Defining the Work Done on an Electromagnetic Field

A. E. Allahverdyan and D. Karakhanyan
Yerevan Physics Institute, Alikhanian Brothers Street 2, Yerevan 375036, Armenia

(Received 30 June 2018; revised manuscript received 3 October 2018; published 14 December 2018)

The problem of defining work done on an electromagnetic field (EMF) via moving charges does not have a ready solution, because the standard Hamiltonian of an EMF—whose time derivative should define the work according to the first law—is not gauge invariant. This limits applications of statistical mechanics to an EMF. We obtained a new, explicitly gauge-invariant Hamiltonian for an EMF that depends only on physical observables. This Hamiltonian allows us to define work and to formulate the second law for an EMF. It also leads to a direct link between this law and the electrodynamic arrow of time, i.e., choosing retarded, and not advanced solutions of wave equations. Measuring the thermodynamic work can determine whether the photon mass is small but nonzero.

DOI: 10.1103/PhysRevLett.121.240602

Introduction.—Hamiltonian dynamics is essential for statistical mechanics and thermodynamics [1–4]. Basic distribution functions of statistical mechanics (e.g., canonical or microcanonical) are formulated in the phase space and are based on the conservation of energy and of the phase-space volume (the Liouville's theorem) [1–4]. Also the basic quantities of thermodynamics—energy, work, and heat—are defined via the Hamiltonian of the system; e.g., the change of the time-dependent Hamiltonian defines the work [4]. The first law divides energy into work and heat [4], while the second law limits work extraction via cyclic processes [5]. The third law studies work as a resource for cooling [6].

Our aim is to understand an electromagnetic field (EMF) as a thermodynamic and hence Hamiltonian system; the research done on EMF from various angles (field-theoretic, quantum, statistical, etc.) is reflected, e.g., in Refs. [7–14]. To this end, we need to understand the work done on EMF via moving charges. We stress that thermodynamics and electrodynamics share at least two structural features. (1) Both study systems with many degrees of freedom. (2) Both need specific subsystems (work sources) whose motion is prescribed in the sense that the backreaction on them is partially neglected. For thermodynamics these are, e.g., vessels of a gas [4], while for EMF these are moving charges [7]. Note that neglecting backreaction does not mean neglecting the energy transfer which is the very point of defining work [4].

Given these similarities, the work is to be defined via the Hamiltonian of an EMF. But it appears that (for nonstationary charges) the *change* of the standard Hamiltonian for an EMF is not gauge invariant. Hence we cannot apply it for defining work. After discussing this issue, we determine an explicitly gauge-invariant Hamiltonian of EMF that (i) generates Maxwell's equations via Hamiltonian equations, (ii) reduces to the standard expression for the free EMF, (iii) allows us to define work. This work consists of electrostatic and vortical contributions. (iv) If the work done on EMF is measured independently, it can indicate on whether the mass m of the photon holds $m = 0$ or $m > 0$. (v) The definition of work demonstrates an explicit relation between thermodynamic arrow of time (i.e., the second law) and the electrodynamic arrow of time. Despite opinions expressed since the Ritz-Einstein debate [15], the two arrows are so far regarded to be different from each other [16–19].

The Lagrangian of a classical EMF.—for a given motion of charged matter with density ρ and current J_i, the Lagrangian reads [7]

$$L_D = \int d^3x \mathcal{L}_D, \qquad \mathcal{L}_D = \frac{E_i^2}{2} - \frac{B_i^2}{2} - \rho\phi + J_i A_i, \tag{1}$$

$$E_i = -\partial_i \phi - \dot{A}_i, \qquad i = 1, 2, 3, \tag{2}$$

$$B_i = \epsilon_{ijk}\partial_j A_k, \qquad \epsilon_{ikl}B_l = \partial_i A_k - \partial_k A_i, \tag{3}$$

where E_i and B_i are (respectively) electric and magnetic fields, ϕ and A_i are (respectively) scalar and vector potential. We took $c = 1$ and denoted the 3D coordinate as $x = (x_1, x_2, x_3)$ [e.g., $E_i = E_i(x, t)$]. Repeated space indices imply summation, $\partial_i \equiv \partial/\partial x_i$, $\dot{A}_i \equiv \partial_t A_i$, and ϵ_{ijk} is the totally asymmetric factor with $\epsilon_{123} = 1$.

L_D refers to coordinates $\phi(x, t)$ and $A_i(x, t)$ and velocities $\dot{A}_i(x, t)$ that are parametrized by a continuous index x and discrete index i. Lagrange equations have the usual form, but with variational derivatives

$$\frac{d}{dt}[\delta L_D/\delta \dot{A}_j(y)] = \delta L_D/\delta A_j(y), \quad \delta L_D/\delta\phi(y) = 0. \tag{4}$$

Note that $\mathcal{L}_D$ does not contain $\dot{\phi}$, hence the last equation in Eq. (4). When working out Eq. (4) we standardly assume that ρ, J_i, E_i, and B_i decay to zero at the spatial infinity, apply integration by parts, and employ known formulas of variational calculus, e.g., $\delta A_i(x)/\delta A_j(y) = \delta_{ij}\delta(x-y)$ with Kronecker and Dirac's deltas, respectively. Hence we get from Eq. (4) equations of motion:

$$\partial_k\dot{\phi} + \ddot{A}_k = \Delta A_k - \partial_k(\partial_i A_i) + J_k, \tag{5}$$

$$\Delta\phi = -\rho - \partial_i \dot{A}_i, \tag{6}$$

where $\Delta = \partial_i\partial_i$ is the Laplace operator. Equations (2) and (3) show that Eqs. (5) and (6) become the Maxwell's equations

$$\dot{E}_i = \epsilon_{ijk}\partial_j B_k - J_i, \qquad \partial_i E_i = \rho. \tag{7}$$

Equations (5) and (6) also imply the conservation of charge:

$$\dot{\rho} + \partial_k J_k = 0. \tag{8}$$

The standard Hamiltonian of an EMF.—is constructed from Eq. (1). An EMF is a singular system, since $\mathcal{L}_D$ does not contain $\dot{\phi}$ [20,21]. This singularity can be dealt with in various equivalent ways, also via the full Dirac's formalism [20,21]. But the simplest way is to carry out the Legendre transformation with respect to $\dot{A}_i$ only [20,21]:

$$H_D = \int d^3x \mathcal{H}_D, \qquad \mathcal{H}_D = p_i\dot{A}_i - \mathcal{L}_D, \tag{9}$$

where the canonic momentum p_i is defined from

$$\delta H_D/\delta\dot{A}_k(y) = 0 \quad \text{or} \quad \dot{A}_i = p_i - \partial_i\phi. \tag{10}$$

Putting Eq. (10) into $\mathcal{H}_D$, and making integration by parts we arrive at [20,21]:

$$\mathcal{H}_D = \frac{1}{2}p_i^2 + \frac{1}{2}B_i^2 - J_iA_i + \phi(\partial_i p_i + \rho), \tag{11}$$

where ϕ is now the Lagrange multiplier for the constraint $\partial_i p_i + \rho = 0$ [given also by Eqs. (4), (6), (10)]. Hamilton equations of motion are read from Eq. (11) with canonic coordinates A_i, momenta p_i, and the Lagrange factor ϕ [20,21]:

$$\dot{A}_i = \delta H_D/\delta p_i, \qquad \dot{p}_i = -\delta H_D/\delta A_i, \qquad \delta H_D/\delta\phi = 0. \tag{12}$$

Equations (12) bring back Eqs. (5) and (6). On the solutions of Eqs. (5) and (6)—where we have $E_i = -p_i$ from Eqs. (2) and (10)—Hamiltonian $\mathcal{H}_D = \frac{1}{2}E_i^2 + \frac{1}{2}B_i^2 - J_iA_i$ reduces for $J_i = 0$ to the known Poynting energy of a free EMF [7]

$$\mathcal{E}_{\rm free} = \frac{1}{2}\int d^3x[E_i^2 + B_i^2]. \tag{13}$$

$\mathcal{E}_{\rm free}$ is energy, not a Hamiltonian, since it is not written in canonical coordinates. Equation (13) includes the case of free (and generically space localized) EMF fields.

Now H_D is generally time dependent due to ρ and J_i. As for any time-dependent Hamiltonian, we have

$$\dot{H}_D = \int d^3x\left(\dot{A}_i\frac{\delta H_D}{\delta A_i} + \dot{p}_i\frac{\delta H_D}{\delta p_i} + \dot{\phi}\frac{\delta H_D}{\delta\phi}\right) \tag{14}$$

$$+ \int d^3x(\phi\dot{\rho} - \dot{J}_iA_i). \tag{15}$$

Now Eq. (14) nullifies due to Eq. (12), so $\dot{H}_D$ is determined by Eq. (15). Hence H_D is conserved if $\dot{\rho} = \dot{J}_i = 0$, where the Lagrangian Eq. (1) is time-translation invariant. Equation (15) could be guessed directly from Eq. (1).

But we cannot apply H_D and Eq. (15) for calculating energy change. Recall that equations of motion Eqs. (5)–(7) are invariant with respect to gauge change

$$\phi \to \phi + \dot{\kappa}, \qquad A_k \to A_k - \partial_k\kappa, \tag{16}$$

where $\kappa(x,t)$ is arbitrary. This invariance relates to the zero mass of an EMF [10]. Because of Eq. (8), the Lagrangian Eq. (1) changes under Eq. (16) by a full time-derivative: $L_D \to L_D - (d/dt)\int d^3x\rho\kappa$. Equation (15) also changes by a full time-derivative under the gauge change (16)

$$\dot{H}_D \to \dot{H}_D + \frac{d}{dt}\int d^3x\dot{\rho}\kappa, \tag{17}$$

where we used Eq. (8). For a Lagrangian a shift by full time-derivatives is allowed [22], but for a Hamiltonian it is a problem, since it alters the energy change $\int_{t_1}^{t_2}\dot{H}_D dt$ between t_1 and t_2. Now $\dot{H}_D$ is gauge invariant for a particular case $\dot{\rho}(x,t_1) = \dot{\rho}(x,t_2) = 0$ for all x. This is too restrictive for the definition of the energy change and work. Indeed, in a standard task of thermodynamics a many-body system (e.g., an EMF) is employed as an energy storage; i.e., the time-dependent parameters are driven by different sources that exchange work through the system. For such cases it is simply necessary to calculate the energy change up to a given time, because this is the work that goes to one of the work sources.

The gauge variant $\dot{H}_D$ is not suitable for defining work.

Gauge-invariant Hamiltonian.—The following gauge-invariant method starts with solving Eq. (6) via the inverse Laplacian Δ^{-1} (see Sec. 1 of Ref. [25]):

$$\phi = -\Delta^{-1}(\rho + \partial_i\dot{A}_i) \equiv \frac{1}{4\pi}\int d^3y\frac{\rho(y) + \partial_i\dot{A}_i(y)}{|x-y|}. \tag{18}$$

We put back Eq. (18) into $\mathcal{L}_D$ thereby eliminating ϕ. In subsequent calculations, see Sec. 1 of Ref. [25], we neglect one full time-derivative (allowed for a Lagrangian), and also full space derivatives, due to boundary conditions. We then get a new Lagrangian that depends on the magnetic field B_i and on ρ:

$$L = \int d^3x \mathcal{L},$$

$$\mathcal{L} = \frac{1}{2}\rho\Delta^{-1}\rho - \frac{1}{2}\dot{B}_i\Delta^{-1}(\dot{B}_i) - \frac{1}{2}B_iB_i - B_i\Delta^{-1}(R_i), \quad (19)$$

$$R_i \equiv \epsilon_{ijk}\partial_j J_k, \quad \text{i.e.,} \vec{R} = \mathrm{rot}\vec{J}. \quad (20)$$

Equation (19) comes with a constraint that follows from Eq. (3)

$$\partial_j B_j = 0, \quad (21)$$

and confirms that an EMF has two independent coordinates.

In equations of motion $(d/dt)(\delta L/\delta\dot{B}_k(y)) = (\delta L/\delta B_k(y))$ we use

$$\delta L/\delta\dot{B}_k(y) = -\Delta^{-1}(\dot{B}_k)(y) \equiv \Pi_k(y). \quad (22)$$

This leads to autonomous equations for B_i that can be also derived from the Maxwell's equations, Eq. (7),

$$\ddot{B}_i - \Delta B_i - R_i = 0. \quad (23)$$

Using Eq. (22) we introduce the canonical momentum Π_k via Eq. (22) and construct from Eq. (19) the Hamiltonian via the usual Legendre transformation

$$H = \int d^3x[\Pi_k\dot{B}_k - \mathcal{L}] = H_B + \mathcal{E}_S, \quad (24)$$

$$\mathcal{E}_S \equiv -\frac{1}{2}\int d^3x\rho\Delta^{-1}\rho = \int\frac{d^3x}{2}(\Delta^{-1}[\partial_i\rho])^2 \geq 0, \quad (25)$$

$$H_B = \int d^3x\left[-\frac{\Pi_i\Delta\Pi_i}{2} + \frac{B_iB_i}{2} + B_i\Delta^{-1}(R_i)\right], \quad (26)$$

where constraint Eq. (21) is implied. Thus Hamiltonian H amounts to the electrostatic part $\mathcal{E}_S$ (see Sec. 2 of Ref. [25]) and the magnetic part H_B. The latter consists of the free magnetic part and the interaction term: $\int d^3xB_i\Delta^{-1}(R_i)$. Hence ρ and R_i emerged as time-dependent parameters of H. We emphasize that they can be given independently, in contrast to ρ and J_i that relate to each other via Eq. (8). Indeed, any J_i is represented as $J_i = \partial_i\psi + \epsilon_{ijk}\partial_j Q_k$ (Helmholtz's decomposition; see Sec. 3 of Ref. [25]). Then ρ relates to ψ via Eq. (8): $\dot{\rho} + \Delta\psi = 0$, while R_i relates to Q_i only: $\vec{R} = \mathrm{rot}\,\mathrm{rot}\vec{Q}$. Once ψ and Q_k can be independent from each other, so are ρ and R_i.

(i) Equation (23) can be reproduced from Eq. (26) via Hamilton equations $\dot{\Pi}_k = (\delta H/\delta B_k)$ and $\dot{B}_k = -(\delta H/\delta\Pi_k)$. Note that $\mathcal{E}_S$ drops out from equations of motion. Hence for $\dot{R}_i = 0$, the magnetic Hamiltonian is conserved $\dot{H}_B = 0$; cf. Eq. (26). A related law: H in Eq. (24) is conserved, $\dot{H} = 0$, if $\dot{J}_k = 0$ and ρ is always bounded; see Sec. 3 of Ref. [25].

(ii) Let us relate Eq. (24) to Poynting's energy; cf. Eq. (13) and see Sec. 6 of Ref. [25] for details. For general discussions on Poynting's energy; see Ref. [26].

Apply $\epsilon_{nmi}\partial_m$ to both sides of the Maxwell's equation $\epsilon_{ijk}\partial_j E_k = -\dot{B}_i$ [deduced from Eqs. (2) and (3)] and employ there Eq. (7). Then we can express E_i via $\dot{B}_k$ and $\partial_i\rho$:

$$E_i = \Delta^{-1}(\partial_i\rho + \epsilon_{ijk}\partial_j\dot{B}_k). \quad (27)$$

We put Eq. (27) into Eq. (13), and integrate by parts using Eq. (21):

$$\int d^3x\frac{E_i^2 + B_i^2}{2} = H|_{R_i=0} = \mathcal{E}_S + H_B|_{R_i=0}. \quad (28)$$

Under $\dot{R}_i = 0$, H_B is conserved in time, and hence the change in time of Poynting's energy $\frac{1}{2}\int d^3x(E_i^2 + B_i^2)$ reduces to the change of $\mathcal{E}_S$. Note the difference between Eqs. (13) and (28): Eq. (13) assumes $J_i = 0$; hence it allows only electrostatics ($\dot{\rho} = 0$). But Eq. (28) uses only $R_i = 0$; hence it does allow for motion of charges.

(iii) We study the thermally isolated case, where all canonic coordinates B_i and momenta Π_i are kept in the description; i.e., no system-environment division is made. Then the work is defined via $\dot{H}$, which is the statement of the first law [1–4]. Using equations of motion $\dot{\Pi}_k = (\delta H_B/\delta B_k)$ and $\dot{B}_k = -(\delta H_B/\delta\Pi_k)$ we get from Eq. (26) [cf. Eq. (14)]:

$$\dot{H} = \dot{\mathcal{E}}_S + \dot{H}_B = \dot{\mathcal{E}}_S + \int d^3xB_i\Delta^{-1}(\dot{R}_i). \quad (29)$$

$\dot{H}$ consists of two parts: electrostatic $\dot{\mathcal{E}}_S[\rho]$ and magnetic $\dot{H}_B(R_i)$; cf. Eq. (20). $\dot{\mathcal{E}}_S$ does not depend on fields, it depends only on the externally controlled $\rho(x,t)$. Hence it is *always reversible*, e.g., $\mathcal{E}_S(t_1) = \mathcal{E}_S(t_2)$ for cyclic changes of ρ: $\rho(x,t_1) = \rho(x,t_2)$. Hamiltonians containing time-dependent, nondynamic terms were discussed in Ref. [27]. Generally, such terms cannot be omitted, since they contribute to the reversible part of work.

(iv) What if a photon has a small but nonzero mass m? Because of its foundational importance, has been pondered in physics for decades [10–13]. Experiments put stringent bounds on m [10], but they cannot show that $m = 0$. Even within such bounds $m > 0$ can be relevant, e.g., in cosmology [12,13]. We show that $m > 0$ leads to a different definition of work. Recall that massive electrodynamics is a consistent theory [10,11] (see Sec. IV of Ref. [25]) that amounts to adding to $\mathcal{L}_D$ in Eq. (1) the massive term

$(m^2/2)(\phi^2 - A_i^2)$. This changes equations of motion Eqs. (5) and (6) by adding $-m^2 A_k$ to the rhs of Eq. (5) and $m^2\phi$ to the rhs of Eq. (6). New equations produce $\dot{\rho} + \partial_k J_k = m^2(\dot{\phi} + \partial_k A_k)$. Hence the charge conservation Eq. (8) and $m > 0$ lead to the Lorenz gauge $\dot{\phi} + \partial_k A_k = 0$ [10]. Then Eq. (15) still applies for the change of the total Hamiltonian, but now no gauge-transformation Eq. (16) can be made. Hence Eq. (15) is consistent for $m > 0$. Moreover, for $m > 0$ the method of Eq. (18) can be implemented, but it does not lead to a Lagrangian (or Hamiltonian) description of A_i; see Sec. 4 of Ref. [25]. Hence Eqs. (15) and (29) provide consistent *and different* definitions of work for (respectively) $m > 0$ and $m = 0$. If the work done on an EMF can be measured independently, this will show whether or not a photon has mass.

Arrows of time.—Given the Hamiltonian system Eq. (26), we can develop for it statistical mechanics, e.g., assuming Gibbsian initial distribution for coordinates B_i and momenta Π_i. Here we focus on a specific, but important situation of the second law: let R_i be switched on at some initial time, *and* there is no magnetic field before that time: $R_i(x,t) = B_i(x,t) = \Pi(x,t) = 0$ for $t \leq 0$. This implies the zero-temperature initial Gibbs distribution.

With these initial conditions, Eq. (23) shows that $B_i(x,t)$ for $t > 0$ relates to R_i via the *retarded* solution [7]

$$B_i(x,t) = \frac{1}{4\pi}\int \frac{d^3y}{|x-y|} R_i(y, t-|x-y|). \qquad (30)$$

We get from Eqs. (29) and (30), for the magnetic work,

$$\dot{H}_B = \int d^3x B_i \Delta^{-1}(\dot{R}_i)(x,t) = -\frac{1}{(4\pi)^2}\int d^3x \times \int \frac{d^3y R_i(y,t-|x-y|)}{|x-y|}\int \frac{d^3z \dot{R}_i(z,t)}{|x-z|}. \qquad (31)$$

For illustration we calculate Eq. (31) for well-localized $R_i(y,t)$, e.g., $R_i(y,t) \simeq f_i(t)\delta(y)$; cf. Sec. 5 of Ref. [25]. Using this in Eq. (31), and going to spherical coordinates in $\int d^3x$, we end up with

$$\dot{H}_B = -\frac{\chi_i(t)\ddot{\chi}_i(t)}{4\pi}, \quad \chi_i(t) \equiv \int d^3x \int_0^t ds R_i(x,s),$$
$$H_B(t) - H_B(0) = \int_0^t \frac{ds}{4\pi}[\dot{\chi}_i(s)]^2 - \frac{1}{4\pi}\chi_i(t)\dot{\chi}_i(t). \qquad (32)$$

If we impose cyclicity assuming that besides $R_i(y,t) = 0$ for $t \leq 0$, it also holds $R_i(y,\tau) = 0$, then $\dot{\chi}_i(\tau) = 0$ in Eq. (32). Hence we get the statement of the second law (Thomson's formulation) for cyclic processes [5]

$$H_B(\tau) - H_B(0) \geq 0 \quad \text{if } R_i(x,t) = B_i(x,t) = 0 \quad \text{for } t \leq 0$$
$$\text{and} \quad R_i(x,t) = 0 \quad \text{for } t > \tau; \qquad (33)$$

i.e., an EMF gains energy in cyclic processes. Equation (33) holds for any cyclic variation; see Sec. 5 of Ref. [25] that cites Ref. [28]. The usage of a localized $R_i(x,t)$ in Eq. (32) was for illustration only.

The above derivation was done assuming initial conditions. Alternatively, we can employ final conditions assuming that $R_i(x,t) = B_i(x,t) = 0$ for $t > \tau$. Then the connection between $R_i(x,t)$ and $B_i(x,t) = 0$ for $t < \tau$ is to be given via the advanced solution of Eq. (23):

$$B_i^{[\mathrm{ad}]}(x,t) = \frac{1}{4\pi}\int \frac{d^3y}{|x-y|} R_i(y, t+|x-y|). \qquad (34)$$

The fact that normally one employs a retarded solution, Eq. (30), via initial conditions, and not the advanced solution Eq. (34) via final conditions, amounts to the electrodynamic arrow of time [16–19].

Repeating the above steps and imposing the cyclicity condition $R_i(x,t) = 0$ for $t < 0$, we get instead of Eq. (32): $H_B(\tau) - H_B(0) = -\int_0^\tau (ds/4\pi)[\dot{\chi}_i(s)]^2$. Now instead of the second law we got its opposite: the energy is extracted from an EMF; again this holds more generally, see Sec. 5 of Ref. [25]. Hence we linked the thermodynamic arrow of time (second law in Thomson's formulation) and the electrodynamic arrow. Relations between the cosmological and thermodynamical arrows were explored in Ref. [29].

Work vs radiation.—We emphasize that Eq. (33) does not describe *all* forms of irreversibility. For example, taking in Eq. (29) a rectilinear motion (along x_1 axes) of charges with time-dependent acceleration, $\dddot{J}_1(x_1,t) \neq 0$ and $J_2 = J_3 = 0$, we get $\dot{H}_B = 0$ due to $R_i = 0$; cf. Eq. (20). But in this case there is radiation [7], which is an irreversible phenomenon. It is natural that we do not see it on the considered level of full energies. Once we consider finite times and fields nullifying at infinity, all the radiation is at finite distances from its sources, and hence Eq. (26)—defined via an integral over the whole space—also contains the (approximately) spherical wave that is responsible for radiation. Indeed, for all cases, where $R_i(x,t) = 0$ at all times, but radiation is present, we get from Eq. (26) that H amounts to the conserved magnetic energy H_B plus the reversible electrostatic energy $\mathcal{E}_S$; see Eq. (25). Thus H does not show irreversibility for those cases.

On the other hand, if in Eq. (33) $H_B(\tau) - H_B(0) > 0$ due to a cyclic $R_i(x,t)$, then there generically will be radiation. But we yet do not known whether this is *always* the case (not just generically), since there are specific configurations with $\dot{R}_i(x,t) \neq 0$ that do not radiate [30]. Their implications are to be clarified.

In conclusion, we found a new gauge-invariant Hamiltonian, Eqs. (24)–(26), for an electromagnetic field that holds desiderata for defining work. In particular, it leads to the second law, relates it with the electrodynamic arrow of time, and differs from the Hamiltonian obtained in the limit of vanishing photon mass. This Hamiltonian should help for constructing the nonequilibrium thermodynamics of an EMF. Several problems are left open: (1) Relations between work and radiation; see above. (2) The local (finite-volume) version of energy given by H_B; cf. Ref. [31]. (3) Physical meaning of the canonic momentum Eq. (22). (4) More general (including thermal) initial states from which one can deduce the second law for the work done on an EMF and relate it with the electrodynamic arrow. (5) Backreaction from the charged matter.

We thank V. G. Gurzadyan and S. G. Babajanyan for discussions. The work was supported by SCS of Armenia, Grants No. 18RF-002, No. 18RF-015, and No. 18T-1C153.

[1] J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, (Scribner and Sons, New York, 1902).
[2] R. C. Tolman, *The Principles of Statistical Mechanics* (Clarendon Press, Oxford, 1938); *Reprint* (Dover, New York, 1979).
[3] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1* (Pergamon Press, New York, 1980).
[4] R. Balian, *From Microphysics to Macrophysics* (Springer, New York, 1992), Vol. 1.
[5] G. Lindblad, *Non-Equilibrium Entropy and Irreversibility* (D. Reidel, Dordrecht, 1983).
[6] A. E. Allahverdyan, K. V. Hovhannisyan, D. Janzing, and G. Mahler, Phys. Rev. E **84**, 041109 (2011).
[7] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975).
[8] M. Ribaric and L. Sustersic, *Conservation Laws and Open Questions of Classical Electrodynamics* (World Scientific, Singapore 1990).
[9] *The Nature of Light: What is a Photon?*, edited by C. Roychoudhuri, A. F. Kracklauer, and K. Creath (CRC Press, Boca Raton, FL, 2008).
[10] L-C. Tu, J. Luo, and G. T. Gillies, Rep. Prog. Phys. **68**, 77 (2005).
[11] H. Ruegg and M. Ruiz-Altaba, Int. J. Mod. Phys. A **19**, 3265 (2004).
[12] A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. **82**, 939 (2010).
[13] T. Prokopec and R. P. Woodard, Am. J. Phys. **72**, 60 (2004).
[14] From the huge literature on electromagnetism we selected references that stress its evolving, open-ended nature.
[15] W. Ritz and A. Einstein, Phys. Z. **10**, 323 (1909).
[16] J. G. Cramer, Found. Phys. **18**, 1205 (1988).
[17] R. S. Fritzius, Phys. Essays **3**, 371 (1990).
[18] D. Zeh, *The Physical Basis of the Direction of Time* (Springer-Verlag, New York, 2001).
[19] M. Frisch, *Inconsistency, Asymmetry, and Non-Locality: A Philosophical Investigation of Classical Electrodynamics* (Oxford University Press, Oxford, 2005).
[20] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
[21] D. M. Gitman and I. V. Tyutin, Nucl. Phys. **B630**, 509 (2002).
[22] Two Lagrangians that differ by a full time-derivative are equivalent [7]. But the inverse does not hold: Two Lagrangians (also for EMF [23]) can generate the same equations of motion without differing from each other by a full time-derivative [23,24].
[23] A. Sudbery, J. Phys. A **19**, L33 (1986).
[24] S. Okubo, Phys. Rev. D **22**, 919 (1980).
[25] See Supplementary Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.240602 for more details on various sections. Section 1 derives in detail our main result given by Eqs. (19), (24). Section 2 reminds how to calculate the electrostatic energy for point charges. Section 3 discusses Helmholtz's theorem. The massive electrodynamics is studied in Sec. 4. The formulation of the second law given in the main text is generalized in Sect. 5. The last Sec. 6 relates our results to Poynting's energy.
[26] J. Slepian, J. Appl. Phys. **13**, 512 (1942); C. S. Lai, Am. J. Phys. **49**, 841 (1981); P. C. Peters, Am. J. Phys. **50**, 1165 (1982); R. H. Romer, Am. J. Phys. **50**, 1166 (1982); D. H. Kobe, Am. J. Phys. **50**, 1162 (1982); U. Backhaus and K. Schafer, Am. J. Phys. **54**, 279 (1986); C. J. Carpenter, IEEE Proc. **136**, 55 (1989); C. Jeffries, SIAM Rev. **34**, 386 (1992).
[27] L. Peliti, J. Stat. Mech. (2008) P05002.
[28] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd ed. (Butterworth-Heinemann, Oxford, 1982). Vol. 4.
[29] A. E. Allahverdyan and V. G. Gurzadyan, Phys. Rev. E **93**, 052125 (2016).
[30] G. H. Goedecke, Phys. Rev. **135B**, 281 (1964).
[31] A. E. Allahverdyan and S. G. Babajanyan, J. Phys. A **49**, 285001 (2016).