Coulomb Blockade of a Nearly Open Majorana Island

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We consider the ground-state energy and the spectrum of the low-energy excitations of a Majorana island formed of topological superconductors connected by a single-mode junction of arbitrary transmission. Coulomb blockade results in e-periodic modulation of the energies with the gate-induced charge. We find the amplitude of modulation as a function of reflection coefficient $\mathcal{R}$. The amplitude scales as $\sqrt{\mathcal{R}}$ in the limit $\mathcal{R} \to 0$. At larger $\mathcal{R}$, the dependence of the amplitude on the Josephson and charging energies is similar to that of a conventional-superconductor Cooper-pair box. The crossover value of $\mathcal{R}$ is small and depends on the ratio of the charging energy to superconducting gap.

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The Coulomb blockade phenomenon is associated with the localization of charge in a small conductor with appreciable charging energy. The Coulomb blockade results in the observable quantities being periodic functions of the charge induced by an applied gate voltage. For a normal system, this periodicity in the induced charge is e, while for an island of conventional (s-wave) superconductor, a so-called Cooper-pair box, the periodicity is $2e$.

With a junction between the island and a lead, charging effects are smeared by delocalization of the electrons. Remarkably, the Coulomb blockade is fully suppressed by the presence of even a single reflectionless channel in the junction [1]. The way oscillations vanish depends on the relevant low-energy excitations. For normal-state conductors, the spectrum is continuous and gapless; the effect of weak reflection can be read off from known results for a quantum impurity in a Luttinger liquid [2,3]. When the island and the lead are s-wave superconductors, the ground state is nondegenerate and separated from the continua by gaps. In this case, the destruction of the Coulomb blockade is described by an imaginary-time version of the Landau-Zener diabatic crossing of two in-gap levels, with the off-diagonal matrix element being proportional to the backscattering amplitude [4].

In this Letter, we elucidate the nature of the suppression of Coulomb blockade in a nearly open system made of topological superconductors, illustrated in Fig. 1. The topological superconductors are characterized by a finite gap in the energy spectrum, coexisting with a nontrivial degeneracy of the ground state, which causes the periodicity in the induced charge to be e and not $2e$. This difference in the states and spectra from both conventional superconductors and normal metals results in a different underlying physics of the disappearance of Coulomb blockade oscillations at perfect transmission. We show that it is related to the physics of diabatic transitions between a discrete state and a continuum of itinerant states, and we formulate a quantitative theory valid for the crossover from a regime where the amplitude of Coulomb blockade oscillations is proportional to the reflection amplitude to a regime where the physics is similar to a conventional Cooper-pair box [5].

The system shown in Fig. 1 has become experimentally relevant since the appearance of viable theoretical models of one-dimensional topological superconductors [6–9]. Several recent experiments reported data consistent with topological superconductivity in Coulomb blockade devices [10–12], thus opening a perspective for the experimental study of the quantum charge fluctuations considered here. Moreover, topological superconducting islands have

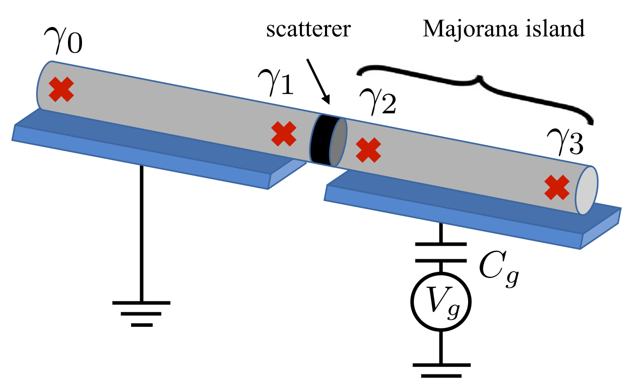

FIG. 1. Two topological superconductors, hosting Majorana zero modes γ_i, are connected by a single-channel junction with reflection coefficient $\mathcal{R}$. Capacitively coupled gate induces average charge bias $e\mathcal{N}_g = C_g V_g$.

been the basis for several proposals for Majorana-based qubits [13–16], some of which [13,14] use control of the charging energy to lift the ground-state degeneracy. The theory of such control is another application of our work.

The conventional transmon qubit is a Cooper-pair box with the charging energy much smaller than the Josephson energy. This arrangement is chosen to suppress charge fluctuations and increase the coherence time of the qubit. In the present work we focus on the case where the charging energy E_C is relatively small, $E_C \ll \Delta$ (here Δ is the superconducting gap in the topological phase; it also fixes the scale of the Josephson energy in the single-channel junction), which is also the limit considered for a conventional transmon [5]. We find that the gate-induced charge $e\mathcal{N}_g$ modulates the energy levels of the topological transmon,

$$\delta E_m(\mathcal{N}_g) = (-1)^{m+1}\frac{\epsilon_m}{2}\cos(2\pi\mathcal{N}_g), \tag{1}$$

where m labels the energy levels, with $m = 0$ being the ground state [17]; unlike the conventional transmon, the modulation period is e. The charge sensitivity comes from the Aharonov-Casher effect [18] in tunneling of the phase variable φ between the classically equivalent minima ($\varphi = 0, 4\pi$ in Fig. 2). The modulation amplitude ϵ_m is

$$\epsilon_m = F(h)E_C\frac{2^{4m+3}}{m!}\sqrt{\frac{2}{\pi}}\left(\frac{E_M}{E_C}\right)^{(2m+3)/4}e^{-4\sqrt{E_M/E_C}}. \tag{2}$$

Here $E_M = \Delta\sqrt{1-\mathcal{R}}$ is the height of the barrier separating the two minima of the ground-state energy in the absence of charging, and $\mathcal{R}$ is the reflection coefficient. Apart from the function $F(h)$, Eq. (2) closely resembles the respective formula [5] for a conventional transmon. It is valid if the electron system is able to adjust to the instantaneous values of φ in the course of tunneling. Such adiabaticity requires a sufficiently large value of the reflection coefficient $\mathcal{R}$. The function $F(h)$ describes the crossover between the diabatic and adiabatic regimes:

$$F(h) = \frac{3^{1/6}}{2^{2/3}}\Gamma(2/3)h \approx 1.02h, \qquad h \ll 1, \tag{3}$$

$$F(h) = 1 - \frac{\pi}{8}h^{-3} \approx 1 - 0.39h^{-3}, \qquad h \gg 1. \tag{4}$$

It depends on a single variable:

$$h = \left(\frac{\Delta}{16E_C}\right)^{1/6}\sqrt{\mathcal{R}}. \tag{5}$$

We first note that $F(0) = 0$, i.e., in the absence of reflection $\delta E_m = 0$, in agreement with the general properties [2–4,19,20] of the Coulomb blockade effect discussed in the introduction. Below, we derive Eqs. (1)–(5) and

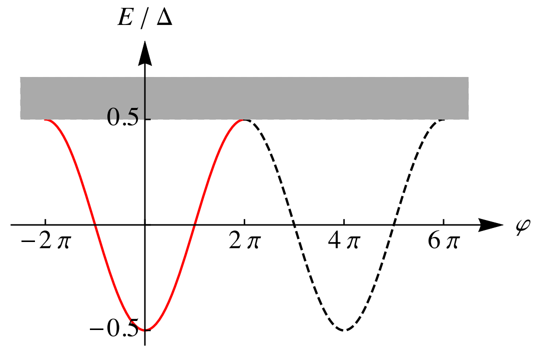

FIG. 2. Energy spectrum of a topological junction in the absence of backscattering. At $\mathcal{R} = 0$, the bound states are degenerate at $\varphi = 2\pi$ mod 4π with the edge of continuum (shaded area).

show that the entire crossover from $F(h)\to 0$ to $F(h)\to 1$ occurs in a narrow region of reflection coefficients, $\mathcal{R} \sim (16E_C/\Delta)^{1/3} \ll 1$ [21].

At zero charging energy, phase φ across the junction is a good quantum number. Assuming that only one pair of helical modes propagates across a short junction, the phase-dependent part of the ground-state energy in the sector with an even number of electrons takes the form [6,22]

$$E_G(\varphi) = -\frac{1}{2}E_M\cos(\varphi/2). \tag{6}$$

Here the sign is fixed by the total parity, which we assume to be conserved. Furthermore, in a ballistic junction ($\mathcal{R} = 0$), the momentum associated with the propagating modes is conserved. The bound states are formed out of states of one chirality: these are, respectively, the right movers at $0 < \varphi < 2\pi$ and left movers at $2\pi < \varphi < 4\pi$, cf. the solid (red) and bold dashed (black) curves in Fig. 2. The two bound states become degenerate with each other *and* with the edge of the continuum at $\varphi = 2\pi$. In the presence of backscattering induced by any finite $\mathcal{R}$, both left and right movers participate in the formation of the continuum and bound states. As a result, the degeneracy is lifted, and the gap between the ground state and continuum, $\frac{1}{2}(\Delta - E_M)$, is finite at $\varphi = 2\pi$.

Finite charging energy endows the phase with quantum dynamics; the same-parity, classically distinguishable states corresponding to $\varphi = 0, 4\pi, \ldots$ may hybridize. The hybridization does not occur at $\mathcal{R} = 0$, as these states are protected by the movers' momentum conservation, but they do hybridize at $\mathcal{R} \neq 0$. At small charging energy, $E_C \ll \Delta$, one may view the hybridization as the result of phase tunneling between the nearest minima ($\varphi = 0, 4\pi$ in Fig. 2).

If the gap $\frac{1}{2}(\Delta - E_M)$ is large enough, phase tunneling occurs in the adiabatic regime and is governed by the Hamiltonian

$$H_0 = E_C(-2i\partial_\varphi - \mathcal{N}_g)^2 + E_G(\varphi) \tag{7}$$

acting in the space of 4π-periodic functions. Here $\hat{N} = -2i\partial/\partial\varphi$ is the operator for the electron number of the island. To find the energy spectrum of H_0 as a function of $\mathcal{N}_g$, we map the problem onto the known one for the conventional transmon [5] and find Eq. (2) with $F(h)$ replaced by 1 (see Secs. I and VIII of Ref. [23] for details).

The adiabatic approximation fails if the gap $\frac{1}{2}(\Delta - E_M)$ is small. The corresponding quantum dynamics of the many-body state in the topological case is very different from that in the conventional s-wave case [4]. Disregarding for a moment the difference between driving the variable φ classically and allowing it to tunnel, one may say that the conventional problem is related to the Landau-Zener passage of an avoided crossing between two discrete many-body states. On the contrary, Coulomb blockade in the topological junction is related to a Demkov-Osherov process involving a discrete state and continuum [24].

We may estimate $\mathcal{R}$ at which adiabaticity is violated by a qualitative consideration that ignores the difference between the real-time evolution and tunneling of the phase (i.e., "imaginary-time" evolution) across the $\varphi = 2\pi$ point. The separation $E_{\rm ex}(\theta)$ of the bound state energy from continuum is small at $\mathcal{R} \ll 1$ and $|\varphi - 2\pi| \ll 1$; using Eq. (6), we find (hereinafter, $\theta = \varphi - 2\pi$)

$$E_{\rm ex}(\theta) = \frac{1}{4}\left(\mathcal{R} + \frac{\theta^2}{4}\right)\Delta. \tag{8}$$

The energy $E_{\rm ex}(\theta)$ can be estimated as $E_{\rm ex}(\theta^*) \sim \mathcal{R}\Delta$ everywhere within the interval $|\theta| \lesssim \theta^*$, where $\theta^* = \sqrt{\mathcal{R}}$. In the (imaginary) time domain, it takes time $\tau(\theta^*) \sim \theta^*/\omega_P$ to pass this interval; here $\omega_P = \sqrt{E_C E_M} \approx \sqrt{E_C \Delta}$ is the Josephson plasma frequency which determines the timescale for both oscillations and tunneling of the phase. The phase is passing the point $\theta = 0$ adiabatically if $E_{\rm ex}(\theta^*)\tau(\theta^*) \gg 1$. Under that condition, the electron system adjusts to the instantaneous value of φ and the use of the Hamiltonian Eq. (7) at any φ is justified. Expressing $E_{\rm ex}(\theta^*)$ and $\tau(\theta^*)$ in terms of $\mathcal{R}$ and utilizing the definition Eq. (5), we find that the adiabaticity is violated at $h \sim 1$, which indeed is the crossover scale for the function $F(h)$, cf. Eq. (2).

To quantify the crossover behavior, we notice that Eq. (7) determines the dynamics of the many-body state in the Born-Oppenheimer (adiabatic) approximation with φ being the slow variable. In that approximation, the eigenfunction of the system is factorized, $\Psi(\{x_i\}, \varphi) \approx \Psi_\varphi(\{x_i\})\psi(\varphi)$. The first factor here is the many-body BCS wave function of the electron ground state at a given phase φ. The phase-dependent part of the corresponding energy, $E_G(\varphi)$, appears in Eqs. (6) and (7). The single-particle states comprising $\Psi_\varphi(\{x_i\})$ are defined by the Bogoliubov–de Gennes equations where φ is treated as a parameter. The second factor, $\psi(\varphi)$, is an eigenfunction of Eq. (7). If $\mathcal{R} \gg (E_C/\Delta)^{1/3}$ (i.e., $h \gg 1$), then the Born-Oppenheimer wave function is a good leading-order approximation at all φ. In the opposite case, $h \ll 1$, we use the condition $E_{\rm ex}(\theta)\tau(\theta) \gtrsim 1$ to determine the range of φ (within the period $[0, 4\pi]$), where the adiabatic approximation is applicable. That yields $|\varphi - 2\pi| \gtrsim (E_C/\Delta)^{1/6}$. Our strategy is to find $\Psi(\{x_i\}, \varphi)$ in the region $|\varphi - 2\pi| \ll 2\pi$ by a method inspired by the Demkov-Osherov approach [24] and then match the found $\Psi(\{x_i\}, \varphi)$ with the Born-Oppenheimer wave function in the common region of applicability $(E_C/\Delta)^{1/6} \lesssim |\varphi - 2\pi| \ll 2\pi$. Knowing the wave functions in the entire interval $[0, 4\pi]$ allows us to find the dependence of energy spectrum on $\mathcal{N}_g$.

To illustrate the strategy, we concentrate on finding $\delta E_0(0)$, cf. Eq. (1). In the vicinity of $\varphi = 0$, the function $\psi(\varphi)$ is well approximated by the eigenstate of a harmonic oscillator:

$$\psi(\varphi) = \frac{(\Delta/E_C)^{1/8}}{(8\pi)^{1/4}} \exp\left(-\frac{\varphi^2}{16}\sqrt{\frac{\Delta}{E_C}}\right). \tag{9}$$

Next we extend Eq. (9) to the apex of the classically forbidden region, $2\pi \gg 2\pi - \varphi \gg \max[\sqrt{\mathcal{R}}, (E_C/\Delta)^{1/6}]$, by using the WKB approximation. This yields

$$\psi(\theta) = \frac{(\Delta/E_C)^{1/8}}{(2\pi)^{1/4}} e^{-2\sqrt{\Delta/E_C}} \exp\left(-\frac{\theta - \theta^3/96}{2\sqrt{E_C/\Delta}}\right). \tag{10}$$

Clearly, the exponentially small factor in Eq. (10) does not affect the normalization factor in Eq. (9). The extension of Eqs. (9) and (10) to arbitrary $\mathcal{N}_g$ and for the entire classically forbidden region is given in Secs. I–III of Ref. [23].

Finding the many-body state is simplified by the observation that the phase-dependent energy $E_G(\varphi)$ of a short junction comes from one single-particle bound state (the latter is formed by two Majorana states γ_2, γ_3 hybridized across the junction; see Fig. 1). That allows us to replace $\{x_i\}$ by a single generalized coordinate, $\Psi(\{x_i\}, \varphi) \to \Psi(x, \theta)$. In the vicinity of $\theta = 0$, the activation energy of the bound state becomes small; see Eq. (8). That further simplifies the problem, as the relevant states are linear combinations of quasiparticle wave functions with energies close to Δ. Similar to the effective mass approximation in the theory of semiconductors [25], we construct an effective Hamiltonian [26,27],

$$H_{\rm eff} = 4E_C(-i\partial_\theta - \mathcal{N}_g/2)^2 + \frac{1}{2}\left[\frac{v_F^2}{2\Delta}(-i\partial_x)^2 - v_F\left(\frac{\theta}{2}\hat{\sigma}_z + \sqrt{\mathcal{R}}\hat{\sigma}_x\right)\delta(x)\right] + \frac{\Delta}{2}; \tag{11}$$

here $\hat{\sigma}_{x,y,z}$ are Pauli matrices in the space of right- or left-propagating states and v_F is the Fermi velocity (it drops out from final results). The divergent-at-the-gap density of states and energy $E_{\mathrm{ex}}(\theta)$ are correctly described by H_{eff}; see Sec. IV in Ref. [23]. Note that $[\hat{\sigma}_z, H_{\mathrm{eff}}] = 0$ at $\mathcal{R} = 0$, and the bound states at $\theta > 0$ and $\theta < 0$ belong to orthogonal subspaces. Therefore, at $\mathcal{R} = 0$ there is no tunneling between the $\varphi = 0, 4\pi$ minima, consistent with momentum conservation.

As we are interested in states with energy $E \approx -\Delta/2$ (see Fig. 2), the problem can be further simplified by factoring out the leading (linear in θ) exponential term in the wave function and replacing x and θ by dimensionless variables y and z:

$$\Psi(x,\theta) = \exp(-\sqrt{\Delta/4E_C}\theta)\Psi(y,z),\\ x = 2^{-2/3}(\Delta/E_C)^{1/6}(v_F/\Delta)y,\\ \theta = 2^{5/3}(E_C/\Delta)^{1/6}z. \tag{12}$$

In the new variables, the Schrödinger equation for $\Psi(y,z)$ at $\mathcal{N}_g = 0$ depends on a *single* parameter h given by Eq. (5) (see also Sec. V of Ref. [23]):

$$\left[\partial_z - \frac{1}{2}\partial_y^2 - (z\hat{\sigma}_z + h\hat{\sigma}_x)\delta(y)\right]\Psi(y,z) = 0. \tag{13}$$

Its solution in the Born-Oppenheimer approximation,

$$\Psi^{(0)}(y,z) = \psi_z^{(0)}(y)g^{(0)}(z)\hat{U}(z)\chi,\\ \psi_z^{(0)}(y) = 2^{1/3}\left(\frac{E_C}{\Delta}\right)^{1/12}\left(\frac{\Delta}{v_F}\kappa_z\right)^{1/2}e^{-\kappa_z|y|},\\ g^{(0)}(z) = \frac{(\Delta/E_C)^{1/8}}{(2\pi)^{1/4}}e^{-2\sqrt{\Delta/E_C}}\exp\left(\frac{1}{2}\int_0^z dz'\kappa_{z'}^2\right), \tag{14}$$

reproduces Eq. (10) in its region of validity [upon returning from $g^{(0)}(z)$ to $\psi(\theta)$]. Here $\kappa_z = (z^2 + h^2)^{1/2}$, pseudospinor χ is an eigenvector, $\hat{\sigma}_z\chi = \chi$, and the unitary operator,

$$\hat{U}(z) = \exp\left[-\frac{i}{2}\cot^{-1}\left(-\frac{z}{h}\right)\hat{\sigma}_y\right], \tag{15}$$

rotates it to align with the z-dependent quantization axis.

The rotation rate in Eq. (15) scales as $1/h$; obviously, the adiabatic approximation fails at $h \ll 1$. We develop perturbation theory in h to find the energy eigenvalues in this limit. At $h = 0$, we can take advantage [24] of the linear z dependence of a coefficient in Eq. (13) and solve the partial differential equations for $\sigma_z = \pm 1$ analytically. For that, we apply the Fourier transformation to Eq. (13),

$$(ip + k^2/2)\psi_{\sigma_z}(k,p) = -\sigma_z i\partial_p F_{\sigma_z}(p),\\ F_{\sigma_z}(p) \equiv \int_{-\infty}^{\infty}\frac{dk}{2\pi}\psi_{\sigma_z}(k,p), \tag{16}$$

which allows us to obtain a closed first-order differential equation for $F_{\sigma_z}(p)$,

$$-i\sigma_z[e^{-i\pi/4}/(2p)^{1/2}]\partial_p F_{\sigma_z}(p) = F_{\sigma_z}(p) \tag{17}$$

($p^{1/2} > 0$ for $p > 0$). Solution of Eq. (17) followed by inverting the Fourier transform $\psi_{\sigma_z}(k,p)$ of Eq. (16) yields

$$\psi_{-1}(y,-z)\\ = \psi_1(y,z)\\ = 2^{7/12}\pi^{1/4}e^{-2\sqrt{\Delta/E_C}}(\Delta/E_C)^{1/24}(\Delta/v_F)^{1/2}\\ \times\int_{-\infty}^{\infty}\frac{dp}{2\pi}\exp\left(ipz - (2ip)^{1/2}|y| + \frac{2}{3}i(i+1)p^{3/2}\right). \tag{18}$$

The constant of integration here is found by matching the $|z| \gg 1$, $z < 0$ asymptote of Eq. (18) with the Born-Oppenheimer limit, Eq. (14). Knowing the wave functions Eq. (18) at $h = 0$, we may express the first-order correction to energy in terms of the matrix element of perturbation, $\langle\psi_{-1}(y,z)|h\hat{\sigma}_x\delta(y)|\psi_1(y,z)\rangle$:

$$\epsilon_0 = 2^{8/3}v_F\sqrt{\mathcal{R}}(E_C/\Delta)^{1/6}\int_{-\infty}^{\infty}dz\psi_1^*(0,z)\psi_{-1}(0,z). \tag{19}$$

Performing the integration with the help of Eq. (18), we arrive at the asymptote Eq. (3); see also Sec. VI of Ref. [23].

In the opposite case, $h \gg 1$, we find correction Eq. (4) by perturbing away from the adiabatic limit, Eq. (14). The correction stems from the perturbations $\partial_z\hat{U}(z), \partial_z\psi_z^{(0)} \propto 1/h$ appearing in Eq. (13) upon substitution of Eqs. (14) and (15) in it. We are interested in the correction which vanishes at $z \to -\infty$ and modifies the asymptote of the adiabatic, localized in y, solution at $z \gg 1$. The perturbations, effective in the region $|z| \lesssim h$, mix the localized state with the itinerant ones, differing in energy by $\sim h^2$. Therefore, the modification of the localized state $\Psi^{(0)}(y,z)$ appears in the second-order perturbation theory. The power counting thus gives $+1$ from the term in the Hamiltonian, -2 from the second-order perturbation theory, and -2 from the energy cost giving the correction $\propto 1/h^3$. The evaluation of the numerical coefficient appearing in Eq. (4) is presented in Sec. VII of Ref. [23].

The interpolation between the diabatic and adiabatic asymptotes of $F(h)$ is shown in Fig. 3. It is obtained by generalizing H_{eff} to arbitrary phases with the help of substitution $\theta/2 \to 2\sin(\theta/4)$ in Eq. (11). The generalized Hamiltonian, being projected at $\mathcal{R} \ll 1$ on its low-energy sector, reproduces Eq. (7) in the region of phases $|\theta| \gg (E_C/\Delta)^{1/6}$. By finding numerically the energy spectrum of that Hamiltonian, we get the relative amplitude of

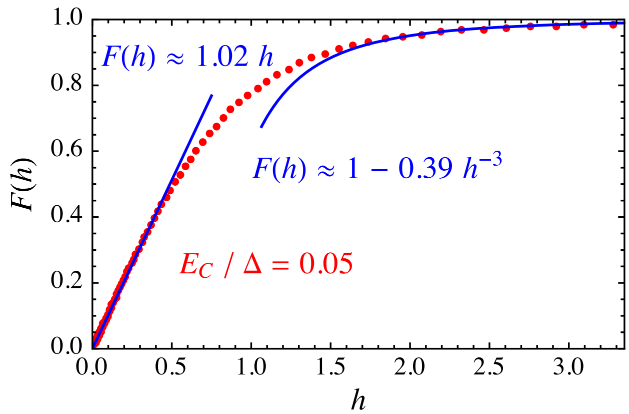

FIG. 3. Full crossover function $F(h)$; see Eq. (2). Dots show numerical solution of the eigenvalue problem at $E_C/\Delta = 0.05$ and varying $\mathcal{R}$, expressed in terms of h given by Eq. (5); lines show analytically found asymptotes Eqs. (3) and (4).

the gate modulation F as a function of two parameters $\mathcal{R}$ and E_C/Δ (see details in Sec. IX of Ref. [23]). The results at the lowest values of E_C/Δ are compatible with F depending on a single parameter, $\sqrt{\mathcal{R}}(\Delta/E_C)^{1/6} \propto h$, and having asymptotes Eqs. (3) and (4).

To conclude, we addressed the problem of the crossover from a pronounced charging effect to its full absence in a topological superconducting junction upon reduction of the reflection coefficient $\mathcal{R}$. The many-body problem was reduced to that of tunneling of a system with a few degrees of freedom—charge and coordinate of an effective particle fluctuating between the state localized in the junction and scattering states in the continuum. The reduction allowed us to find the full crossover function $F(h)$. The control parameter h depends weakly on Δ/E_C, so that $h \approx (0.6\text{–}1.1)\sqrt{\mathcal{R}}$ for $\Delta/E_C = 1\text{–}25$. The function $F(h)$ is well approximated by a linear dependence for $F \lesssim 0.5$; in this range, $F(h) \sim \sqrt{\mathcal{R}}$ for typical values of Δ/E_C.

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[22] For definiteness, hereinafter we assume the device is in an even total-parity state; the sign of the right-hand side is opposite in the odd-parity state.
[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.122.016801 for details of the derivation of asymptotes, Eqs. (3) and (4), and the numerics can be found in the SM.
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