a Lehmann representation, which gives (the states $|b\rangle$, $|c\rangle$, $|d\rangle$, $|e\rangle$ are all eigenstates)

$$F(t)=\sum_{c,d,e}\langle b|\hat{A}_j|c\rangle\langle c|\hat{B}_{j+r}|d\rangle\langle d|\hat{A}_j|e\rangle\langle e|\hat{B}_{j+r}|b\rangle \times \exp\left[i(E_b+E_d-E_c-E_e)t\right], \tag{5}$$

and let us consider the long-time limit of the OTOC $F(\infty)\coloneqq\lim_{T\to\infty}(1/T)\int_0^T \mathrm{d}t' F(t')$ [54]. The terms with nontrivial dynamics $E_b+E_d-E_c-E_e\neq 0$ will oscillate and average to zero in the long-time limit, leading to a decay of $F(t)$ from its initial value $F(0)=1$ [equivalently an increase of $C(t)$ from zero]. We now discuss the criteria for nontrivial terms to have finite matrix elements, and hence for $C(t)$ to be nonzero.

Since all our systems are spectrally equivalent to the noninteracting Hamiltonian (2), we can label energy eigenstates by their single-particle occupation numbers $\langle b|\hat{a}_n^\dagger\hat{a}_n|b\rangle \eqqcolon \eta_n^{(b)}$. Terms with nontrivial dynamics satisfy $\sum_n(\eta_n^{(b)}+\eta_n^{(d)}-\eta_n^{(c)}-\eta_n^{(e)})\epsilon_n\neq 0$ for single-particle energies ϵ_n. For a finite system, we assume that no two single-particle energies are commensurate, this quantity is only zero if $\tau_n^{(b)}+\tau_n^{(d)}-\tau_n^{(c)}-\tau_n^{(e)}=0$ for all n (i.e., there are no "accidental" cancellations of the incommensurate ϵ_n).

Therefore, we seek terms where $\eta_n^{(b)}+\eta_n^{(d)}-\eta_n^{(c)}-\eta_n^{(e)}\neq 0$ for at least one n. From (5), one sees that, if $\hat{A}_j$ has no overlap with the excitation operators $\hat{a}_n^\dagger$ and/or $\hat{a}_n$ (i.e., $\hat{A}_j$ cannot cause a transition in the value of η_n), then we must have $\eta_n^{(b)}=\eta_n^{(c)}$ and $\eta_n^{(d)}=\eta_n^{(e)}$, so the term will be static; the same holds for $\hat{B}_{j+r}$. Therefore, nonzero terms only arise when $\hat{A}_j$ and $\hat{B}_{j+r}$ can excite or deexcite the same single-particle orbital.

In generic AL systems, $\hat{A}_j$ will only be able to excite orbitals "near" site j, and similarly, $\hat{B}_{j+r}$ acts only near site $(j+r)$. For sufficiently large r, it will not be possible for $\hat{A}_j$ and $\hat{B}_{j+r}$ to simultaneously act on the same orbital without incurring a factor of $e^{-r/\xi}$, where ξ is the single-particle localization length; hence, the OTOC will not spread beyond the length ξ. Additionally, for small r, only an $\mathcal{O}(1)$ number of orbitals can participate in the nonzero terms, so the time at which the OTOC saturates to its long-time limit will also be $\mathcal{O}(1)$. This explains the fast saturation and spatial decay of the OTOC in Fig. 2(a).

However, in systems (1) and (3), the elementary excitations described by $\hat{a}_j^{(\dagger)}$ are nonlocal. This allows for $\hat{A}_j$ and $\hat{B}_{j+r}$ to act on the same orbital even when r is large. Indeed, when one expresses the OTOC in the free-fermion $\hat{f}_j$ basis, noncanceling JW strings appear between sites j and $(j+r)$, so all orbitals in this range can participate in the contributing terms. This leads to the long-time spreading of OTOCs beyond the static single-particle localization length. The number of participating single-particle energies ϵ_n is $\mathcal{O}(r)$, so the time taken to approach the long-time limit will also increase with r, since there will be more nearly canceling terms with slow dynamics in (5). This explains the qualitative aspects of the OTOC growth seen in Fig. 2(b). We expect that similar arguments hold for more general systems which also have nonlocal string correlations, leading to slow growth of OTOCs.

Note that OTOC operators $\hat{A}_j$ and $\hat{B}_{j+r}$ with canceling JW strings, e.g., $\hat{A}_j = (\hat{c}_j - \hat{c}_j^\dagger)(\hat{c}_{j+1} + \hat{c}_{j+1}^\dagger)$, would not be sensitive to the nonlocality of our systems, and we would see the same behavior as in Fig. 2(a). This sensitivity to the choice of OTOC has been reported in the clean TFIC in Ref. [44].

We note that fast oscillations in the OTOC for individual disorder realizations are expected even in the infinite-time limit due to the persistence of single-particle recurrences. The above arguments and the data shown in Fig. 2 characterize the long-time average over many disorder distributions; however, the variance in the data is large, as one would expect.

We thank Jens Bardarson, Markus Heyl, and Frank Pollmann for enlightening discussions, and Adam Smith for useful comments on the manuscript. M. M. acknowledges support from the EPSRC. A. N. holds a University Research Fellowship from the Royal Society and acknowledges support from the Winton Programme for the Physics of Sustainability.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] B. Kramer and A. MacKinnon, Rep. Prog. Phys. **56**, 1469 (1993).
[3] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. (Amsterdam) **321**, 1126 (2006).
[4] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[5] E. Altman and R. Vosk, Annu. Rev. Condens. Matter Phys. **6**, 383 (2015).
[6] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, arXiv:1804.11065.
[7] M. Žnidarič, T. Prosen, and P. Prelovšek, Phys. Rev. B **77**, 064426 (2008).
[8] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[9] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013).
[10] Y. Huang, Y.-L. Zhang, and X. Chen, Ann. Phys. (Amsterdam) **529**, 1600318 (2017).
[11] B. Swingle and D. Chowdhury, Phys. Rev. B **95**, 060201 (2017).
[12] R. Fan, P. Zhang, H. Shen, and H. Zhai, Sci. Bull. **62**, 707 (2017).
[13] D.-L. Deng, X. Li, J. H. Pixley, Y.-L. Wu, and S. Das Sarma, Phys. Rev. B **95**, 024202 (2017).
[14] X. Chen, T. Zhou, D. A. Huse, and E. Fradkin, Ann. Phys. (Amsterdam) **529**, 1600332 (2017).
[15] D. J. Luitz and Y. Bar Lev, Phys. Rev. B **96**, 020406 (2017).
[16] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science **349**, 842 (2015).
[17] J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science **352**, 1547 (2016).
[18] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Nat. Phys. **12**, 907 (2016).
[19] A. J. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, Phys. Rev. Lett. **109**, 020505 (2012).
[20] R. Islam, R. Ma, P. M. Preiss, M. Eric Tai, A. Lukin, M. Rispoli, and M. Greiner, Nature (London) **528**, 77 (2015).
[21] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A **94**, 040302 (2016).
[22] G. Zhu, M. Hafezi, and T. Grover, Phys. Rev. A **94**, 062329 (2016).
[23] A. M. Kaufman, M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss, and M. Greiner, Science **353**, 794 (2016).
[24] M. Gärttner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Nat. Phys. **13**, 781 (2017).
[25] L. García-Álvarez, I. L. Egusquiza, L. Lamata, A. del Campo, J. Sonner, and E. Solano, Phys. Rev. Lett. **119**, 040501 (2017).
[26] K. A. Landsman, C. Figgatt, T. Schuster, N. M. Linke, B. Yoshida, N. Y. Yao, and C. Monroe, arXiv:1806.02807.
[27] A. Lukin, M. Rispoli, R. Schittko, M. E. Tai, A. M. Kaufman, S. Choi, V. Khemani, J. Léonard, and M. Greiner, arXiv: 1805.09819.
[28] P. Fendley, J. Phys. A **49**, 30LT01 (2016).
[29] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).
[30] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).

[31] J.-J. Miao, H.-K. Jin, F.-C. Zhang, and Y. Zhou, Phys. Rev. Lett. **118**, 267701 (2017).
[32] X. Yu and M. Müller, Ann. Phys. (Amsterdam) **337**, 55 (2013).
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.020603 for details of the solutions of models (1) and (3); derivations of closed-form expressions for the entanglement entropy and the OTOC; a discussion of the toy model used to derive logarithmic entanglement growth; and a disucssion of the locality of l-bits in the physical bases, which contains references [34–40].
[34] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).
[35] M. Fagotti and P. Calabrese, J. Stat. Mech. (2010) P04016.
[36] I. Peschel, J. Phys. A **36**, L205 (2003).
[37] V. Eisler, F. Maislinger, and H. G. Evertz, SciPost Phys. **1**, 014 (2016).
[38] I. L. Aleiner, L. Faoro, and L. B. Ioffe, Ann. Phys. (Amsterdam) **375**, 378 (2016).
[39] S. Bravyi and R. König, Commun. Math. Phys. **316**, 641 (2012).
[40] J. I. Latorre, E. Rico, and G. Vidal, Quantum Inf. Comput. **4**, 48 (2004).
[41] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014).
[42] D. Rossini, A. Silva, G. Mussardo, and G. E. Santoro, Phys. Rev. Lett. **102**, 127204 (2009); D. Rossini, S. Suzuki, G. Mussardo, G. E. Santoro, and A. Silva, Phys. Rev. B **82**, 144302 (2010).
[43] P. Calabrese, F. H. L. Essler, and M. Fagotti, Phys. Rev. Lett. **106**, 227203 (2011); J. Stat. Mech. (2012) P07016.
[44] C.-J. Lin and O. I. Motrunich, Phys. Rev. B **97**, 144304 (2018).
[45] M. McGinley, J. Knolle, and A. Nunnenkamp, Phys. Rev. B **96**, 241113 (2017).
[46] P. Fendley, J. Stat. Mech. (2012) P11020.
[47] A. Amir, Y. Oreg, and Y. Imry, Annu. Rev. Condens. Matter Phys. **2**, 235 (2011).
[48] A. I. Larkin and Y. N. Ovchinnikov, Sov. Phys. JETP **28**, 1200 (1969).
[49] S. H. Shenker and D. Stanford, J. High Energy Phys. 03 (2014) 67.
[50] A. Kitaev, A simple model of quantum holography, http://online.kitp.ucsb.edu/online/entangled15/kitaev/.
[51] B. Dóra and R. Moessner, Phys. Rev. Lett. **119**, 026802 (2017).
[52] G. D. Chiara, S. Montangero, P. Calabrese, and R. Fazio, J. Stat. Mech. (2006) P03001.
[53] H. Abdul-Rahman, B. Nachtergaele, R. Sims, and G. Stolz, Ann. Phys. (Amsterdam) **529**, 1600280 (2017).
[54] Y. Huang, F. G. S. L. Brandao, and Y.-L. Zhang, arXiv:1705.07597.