Swarming in the Dirt: Ordered Flocks with Quenched Disorder

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The effect of quenched (frozen) disorder on the collective motion of active particles is analyzed. We find that active polar systems are far more robust against quenched disorder than equilibrium ferromagnets. Long-ranged order (a nonzero average velocity $\langle \mathbf{v} \rangle$) persists in the presence of quenched disorder even in spatial dimensions $d = 3$; in $d = 2$, quasi-long-ranged order (i.e., spatial velocity correlations that decay as a power law with distance) occurs. In equilibrium systems, only quasi-long-ranged order in $d = 3$ and short-ranged order in $d = 2$ are possible. Our theoretical predictions for two dimensions are borne out by simulations.

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Introduction.—A great deal of the immense current interest in "active matter" focuses on coherent collective motion, i.e., "flocking" [1–7], or "swarming" [8,9]. Such coherent motion occurs over a wide range of length scales: from macroscopic organisms to mobile macromolecules in living cells [8–11] and synthetic active particles [12,13] and in the presence of complex environments [14,15]. Such coherent motion is possible even in $d = 2$ [2], in apparent violation of the Mermin-Wagner theorem [16]. This has been explained by the "hydrodynamic" theory of flocking [3–7], which shows that, unlike equilibrium "pointers", nonequilibrium "movers" *can* spontaneously break a continuous symmetry (rotation invariance) by developing long-ranged orientational order (as they must to have a nonzero average velocity $\langle \mathbf{v}(\mathbf{r}, t) \rangle \neq \mathbf{0}$), even in noisy systems with only short-range interactions in dimension $d = 2$, and in flocks with birth and death [17].

In equilibrium systems, even *arbitrarily weak* quenched random fields destroy long-ranged ferromagnetic order in all spatial dimensions $d \leq 4$ [18–21]. This raises the question: can the nonlinear, nonequilibrium effects that make long-ranged order possible in 2D flocks without quenched disorder stabilize them when random field disorder is present? Simulations of flocks with quenched disorder [22,23] find quasi-long-ranged order in $d = 2$; that is,

$$\overline{\mathbf{v}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}', t)} \propto |\mathbf{r} - \mathbf{r}'|^{-\eta}, \tag{1}$$

where the exponent η is nonuniversal (that is, system dependent), and the over bar denotes an average over $\mathbf{r}$ with fixed $\mathbf{r} - \mathbf{r}'$.

In this Letter and the companion paper [24], we address this problem analytically and by simulations. The analytical approach (the focus of this Letter) extends the hydrodynamic theory of flocking developed in Refs. [3–7] to include quenched disorder. Both approaches confirm that flocks are more robust against quenched disorder than ferromagnets. Specifically, we find that flocks *can* develop long-ranged order in three dimensions, and quasi-long-ranged order in two dimensions, due to strong nonlinear effects, in contrast to the equilibrium case, in which only short-ranged order is possible in two dimensions [18–21], and only quasi-long-ranged order in three dimensions. We also determine exact scaling laws for velocity fluctuations for one range of hydrodynamic parameters in $d = 3$.

Hydrodynamic theory.—To study the effects of quenched disorder for flocking, we use the hydrodynamic theory of Refs. [3–7], modified only by the inclusion of a quenched random force $\mathbf{f}$. In the thermodynamic limit, the annealed noise is irrelevant for determining the stability and scaling of the flocking phase in the presence of the quenched noise, and is thus neglected in the current study. In the ordered phase with average speed v_0 and an average density ρ_0, the velocity ($\mathbf{v}$) and density (ρ) fields can be written as $\mathbf{v} \approx v_0 \mathbf{e}_\parallel + \mathbf{v}_\perp$, $\rho = \rho_0 + \delta\rho$, where $\mathbf{e}_\parallel$ is the unit vector in the direction of mean flock motion. Plugging these in the original hydrodynamic equations (shown in Ref. [24], see also the original papers Refs. [3–7]), we obtain the following pair of coupled equations of motion for the fluctuation $\mathbf{v}_\perp(\mathbf{r}, t)$ of the local velocity of the flock perpendicular to $\mathbf{e}_\parallel$, and the departure $\delta\rho(\mathbf{r}, t)$ of the density from its mean value ρ_0:

$$\begin{aligned}
&\partial_t \mathbf{v}_\perp + \gamma \partial_\parallel \mathbf{v}_\perp + \lambda(\mathbf{v}_\perp \cdot \nabla_\perp)\mathbf{v}_\perp \\
&\quad = -g_1 \delta\rho \partial_\parallel \mathbf{v}_\perp - g_2 \mathbf{v}_\perp \partial_\parallel \delta\rho - g_3 \mathbf{v}_\perp \partial_t \delta\rho - \frac{c_0^2}{\rho_0} \nabla_\perp \delta\rho \\
&\qquad - g_4 \nabla_\perp(\delta\rho^2) + D_B \nabla_\perp(\nabla_\perp \cdot \mathbf{v}_\perp) + D_T \nabla_\perp^2 \mathbf{v}_\perp \\
&\qquad + D_\parallel \partial_\parallel^2 \mathbf{v}_\perp + \nu_t \partial_t \nabla_\perp \delta\rho + \nu_\parallel \partial_\parallel \nabla_\perp \delta\rho + \mathbf{f}_\perp,
\end{aligned} \tag{2}$$

$$\partial_t \delta\rho + \rho_o \nabla_\perp \cdot \mathbf{v}_\perp + \lambda_\rho \nabla_\perp \cdot (\mathbf{v}_\perp \delta\rho) + v_2 \partial_\parallel \delta\rho = D_{\rho\parallel} \partial_\parallel^2 \delta\rho + D_{\rho v} \partial_\parallel (\nabla_\perp \cdot \mathbf{v}_\perp) + \phi \partial_t \partial_\parallel \delta\rho + \partial_\parallel (w_1 \delta\rho^2 + w_2 |\mathbf{v}_\perp|^2), \tag{3}$$

where λ and λ_ρ are dimensionless coefficients for the nonlinear convective terms, $D_{\mathrm{Beff},T,\parallel,\rho\parallel,\rho v}$, $\nu_{t,\parallel}$ are coefficients for the linear terms (e.g., diffusion terms), $g_{1,2,3,4}$, $w_{1,2}$ are nonlinear coupling constants, c_0 sets the scale of the sound speed, and ϕ sets the diffusion length scale. There are two parameters $\gamma = \lambda v_0$ and $v_2 = \lambda_\rho v_0$ that are particularly relevant for our current study; they correspond to the speeds of the velocity and density fluctuations advected by the mean flocking motion along $\mathbf{e}_\parallel$.

To treat quenched disorder, we simply take the random force to be *static*, i.e., to depend *only* on position, $\mathbf{f}(\mathbf{r}, t) = \mathbf{f}(\mathbf{r})$, and not on time t at all, with short-ranged spatial correlations:

$$\overline{f_i^\perp(\mathbf{r}) f_j^\perp(\mathbf{r}')} = \Delta \delta_{ij}^\perp \delta^d(\mathbf{r} - \mathbf{r}'), \tag{4}$$

where the over bar denotes averages over the quenched disorder, and $\delta_{ij}^\perp = 1$ if and only if $i = j \neq \parallel$, and is zero for all other i, j. We will also assume $\mathbf{f}_\perp$ is zero mean, and Gaussian.

Linearized hydrodynamic theory and anisotropic fluctuations.—Our first step in analyzing these equations is to linearize them. We then Fourier transform them in space and time and decompose the velocity $\mathbf{v}_\perp$ along and perpendicular to the projection $\mathbf{q}_\perp$ of $\mathbf{q}$ perpendicular to the mean direction of flock motion: $v_L \equiv \mathbf{v}_\perp \cdot \mathbf{q}_\perp / q_\perp$, $\mathbf{v}_T \equiv \mathbf{v}_\perp - v_L(\mathbf{q}_\perp / q_\perp)$. Note that the "transverse" velocity $\mathbf{v}_T$ does not exist in $d = 2$, where there are no directions that are orthogonal to both $\mathbf{q}_\perp$ and the mean direction of flock motion $\mathbf{e}_\parallel$. This has important consequences, as we will see later.

The set of coupled linear algebraic equations for $\delta\rho$, $\mathbf{v}_T$, and v_L that we thereby obtain can be solved analytically to obtain the strength of the fluctuations (details are given in Ref. [24]):

$$\overline{|v_L(\mathbf{q})|^2} = \frac{(\tilde{\Delta} \cos^2\theta_\mathbf{q}) q^{-2}}{\epsilon^2(\theta_\mathbf{q}) q^2 + (\sin^2\theta_\mathbf{q} - [\gamma v_2 / c_0^2] \cos^2\theta_\mathbf{q})^2}, \tag{5}$$

$$\overline{|\delta\rho(\mathbf{q})|^2} = \frac{[\tilde{\Delta} (\rho_0^2 / v_2^2) \sin^2\theta_\mathbf{q}] q^{-2}}{\epsilon^2(\theta_\mathbf{q}) q^2 + (\sin^2\theta_\mathbf{q} - [\gamma v_2 / c_0^2] \cos^2\theta_\mathbf{q})^2}, \tag{6}$$

$$\overline{|\mathbf{v}_T(\mathbf{q})|^2} = \frac{(d-2)\Delta}{\gamma^2 q^2 [\epsilon_T^2(\theta_\mathbf{q}) q^2 + \cos^2\theta_\mathbf{q}]}, \tag{7}$$

with q the magnitude of the wave vector $\mathbf{q}$ and $\theta_\mathbf{q}$ the angle between $\mathbf{q}$ and the direction $\mathbf{e}_\parallel$ of mean flock motion. In Eqs. (5)–(7), $\tilde{\Delta} \equiv v_2^2 \Delta / c_0^4$ and $\epsilon(\theta_\mathbf{q})$ and $\epsilon_T(\theta_\mathbf{q})$ are the finite direction-dependent damping coefficients (see Ref. [24] for their expressions).

From Eqs. (5)–(7), we immediately see that there is an important distinction between the cases $\gamma v_2 > 0$ and $\gamma v_2 < 0$. In the former case, fluctuations of v_L and ρ are highly anisotropic: they scale like q^{-2} for all directions of $\mathbf{q}$ *except* when $\theta_\mathbf{q} = \theta_c$ or $\pi - \theta_c$, where we have defined a critical angle of propagation $\theta_c \equiv \arctan[\sqrt{\gamma v_2}/c_0]$. The physical significance of θ_c is that it is the direction in which the speed of propagation of longitudinal sound waves in the flock vanishes [3–6]. For these special directions (which only exist if $\gamma v_2 > 0$) both $\overline{|v_L(\mathbf{q})|^2}$ and $\overline{|\delta\rho(\mathbf{q})|^2}$ scale like q^{-4}. On the other hand, when $\gamma v_2 < 0$, fluctuations of v_L and ρ are essentially isotropic: they scale as q^{-2} for *all* directions of $\mathbf{q}$.

Fluctuations of $\mathbf{v}_T$, however, are *always* anisotropic, diverging as q^{-4} for $\theta_\mathbf{q} = \pi/2$, and as q^{-2} for all other directions of $\mathbf{q}$. Of course, there *are* no such fluctuations in $d = 2$, since, as noted earlier, $\mathbf{v}_T$ does not exist in that case, as reflected by the factor of $(d-2)$ in Eq. (7).

These special directions (θ_c and $\pi/2$) dominate the real space fluctuations $\overline{|\mathbf{v}_\perp(\mathbf{r})|^2}$ and $\overline{|\delta\rho(\mathbf{r})|^2}$, which can be obtained by integrating $\overline{|\delta\rho(\mathbf{q})|^2}$, $\overline{|v_L(\mathbf{q})|^2}$, and $\overline{|\mathbf{v}_T(\mathbf{q})|^2}$ over all wave vector $\mathbf{q}$. In particular, we have

$$\overline{|\mathbf{v}_\perp(\mathbf{r})|^2} = \int q^{d-1} dq \int d\Omega_\mathbf{q} (\overline{|\mathbf{v}_T(\mathbf{q})|^2} + \overline{|v_L(\mathbf{q})|^2}), \tag{8}$$

where $\int d\Omega_\mathbf{q}$ denotes an integral over the directions of $\mathbf{q}$. As shown in Ref. [24], this angular integral scales like q^{-3} for the $\mathbf{v}_T$ term in Eq. (8), except, of course, in $d = 2$, where that term does not exist. The v_L term also scales like q^{-3} when $\gamma v_2 > 0$, due to the aforementioned divergence of $\overline{|v_L(\mathbf{q})|^2}$ as $\theta_\mathbf{q} \to \theta_c$. However, it only scales like q^{-2} when $\gamma v_2 < 0$, since $\overline{|v_L(\mathbf{q})|^2}$ does not blow up for any direction of $\mathbf{q}$ in that case.

Hence, if *either* $d > 2$ *or* $\gamma v_2 > 0$, Eq. (8) implies

$$\overline{|\mathbf{v}_\perp(\mathbf{r})|^2} \propto \int q^{d-4} dq, \tag{9}$$

which clearly diverges in the long wavelength (i.e., infrared, or $q \to 0$) limit for $d \leq 3$. Thus, according to the linearized theory, there should be no long-ranged orientational order [a nonzero $\overline{\mathbf{v}(\mathbf{r})}$] for $d \leq 3$, no matter how weak the disorder. In the critical dimension $d = 3$, quasi-long-ranged order (with algebraic decay of velocity correlations in space) should, again according to the *linearized* theory, occur.

However, for the case $\gamma v_2 < 0$ (when $\overline{|v_L(\mathbf{q})|^2}$ has no soft directions) and $d = 2$ (when $\mathbf{v}_T$ does not exist), we have

$$\overline{|\mathbf{v}_\perp(\mathbf{r})|^2} \propto \int q^{d-3} dq, \tag{10}$$

which only diverges in $d \le 2$. In $d = 2$, this divergence is only logarithmic, suggesting quasi-long-ranged order characterized by Eq. (1).

We thus see that there is a significant difference between dimension $d = 2$ and $d > 2$, and between $\gamma v_2 > 0$ and $\gamma v_2 < 0$. Thus there are four distinct cases of physical interest. The linear theory just presented predicts quasi-long-ranged order for three of these four cases: $d = 3$ for both $\gamma v_2 > 0$ and $\gamma v_2 < 0$, and $d = 2$ for the case $\gamma v_2 < 0$. For the remaining case, $d = 2$ and $\gamma v_2 > 0$, the linear theory predicts only short-ranged order.

However, in the full, nonlinear theory, there is true long-ranged order—specifically, a nonzero average velocity $\overline{\mathbf{v}(\mathbf{r}, t)} \neq \mathbf{0}$ for $d = 3$, and quasi-long-ranged order in $d = 2$, in both cases $\gamma v_2 > 0$ and $\gamma v_2 < 0$. Below, we will present the detailed analysis of the full nonlinear model for the simplest case, $\gamma v_2 < 0$ and $d > 2$, and for the case we simulate, $\gamma v_2 > 0$ and $d = 2$. We defer detailed discussion of the other two cases to Ref. [24].

Breakdown of the linearized theory in $d \le 5$.—We now show that the nonlinearities explicitly displayed in the coarse-grained equations of motion radically change the scaling of fluctuations in flocks with quenched disorder for all spatial dimensions $d \le 5$. Furthermore, this change in scaling stabilizes orientational order, i.e., makes it possible for the flock to acquire a nonzero mean velocity ($\bar{\mathbf{v}} \neq 0$) in three dimensions.

We begin by demonstrating this for $d \neq 2$ and $\gamma v_2 < 0$ by power counting. (The same conclusion also holds for $\gamma v_2 > 0$, but we defer the more complicated argument for that case to Ref. [24].) Because of the anisotropy, we rescale coordinates $r_\parallel$ along the direction of flock motion differently from those $\mathbf{r}_\perp$ orthogonal to that direction, and also rescale time and the fields:

$$\mathbf{r}_\perp \to b\mathbf{r}_\perp, \qquad r_\parallel \to b^\zeta r_\parallel, \qquad t \to b^z t,$$
$$\mathbf{v}_\perp \to b^\chi \mathbf{v}_\perp, \qquad \delta\rho \to b^{\chi_\rho}\delta\rho. \tag{11}$$

These rescalings relate the parameters in the rescaled equations (denoted by primes) to those of the unrescaled equations. We will focus on the parameters Δ, γ, and D_T, and the combination of parameters c_0^2/ρ_0, which control the fluctuations in the dominant direction $\theta_\mathbf{q} = \pi/2$ of wave vector $\mathbf{q}$. We easily find

$$\gamma' = b^{z-\zeta}\gamma, \qquad \Delta' = b^{2(z-\chi)+1-d-\zeta}\Delta, \tag{12}$$

$$\left(\frac{c_0^2}{\rho_0}\right)' = b^{\chi_\rho-\chi+z-1}\left(\frac{c_0^2}{\rho_0}\right), \qquad D_T' = b^{z-2}D_T. \tag{13}$$

We can thus keep the scale of the fluctuations fixed by choosing the exponents z, ζ, χ, and χ_ρ to obey

$$z - \zeta = 0, \qquad \chi_\rho - \chi + z - 1 = 0, \qquad z - 2 = 0,$$
$$2(z-\chi) + 1 - d - \zeta = 0. \tag{14}$$

Solving these yields

$$z_{\text{lin}} = \zeta_{\text{lin}} = 2, \qquad \chi_{\text{lin}} = \frac{3-d}{2}, \qquad \chi_{\rho,\text{lin}} = \frac{1-d}{2}. \tag{15}$$

The subscript "lin" in these expressions denotes the fact that we have determined these exponents ignoring the effects of the nonlinearities in the equations of motion Eqs. (2) and (3). We now use them to determine in what spatial dimension d those nonlinearities become important.

Upon the rescalings (11), the non-linear terms λ, and $g_{1,2,3,4}$ in the $\mathbf{v}_\perp$ equation of motion (2) obey

$$\lambda' = b^{z+\chi-1}\lambda = b^{(5-d)/2}\lambda, \tag{16}$$

$$g'_{1,2,3,4} = b^{z+\chi_\rho-\zeta}g_{1,2,3,4} = b^{(1-d)/2}g_{1,2,3,4}. \tag{17}$$

By inspection of Eq. (17), we see that only λ becomes relevant in any spatial dimension $d > 1$; in fact, it becomes relevant for $d \le d_c = 5$. The g_i's are all irrelevant, and can be dropped. Furthermore, if we restrict ourselves to consideration of the transverse modes $\mathbf{v}_T$, which we can do by projecting the spatial Fourier transform of Eq. (2) perpendicular to $\mathbf{q}_\perp$, we see that there is *no* coupling between $\mathbf{v}_T$ and ρ *at all*, even at nonlinear order. Hence, ρ completely drops out of the problem of determining the fluctuations of $\mathbf{v}_T$. And since $\mathbf{v}_T$ is, as we saw in our treatment of the linearized version of this problem, the dominant contribution to the velocity fluctuations when $d > 2$ (so that $\mathbf{v}_T$ actually exists) and $\gamma v_2 < 0$ (so that there is no direction of $\mathbf{q}$ for which the *longitudinal* velocity fluctuations v_L diverge more strongly than $1/q^2$ in the linearized approximation), this means that the long distance scaling of the velocity fluctuations will be the same as in a model with no density fluctuations *at all*, that is, an incompressible model, in which $\nabla_\perp \cdot \mathbf{v}_\perp = 0$.

We now note two useful facts.

(1) The only nonlinearity (the λ term) can be written as a total $\perp$ derivative. This follows from the identity

$$(\mathbf{v}_\perp \cdot \nabla_\perp)v_i^\perp = \partial_j^\perp(v_j^\perp v_i^\perp) - v_i^\perp \nabla_\perp \cdot \mathbf{v}_\perp. \tag{18}$$

The first term on the right-hand side of this expression is obviously a total $\perp$ derivative. The second term vanishes since $\nabla_\perp \cdot \mathbf{v}_\perp = 0$, which implies that the nonlinearity can *only* renormalize terms which involve $\perp$ derivatives (i.e., D_T^0); specifically, there are *no* graphical corrections to either γ or Δ.

(2) There are no graphical corrections for λ either, because the equations of motion Eqs. (2) and (3) have an exact "pseudo-Galilean invariance" symmetry [25];

i.e., they remain unchanged by a pseudo-Galilean transformation,

$$\mathbf{r}_\perp \to \mathbf{r}_\perp - \lambda \mathbf{v}_1 t, \qquad \mathbf{v}_\perp \to \mathbf{v}_\perp + \mathbf{v}_1, \tag{19}$$

for arbitrary constant vector $\mathbf{v}_1 \perp \mathbf{e}_\parallel$. Since such an exact symmetry must continue to hold upon renormalization, with the *same* value of λ, the parameter λ cannot be graphically renormalized.

Taken together, these two facts imply that Eq. (12) and the first equality of Eq. (16) are exact, even when graphical corrections are included. Therefore, to get a fixed point, we must have

$$z-\zeta=0, \quad 2(z-\chi)+1-d-\zeta=0, \quad z+\chi-1=0, \tag{20}$$

which imply

$$z=\frac{d+1}{3}=\zeta, \qquad \chi=\frac{2-d}{3}. \tag{21}$$

The fact that $\chi<0$ for all d in the range $2<d<5$ implies that velocity fluctuations get smaller as we go to longer and longer length scales; this implies the existence of long-ranged order (i.e., a nonzero average velocity $\bar{\mathbf{v}} \neq \mathbf{0}$) in all of those spatial dimensions. The physically realistic case in this range is, of course, $d=3$.

These exponents imply that Fourier transformed velocity correlations take the form

$$\overline{|\mathbf{v}_\perp(\mathbf{q})|^2}=\frac{h[(q_\parallel/\Lambda)/(q_\perp/\Lambda)^\zeta]}{q_\parallel^2} \propto \begin{cases} q_\perp^{-2\zeta}, & (\frac{q_\perp}{\Lambda})^\zeta \gg \frac{q_\parallel}{\Lambda}, \\ q_\parallel^{-2}, & (\frac{q_\perp}{\Lambda})^\zeta \ll \frac{q_\parallel}{\Lambda}, \end{cases} \tag{22}$$

where $h(x)$ is a universal scaling function (up to a nonuniversal overall multiplicative constant), and Λ is an ultraviolet cutoff.

Nonlinear effects for $\gamma v_2>0$, $d=2$.—Now "longitudinal" fluctuations (i.e., $\delta\rho$ and v_L) become important, which causes the g and w nonlinearities in the equations of motion Eqs. (2) and (3) to also become important. This prevents us from making such a compelling argument for exact exponents. However, our experience with the annealed noise problem suggests a way forward. In that annealed case, the *assumption* that below the critical dimension only *one* of the nonlinearities, namely the convective λ term, is relevant makes it possible to determine exact exponents in $d=2$. These exponents agree extremely well with simulations of flocking [3–7]. Thus this assumption appears to be correct for the annealed problem, which suggests that it might also be true in the quenched disorder problem.

If it is, then the two points that we used to determine the exact exponents for the $\gamma v_2<0$, $d\neq 2$ case just considered also hold here. In this case, the λ nonlinearity can be written as a total derivative because $\mathbf{v}_\perp$ has only one component in $d=2$, so $(\mathbf{v}_\perp \cdot \nabla_\perp) v_i^\perp = v_\perp \partial_\perp v_\perp = \partial_\perp (v_\perp^2/2)$. Pseudo-Galilean invariance also applies once λ is the only relevant nonlinearity [25].

Hence, the arguments we made earlier for the exact exponents for the case $\gamma v_2<0$, $d\neq 2$ also apply for $\gamma v_2>0$, $d=2$. This implies that the exponents of Eq. (21) apply here as well, albeit with $d=2$, which implies $z=\zeta=1$, $\chi=0$. The vanishing of χ implies quasi-long-ranged order [Eq. (1)], while the fact that $\zeta=1$ implies that fluctuations scale isotropically. Note that this is in strong contradiction to the linear theory, which predicts extremely *anisotropic* scaling of fluctuations when $\gamma v_2>0$, as it is here.

The physical origin of this restoration of isotropic scaling is that the damping coefficient $\epsilon^2(\theta_{\mathbf{q}})$ is renormalized by nonlinear fluctuation effects by an amount that scales like q^{-2} as $\mathbf{q}\to\mathbf{0}$ and $\theta_{\mathbf{q}}\to\theta_c$, cancelling off the explicit q^2 in Eq. (5), and thereby making the fluctuations scale isotropically. Since this nonlinear effect is caused by disorder-induced fluctuations, we expect the finite $\mathbf{q}\to\mathbf{0}$ limiting value of $q^2\epsilon^2(\theta_{\mathbf{q}})$, which we define as δ [i.e., $\delta \equiv (\lim/q\to 0) q^2\epsilon^2(\theta_c)$], to get very small as the strength Δ of the disorder does. Since our simulations are done at weak disorder, we expect δ to be small, which implies a sharp peak in a plot of $q^2\overline{|\mathbf{v}_\perp(\mathbf{q})|^2}$ versus $\theta_{\mathbf{q}}$. Specifically, our analysis implies

$$q^2\overline{|v_\perp(\mathbf{q})|^2} \propto \frac{\Delta\cos^2(\theta)}{[\sin^2(\theta)-\tan^2(\theta_c)\cos^2(\theta)]^2+\delta}. \tag{23}$$

We have tested some of our analytical predictions by numerical simulations of a modified Vicsek model where a certain number of static particles ("dead birds") are added to the simulation. These stationary particles are placed randomly in space with fixed "pseudovelocity vectors" of length v_0 in random directions. Normal moving particles will align their velocities with these pseudovelocity vectors in their neighborhood. Details of the model and all the simulation results are described in Ref. [24]. Because of space limitations, here we only highlight the velocity correlation function obtained from our simulations. As shown in Fig. 1, the angular dependence of the velocity correlation function agrees well with our theoretical prediction [Eq. (23)]. It should be noted that this result cannot continue to hold down to arbitrarily small $\mathbf{q}$ because quasi-long-ranged order, which our result $\chi=0$ implies, is inconsistent with macroscopic anisotropy. Therefore, at large enough length scales that the velocity correlation function in Eq. (1) becomes $\ll (\overline{\mathbf{v}(\mathbf{r})})^2$, Eq. (23) will break down and isotropy will be restored. However, as in the case of an equilibrium two-dimensional nematic [26], isotropy is restored by slow (logarithmic) effects, which only dominates at an exponentially large length scale that is much larger than our simulation size.

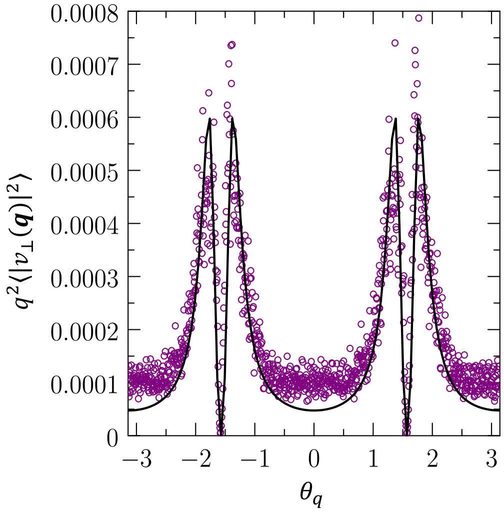

FIG. 1. Fourier space velocity-velocity correlation function of a Vicsek flock as a function of the direction of wave vector $\mathbf{q}$ in the presence of quenched disorder. The solid line is the theoretical prediction from the continuum hydrodynamic theory. The linearized theory predicts that $q^2\overline{|v_\perp(\mathbf{q})|^2}$ diverges as $\mathbf{q} \to \mathbf{0}$ at some nonuniversal critical angle θ_c, while the nonlinear theory predicts that the divergence at θ_c will be cut off, leaving a large, but finite, maximum. In our simulations, $\theta_c \approx 78°$.

Summary.—We have studied a fully nonlinear hydrodynamic equation for flocking in the presence of quenched disorder. We find that the critical dimension for the nonlinear terms to become relevant is $d_c = 5$. For $d < 5$ and $\gamma v_2 < 0$ we can determine all the scaling exponents Eq. (21). These predicted exponents show that flocks with nonzero quenched disorder can still develop long-ranged order in three dimensions, and quasi-long-ranged order in two dimensions, in strong contrast to the equilibrium case, in which any amount of quenched disorder destroys ordering in both two and three dimensions [18–20]. This prediction is consistent with the simulation results of Chepizhko *et al.* [22] and Das *et al.* [23] and our own (see Ref. [24] for more comparisons).

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