Superdiffusion in One-Dimensional Quantum Lattice Models

Enej Ilievski,[1] Jacopo De Nardis,[2] Marko Medenjak,[3] and Tomaž Prosen[3]
[1]*Institute for Theoretical Physics Amsterdam and Delta Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, Netherlands*
[2]*Département de Physique, Ecole Normale Supérieure, PSL Research University, CNRS, 24 rue Lhomond, 75005 Paris, France*
[3]*Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia*

(Received 26 June 2018; revised manuscript received 10 September 2018; published 7 December 2018)

We identify a class of one-dimensional spin and fermionic lattice models that display diverging spin and charge diffusion constants, including several paradigmatic models of exactly solvable, strongly correlated many-body dynamics such as the isotropic Heisenberg spin chains, the Fermi-Hubbard model, and the *t-J* model at the integrable point. Using the hydrodynamic transport theory, we derive an analytic lower bound on the spin and charge diffusion constants by calculating the curvature of the corresponding Drude weights at half-filling, and demonstrate that for certain lattice models with isotropic interactions some of the Noether charges exhibit superdiffusive transport at finite temperature and half-filling.

DOI: 10.1103/PhysRevLett.121.230602

Introduction.—Understanding the microscopic mechanisms for the emergent macroscopic laws in many-body systems poses a fundamental question in condensed matter physics. Despite a long tradition, the question has mostly been pursued by studying certain simple classical dynamical systems [1], such as elastically colliding rigid objects [2,3], whereas much less is known about strongly correlated quantum dynamics.

From the theoretical viewpoint, holographic theories [4,5] and solvable systems in one dimension play an instrumental role in this context thanks to many powerful methods which enable explicit analytical calculations. Exactly solvable models display anomalous transport behavior characterized by singular conductivities [6–18]. In contrast, very little is known about the regular part of DC conductivities that characterize the sub-ballistic timescales, save for a few numerical studies typically suffering from strong finite-size or finite-time effects [19–24]. Exactly solvable interacting models are naturally tailored not only to tackle this problem in a rigorous manner, but moreover permit efficient numerical simulations [22,25–30] and sometimes allow for experimental realizations [31–35]. Yet, even in a very simple interacting system, such as the integrable Heisenberg spin-$1/2$ chain, the status of the spin dynamics on the sub-ballistic scales remains unresolved despite several recent numerical efforts; depending on the choice of parameters, the model shows a wide range of transport phenomena, ranging from ideal transport to diffusion and in some cases even superdiffusion [36–41]. It is thus reasonable to regard the Heisenberg spin-$1/2$ chain and other integrable models as exactly solvable representative models for various universality classes of transport behavior exhibited by generic many-body quantum systems.

In this Letter, we report on a class of quantum spin and electron models which exhibit diverging spin and charge diffusion constants in thermal equilibrium with no charge or spin imbalance (i.e., at half-filling), despite the absence of ideal transport. We build on an earlier proposal of Ref. [42], which relates diffusion constants to the corresponding Drude weights in the vicinity of half-filled thermal states. Here we find a reinterpretation of the diffusion bound and optimize it in the framework of the hydrodynamic linear transport theory developed in Refs. [43,44]. We derive an analytic closed-form expression for the lower bound in the limit of infinite temperatures, and evaluate it for several paradigmatic interacting quantum lattice models.

By explicitly calculating the bound on the diffusion constant, we show that for several models with *isotropic interactions*, invariant under a continuous non-Abelian (and possibly graded) Lie group G, that the conserved Noether charges (e.g., spin or electron charges) belonging to the $SU(2)$ sector of the model exhibit superdiffusive behavior in a half-filled state at any finite temperature. As prototypical examples we will focus on the Heisenberg spin chain and the Fermi-Hubbard chain.

Summary.—The central result of this work is an analytical lower bound on the spin or charge diffusion constants for a family of interacting many-body one-dimensional lattice systems. Let $\hat{Q} = \sum_x \hat{q}_x$ denote a conserved $U(1)$ charge of the model, with density $\hat{q}$ satisfying a local conservation law $\partial_t \hat{q}_x(t) + \partial_x \hat{j}_x(t) = 0$. The corresponding diffusion constant is defined via the Kubo formula

$$D^{(q)}(\beta) = \lim_{T\to\infty} \frac{\beta}{\chi^{(q)}(\beta)} \sum_x \int_0^T dt \langle \hat{j}_x^{(q)}(t)\hat{j}_0^{(q)}(0)\rangle, \quad (1)$$

where $\langle\bullet\rangle$ is the expectation value with respect to the grand-canonical Gibbs ensemble $\hat{\varrho}_{\rm GC}(\beta)\simeq\exp(-\beta\hat{H}+\sum_i 2h_i\hat{N}_i)$ at inverse temperature β, with $\hat{N}_i$ denoting the globally conserved $U(1)$ charges of the model including $\hat{Q}$, $\chi^{(i)}(\beta)=\partial^2 f(\beta)/\partial h_i^2$, denoting the static susceptibilities, and f is the grand-canonical free energy [45].

We shall avoid a general formulation and rather concentrate on two prominent interacting systems that often play a pivotal role in the studies of strongly correlated one-dimensional materials, the anisotropic Heisenberg spin chain and the Fermi-Hubbard model. These exactly solvable systems feature stable interacting particle excitations that undergo a completely elastic scattering. Consequently, the thermal average of the current density generally involves a dissipation-free component, implying a singular dc conductivity characterized by a finite Drude weight

$$\mathcal{D}^{(q)}(\beta)=\lim_{T\to\infty}\frac{\beta}{2T}\sum_x\int_0^T dt\langle\hat{j}_x^{(q)}(t)\hat{j}_0^{(q)}(0)\rangle, \qquad (2)$$

which signals ballistic transport. Drude weights can be efficiently computed within the hydrodynamic approach developed in Refs. [46,47], essentially exploiting the fact that the net effect of interparticle interactions (which are fully accounted for by a two-body scattering amplitude) in the thermodynamic limit manifests itself as the renormalization of particles' bare quantities in the presence of a finite-density many-body background (e.g., a Gibbs thermal state or generalized Gibbs states [48–52]), commonly referred to as *dressing* (see, e.g., Refs. [53–57]). Spectra of solvable models are parametrized in terms of particle excitations. We label them by a discrete index A counting over (typically infinitely many) particle types, and a continuous rapidity variable u encoding their bare momenta $k_A(u)$ and energies $e_A(u)$. The dressing of a bare quantity q_A is expressible as a linear transformation $q_A\mapsto q_A^{\rm dr}$, while the effective velocities of propagation are obtained from the dressed dispersion relations, $v_A^{\rm eff}=\partial\varepsilon_A/\partial p_A=\varepsilon'_A/p'_A$, where $p'_A=(k'_A)^{\rm dr}$ and $\varepsilon_A=(e'_A)^{\rm dr}$, with prime denoting the rapidity derivative. In this picture, the hydrodynamic mode decomposition of the Drude weight reads [44,57]

$$\mathcal{D}^{(q)}(\beta)=\frac{\beta}{2}\sum_A\int du\mathcal{D}_A(u)[q_A^{\rm dr}(u)]^2, \qquad (3)$$

where $\mathcal{D}_A(u)=\rho_A(u)[1-\vartheta_A(u)][v_A^{\rm eff}(u)]^2$ is the "Drude kernel" and $q_A^{\rm dr}(u)$ are the dressed charges of individual excitations with respect to an equilibrium state (defined in Ref. [58]). Dependence on the reference equilibrium state enters through the rapidity distributions $\rho_A(u)$, which are uniquely determined by the mode occupation (filling) functions $\vartheta_A(u)=\rho_A(u)/[2\pi\sigma_A p'_A(u)]$ $\{\sigma_A={\rm sgn}[k'_A(u)]\}$.

Lower bound on diffusion.—In the half-filled equilibrium states, the spin or charge Drude weight vanishes due to the symmetry reasons despite integrability. To characterize transport on sub-ballistic timescales, we exploit a useful relation between the diffusion constant and the curvature of the Drude weight with respect to the filling parameter, proposed in Ref. [42]. Consequentially, the relation provides a nonvanishing lower bound on diffusion, provided the Drude weight vanishes at most quadratically as a function of the filling parameter. This condition is satisfied for the half-filled thermal states in the particle-hole symmetric lattice models considered in this Letter.

To briefly outline the idea of the lower bound, we imagine a small gradient of the charge density imposed across the system and subsequently measure the induced current. The current (initially localized at the origin) spreads only over a finite portion of the system in a finite amount of time due to the Lieb-Robinson causality. This means that, at finite times on the relevant sublattice, the probability of measuring the current in the sector away from half-filling is nonzero. In these sectors the current grows indefinitely with time; however, the probability of the system being away from half-filling vanishes with the system size. The interplay of vanishing probability and diverging conductivity permits us to obtain a lower bound on the diffusion constant, reading as [42]

$$D^{(q)}(\beta)\geq\frac{1}{8\beta\chi^2(\beta)v_{LR}}\partial_h^2\mathcal{D}^{(q)}(\beta,h)|_{h=0}, \qquad (4)$$

where $v_{LR}=\max_{u,A}v_A^{\rm eff}(u)$ is the Lieb-Robinson velocity. In particular, in the high-temperature limit the bound becomes

$$D^{(q)}(0)\geq\lim_{\beta\to 0}\frac{18}{\beta(d^2-1)^2v_{LR}}\partial_h^2\mathcal{D}^{(q)}(\beta,h)|_{h=0}, \qquad (5)$$

where we assumed that the local degrees of freedom carry charge $q\in\{-\frac{1}{2}(d-1),\ldots,\frac{1}{2}(d-1)\}$.

Solving the dressing equations.—The Drude weight and its curvature can be expressed in terms of dressed quantities [Eq. (3)]. Given the full set of equilibrium occupation functions ϑ_A, the dressing equations take the form of coupled linear integral equations, cf. Ref. [58]. Functions ϑ_A are determined by minimizing the free energy as a functional of the densities ρ_A. This requires us to solve a system of nonlinear integral equations, which is only possible numerically using an iteration scheme, except in two extreme cases corresponding to either the ground states or the high-temperature limit. In the latter case, the occupation functions become momentum independent and the dressing transformation becomes an algebraic system. For the class of rotationally symmetric solvable spin and fermion lattice Hamiltonians considered here, the dressing equations admit an analytic group-theoretic solution, as explained in detail in Ref. [58]. This permits us to obtain a closed-form expression for the bound [Eq. (5)]

(when it is finite), and rigorously establish the occurrence of superdiffusion signalled by a divergent bound. Importantly, since the divergence is a result of a particular dependence on the dressed properties of particles with large bare spin or charge, the main statement about the superdiffusive dynamics remains valid even at finite temperatures. We note that in our calculations we take into account the exact dressed dispersion relations of interacting excitations, and our results cannot be accessed with alternative approaches, such as effective field-theoretical methods [59–62] or semiclassical approximations [63], which fail to capture the essential contributions of the bound states.

We subsequently concentrate on the transport of global $U(1)$ charges, such as the total magnetization $\hat{S}^z \equiv \sum_j \hat{S}^z_j$, and/or the total electron charge $\hat{N}_{\rm e} \equiv \frac{1}{2}\sum_{j,\sigma=\uparrow,\downarrow}\hat{c}^\dagger_{j,\sigma}\hat{c}_{j,\sigma}$. We consider the half-filled spin or charge sectors, where the Drude weights vanishes as

$$\mathcal{D}^{(q)}(\beta,h) = \mathcal{C}^{(q)}(\beta)\frac{h^2}{2} + \cdots \quad \text{for } h \sim 0, \tag{6}$$

and evaluate the bound [Eq. (5)]. The Drude weight curvature reads

$$\mathcal{C}^{(q)}(\beta) = \frac{\beta}{2}\sum_A \int du \mathcal{D}_A(u)\partial_h^2[q_A^{\rm dr}(u)]^2|_{h=0}. \tag{7}$$

In exactly solvable interacting quantum lattice models the elementary excitations which carry spin and charge typically form bound states. Let integer s denote their "bare charge" (or "bare mass"), i.e., the number of constituents within a bound state; for instance, in a spin system, such as the Heisenberg spin chain, s pertains to the number of bound magnons in multimagnon excitations, while, in an electron system (e.g., the Fermi-Hubbard model), s can be the number of bound spin-full electrons that form spin singlet states, etc. Moreover, if the Hamiltonian has a global rotational symmetry of a (graded) Lie group $G = SU(N|M)$ (with scattering amplitudes being rational functions of the scattering momenta), the number of distinct bound states is infinite; i.e., s can be arbitrarily large. We found that, for such models, the Drude weight curvature per particle decreases at $\sim 1/s$ for a large s, yielding a (logarithmically) divergent diffusion lower bound after summing over all the particle types.

Anisotropic Heisenberg spin-$1/2$ *chain.*—The simplest model that features several distinct transport regimes is the Heisenberg XXZ spin-$1/2$ chain,

$$\hat{H}_{XXZ} = \sum_{j=1}^{L}(\hat{S}^x_j\hat{S}^x_{j+1} + \hat{S}^y_j\hat{S}^y_{j+1} + \Delta\hat{S}^z_j\hat{S}^z_{j+1}), \tag{8}$$

with gapless (gapped) spectrum for $|\Delta| \le 1$ ($|\Delta| > 1$). The interaction anisotropy has a profound influence on the spin transport, which is shortly summarized below. The hydrodynamic representation of the spin Drude weight curvature reads

$$\mathcal{C}^{(m)}(\beta) = \frac{\beta}{2}\sum_{s\geq 1}\int \frac{du}{2\pi}\vartheta_s(1-\vartheta_s)p'_s[v^{\rm eff}_s]^2 \frac{\partial^2[m^{\rm dr}_s]^2}{\partial h^2}\bigg|_{h=0}, \tag{9}$$

where $m^{\rm dr}_s = \partial_{2h}\log(\vartheta_s^{-1} - 1)$. Our conclusions are that, exactly at the $SU(2)$ isotropic point $\Delta = 1$, the finite-temperature spin diffusion constant $D^{(m)}$ *diverges* in the limit of half-filling $h \to 0$. This can be inferred from the large-s scaling of the dressed spin (magnetization), mode occupation functions, and dressed dispersion relations,

$$m^{\rm dr}_s(h) \simeq \frac{1}{3}[s + \kappa(\beta)]^2 h + \mathcal{O}(h^3), \tag{10}$$

$$\lim_{h\to 0}\vartheta_s(h) \simeq [s + \kappa(\beta)]^{-2}, \tag{11}$$

$$\lim_{h\to 0}\int_{-\infty}^{\infty} du p'_s(u)[v^{\rm eff}_s(u)]^2 \simeq \frac{1}{s^3}, \tag{12}$$

for some (unknown) temperature-dependent function $\kappa(\beta)$. The above large-s asymptotics hold for any finite value of β. The finite-temperature behavior [Eq. (12)] is also confirmed numerically, see Fig. 2. In the $\beta \to 0$ limit however, relations [Eqs. (10) and (11)] have indeed become equalities valid for all values of $s \geq 1$, with $\kappa(0) = 1$.

In the gapped regime, anisotropy $\Delta = \cosh\eta$ breaks the $SU(2)$ symmetry of the interaction to $U(1)$. In the limit of infinite temperature and vanishing chemical potential, the dressed spin and mode occupations functions of bound magnons remain the same as in the isotropic case, cf. Eqs. (10) and (11). Notice that ϑ_s and $m^{\rm dr}_s$ become independent of u for large s. The key difference now is that the bare dispersion of bound magnons become η-dependent functions. In particular, the rapidity-dependent part of Eq. (9) scales as

$$\int_{-\pi/2}^{\pi/2}\frac{du}{2\pi}p'_s(u)[v^{\rm eff}_s(u)]^2 \simeq e^{-\eta s}, \tag{13}$$

i.e., is exponentially suppressed for large bound states. Contrary to the isotropic case, exponential convergence in s results in a finite spin diffusion lower bound [Eq. (5)].

The gapless regime $|\Delta| < 1$ is rather exceptional, with a positive finite-temperature spin Drude weight even in the half-filled sector [9,15,43], with a noncontinuous dependence on Δ. Still, it is interesting to ask whether the subballistic corrections to spin transport are normal, diffusive, or anomalous sub- or superdiffusive. The thermodynamic particle content of the model in this regime is quite involved (see Ref. [64]) and, in distinction to the gapped regime, changes depending on the value of Δ [43,65]. For simplicity we restrict ourselves to discrete points $\Delta = \cos\pi/\ell$, for integer $\ell \geq 3$ (the Drude curvature at $\Delta = 0$ is not positive, consistently with the vanishing diffusion constant at the free fermionic point [66]), where the spectrum consists of ℓ distinct particle types. For $s = 1, \ldots, \ell - 2$, the particles represent bound states of s magnons whose high-temperature dressed spin is given by Eq. (10), which therefore vanishes as $h \to 0$. There is an extra (exceptional) doublet of particles carrying finite dressed spins $m_A^{\mathrm{dr}} = \ell/2 \pm \kappa_\ell h$ ($\kappa_\ell > 0$), for $A = \ell - 1$, ℓ, charged under the nonunitary local conservation laws found in Refs. [12,13], which are responsible for the nonvanishing of spin Drude weight even at half-filling [43]. A finite contribution to the curvature $\mathcal{C}^{(m)}$ is obtained by subtracting a finite Drude weight $\mathcal{D}^{(m)} = \sum_{A=\ell-1,\ell} \int du \rho_A(u)[1 - \vartheta_A(u)][v_A^{\mathrm{eff}}(u)\ell/2]^2$ and expanding the remainder to the second order in h. We find a finite lower bound for all $\ell < \infty$, which diverges as $\ell \to \infty$, namely $\Delta \to 1^-$, as shown in Fig. 1. [67].

Fermi-Hubbard model.—Another class of models of particular importance are lattice models of fermions, the most prominent example being the one-dimensional Fermi-Hubbard model describing spin-full electrons interacting via Coulomb repulsion,

$$\hat{H}_{\mathrm{H}} = -\sum_{j=1}^{L} \sum_{\sigma \in \uparrow,\downarrow} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j+1,\sigma} + \hat{c}_{j+1,\sigma}^\dagger \hat{c}_{j,\sigma} + 4\mathfrak{u} \sum_{j=1}^{L} \hat{V}_j^{\mathrm{H}}, \tag{14}$$

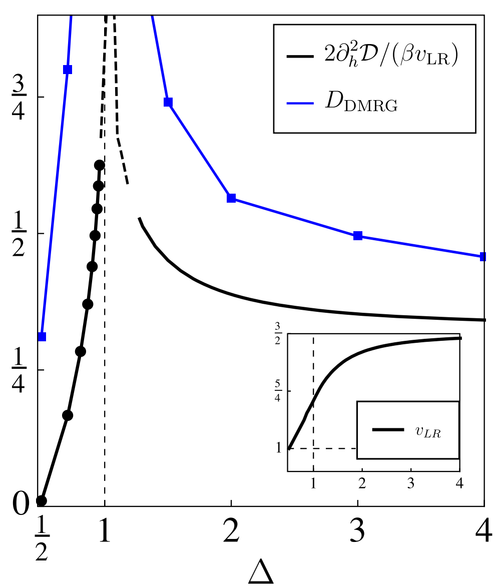

FIG. 1. *XXZ* chain at infinite temperature: the black curve shows diffusion bound [Eq. (5),] $D^{(m)} \geq 2\mathcal{C}^{(m)}(0)/(\beta v_{LR})$, and the blue points display numerical values of the spin diffusion constant obtained by time-dependent density matrix renormalization group in Refs. [21,27]. The logarithmic divergence close to $\Delta = 1$ is indicated by the dashed line. Notice that the bound does not vanish even in the Ising limit $\Delta \to \infty$, in contrast to the dissipative case [68]. Inside the gapless interval we display $\Delta = \cos \pi/\ell$ with $\ell = 3, \ldots, 10$.

with $\hat{V}_j^{\mathrm{H}} = \sum_{j=1}^{L} (\hat{n}_{j,\uparrow} - \frac{1}{2})(\hat{n}_{j,\downarrow} - \frac{1}{2})$. The spin and charge excitations both participate in the formation of bound states. The particle content consists of individual spin-up electrons, spin-singlet compounds of $2a$ electrons with ($a \in \mathbb{N}$) and chargeless bound states of s spin excitations with bare spin ($s \in \mathbb{N}$). Although spin and charge degrees of freedom mutually interact and undergo a nontrivial dressing, the transport of both spin and charge are in qualitative agreement with the isotropic Heisenberg chain: in the vicinity of the half-filled regime $h \to 0$ where $\mathcal{D}^{(m)}(\beta)$ vanishes, the dressed spin and thermal occupation functions scale with s as $m_s^{\mathrm{dr}}(h) \sim hs^2$ and $\lim_{h\to 0} \vartheta_s(h) \sim s^{-2}$, respectively, with no dependence on charge chemical potential μ associated to the conservation of the number of electrons. An analogous reasoning applies for the transport of electron charge, see Ref. [58] for further details. Numerical evaluation shows that the momentum-dependent part of $\mathcal{D}_A$ for the spin-carrying bound states once again scales as in Eq. (11), implying a (logarithmically in s) diverging spin diffusion bound [Eq. (5)].

Higher spins and higher rank symmetries.—We have additionally solved the dressing equations for a family of integrable spin-S isotropic Heisenberg chains, and for the higher-rank $SU(N)$-symmetric lattice models that comprise $N - 1$ species of interacting excitations. The picture, exemplified above for the isotropic $S = 1/2$ Heisenberg model ($N = 2$), is qualitatively unchanged. By virtue of $SU(N)$ invariance however, the statement now holds for all the components of the Noether charges. Explicit results are reported in the Supplemental Material [58], which includes Refs. [69–76].

Other fermionic models.—We have also inspected the $SU(2|2)$-symmetric (EKS model [70]) and $SU(2|1)$-symmetric (t-J model) fermionic lattice models of spin-carrying electrons, where the conclusions do not change provided the conserved $U(1)$ charge $\hat{Q}$ belongs to a bosonic (i.e., even) $SU(2)$ sector. Notice however that, in addition to the conserved total magnetization $\hat{S}^z$, the $SU(2|1)$-invariant integrable t-J model conserves the total electron charge $\hat{N}_e$ that (in distinction to the total spin and charge in the Fermi-Hubbard model) corresponds to a global $U(1)$ charge, which does not belong to the $SU(2)$ sector of the full $SU(2|1)$ symmetry of the Hamiltonian. The absence of particle-hole symmetry for the electron charge implies a finite charge Drude weight in any equilibrium state with a finite density of electrons, and the diffusion bound cannot be employed there. Likewise, in the $SU(2|2)$-symmetric model of spin-full electrons there exists, besides two independent spin $SU(2)$ sectors as in the Hubbard model, the third global $U(1)$ conserved charge (the Hubbard interaction $\hat{V}_{\mathrm{H}}$). The latter also yields a finite Drude weight for all values of chemical potentials (cf. Ref. [58] for additional information).

Conclusion.—We identified and discussed a class of exactly solvable quantum lattice models with isotropic

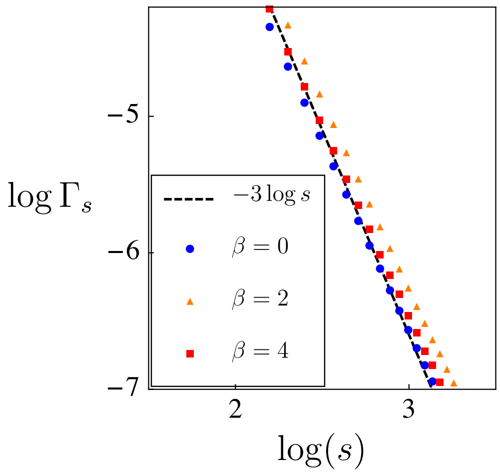

FIG. 2. Large-s scaling of $\log\Gamma_s = \log\int_{-\infty}^{\infty} du p'_s(u)[v_s^{\rm eff}(u)]^2$, confirming the asymptotic of Eq. (12) for the isotropic Heisenberg chain for various temperatures, showing that the large-s scaling is independent of β.

interactions where Noether charges exhibit sub-ballistic transport with divergent diffusion constants. Superdiffusive transport is attributed to the existence of infinitely many bound states of magnons or electrons which behave at any finite temperature [cf. Fig. 2 and Eqs. (11) and (10)] as effective paramagnetic compounds of spins (or electrons): their dressed spin (or charge) grows as $\sim hs^2$ with their bare mass s for small values of chemical potential h, and whose velocities decay proportionally to $1/s$. We wish to stress that an infinite number of bound states in the spectrum is not sufficient for a divergent diffusion constant, as shown explicitly in the gapped regime of the anisotropic Heisenberg spin-$1/2$ chain where, indeed, bound states acquire dressed velocities that get exponentially suppressed with their size.

There are several related aspects to be addressed in future. At the moment it is difficult to estimate the importance of integrability for the observed anomalous behavior. Although we have excluded normal diffusion at half-filling, invoking only a lower bound precludes determining the exact superdiffusive dynamical exponent. Indeed, numerical simulations on the isotropic quantum [37] and classical Heisenberg magnet [77] give a firm indication of dynamical exponent $z = 2/3$—which is consistent with the Kardar-Parisi-Zhang (KPZ) universality [78] {also observed in random unitary circuits in $(1+1)$D [79]}, in contrast to the standard diffusive exponent $z = 1/2$ observed in anisotropic models and strongly dissipative XXZ chains [80]. The hope is that the models exhibiting superdiffusion identified in this Letter can be viewed as representative models for a broad superdiffusive universality class of quantum systems, possibly of the KPZ type, whose precise determination remains an open problem.

The authors thank C. Karrasch for providing time-dependent density matrix renormalization group data, and thank F. Heidrich-Meisner for valuable comments. J. D. N. acknowledges support from LabEx ENS-ICFP: ANR-10-LABX-0010/ANR-10-IDEX-0001-02 PSL*. E. I. is supported by VENI Grant No. 680-47-454 by the Netherlands Organisation for Scientific Research (NWO). M. M. and T P. acknowledge the support from the ERC Advanced Grant No. 694544 OMNES and the Grant No. P1-0044 of Slovenian Research Agency.

[1] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin Heidelberg, 1991).
[2] J. L. Lebowitz, Phys. Rev. **133**, A895 (1964).
[3] J. L. Lebowitz and J. K. Percus, Phys. Rev. **155**, 122 (1967).
[4] S. A. Hartnoll, Nat. Phys. **11**, 54 (2014).
[5] M. Blake, Phys. Rev. Lett. **117**, 091601 (2016).
[6] X. Zotos, F. Naef, and P. Prelovšek, Phys. Rev. B **55**, 11029 (1997).
[7] R. G. Pereira, V. Pasquier, J. Sirker, and I. Affleck, J. Stat. Mech. (2014) P09037.
[8] C. Karrasch, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. **108**, 227206 (2012).
[9] X. Zotos, Phys. Rev. Lett. **82**, 1764 (1999).
[10] J. Herbrych and P. Prelovšek, and X. Zotos, Phys. Rev. B **84**, 155125 (2011).
[11] C. Karrasch, T. Prosen, and F. Heidrich-Meisner, Phys. Rev. B **95**, 060406 (2017).
[12] T. Prosen, Phys. Rev. Lett. **106**, 217206 (2011).
[13] T. Prosen and E. Ilievski, Phys. Rev. Lett. **111**, 057203 (2013).
[14] C. Karrasch, J. Hauschild, S. Langer, and F. Heidrich-Meisner, Phys. Rev. B **87**, 245128 (2013).
[15] E. Ilievski and T. Prosen, Commun. Math. Phys. **318**, 809 (2013).
[16] C. Karrasch, D. M. Kennes, and F. Heidrich-Meisner, Phys. Rev. B **91**, 115130 (2015).
[17] V. Mastropietro and M. Porta, J. Stat. Phys. **172**, 379 (2018).
[18] L. Mazza, J. Viti, M. Carrega, D. Rossini, and A. D. Luca, Phys. Rev. B **98**, 075421 (2018).
[19] T. Barthel, New J. Phys. **15**, 073010 (2013).
[20] R. Steinigeweg, J. Gemmer, and W. Brenig, Phys. Rev. Lett. **112**, 120601 (2014).
[21] C. Karrasch, J. E. Moore, and F. Heidrich-Meisner, Phys. Rev. B **89**, 075139 (2014).
[22] C. Karrasch, D. M. Kennes, and J. E. Moore, Phys. Rev. B **90**, 155104 (2014).
[23] C. Karrasch, R. Ilan, and J. E. Moore, Phys. Rev. B **88**, 195129 (2013).
[24] E. Leviatan, F. Pollmann, J. H. Bardarson, D. A. Huse, and E. Altman, arXiv:1702.08894.
[25] R. J. Sánchez and V. K. Varma, Phys. Rev. B **96**, 245117 (2017).
[26] M. Žnidarič, New J. Phys. **12**, 043001 (2010).
[27] C. Karrasch, New J. Phys. **19**, 033027 (2017).
[28] A. Biella, A. De Luca, J. Viti, D. Rossini, L. Mazza, and R. Fazio, Phys. Rev. B **93**, 205121 (2016).
[29] C. Psaroudaki and X. Zotos, J. Stat. Mech. (2016) 063103.
[30] F. Jin, R. Steinigeweg, F. Heidrich-Meisner, K. Michielsen, and H. De Raedt, Phys. Rev. B **92**, 205103 (2015).

[31] D. Hirobe, M. Sato, T. Kawamata, Y. Shiomi, K.-i. Uchida, R. Iguchi, Y. Koike, S. Maekawa, and E. Saitoh, Nat. Phys. **13**, 30 (2016).

[32] S. Hild, T. Fukuhara, P. Schauß, J. Zeiher, M. Knap, E. Demler, I. Bloch, and C. Gross, Phys. Rev. Lett. **113**, 147205 (2014).

[33] H. Maeter, A. A. Zvyagin, H. Luetkens, G. Pascua, Z. Shermadini, R. Saint-Martin, A. Revcolevschi, C. Hess, B. Büchner, and H.-H. Klauss, J. Phys.: Condens. Matter **25**, 365601 (2013).

[34] M. Boll, T. A. Hilker, G. Salomon, A. Omran, J. Nespolo, L. Pollet, I. Bloch, and C. Gross, Science **353**, 1257 (2016).

[35] S. Scherg, T. Kohlert, J. Herbrych, J. Stolpp, P. Bordia, U. Schneider, F. Heidrich-Meisner, I. Bloch, and M. Aidelsburger, Phys. Rev. Lett. **121**, 130402 (2018).

[36] R. Steinigeweg, Europhys. Lett. **97**, 67001 (2012).

[37] M. Ljubotina, M. Žnidarič, and T. Prosen, Nat. Commun. **8**, 16117 (2017).

[38] J.-M. Stéphan, J. Stat. Mech. (2017) 103108.

[39] G. Misguich, K. Mallick, and P. L. Krapivsky, Phys. Rev. B **96**, 195151 (2017).

[40] M. Collura, A. D. Luca, and J. Viti, Phys. Rev. B **97**, 081111 (2018).

[41] R. J. Sánchez, V. K. Varma, and V. Oganesyan, Phys. Rev. B **98**, 054415 (2018).

[42] M. Medenjak, C. Karrasch, and T. Prosen, Phys. Rev. Lett. **119**, 080602 (2017).

[43] E. Ilievski and J. D. Nardis, Phys. Rev. Lett. **119**, 020602 (2017).

[44] B. Doyon and H. Spohn, SciPost Phys. **3**, 039 (2017).

[45] While in principle one should use the Kubo-Mori inner product, the latter reduces (under certain mild assumptions [15] that do not affect our results) to the commonly used grand-canonical averaging.

[46] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Phys. Rev. Lett. **117**, 207201 (2016).

[47] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Phys. Rev. X **6**, 041065 (2016).

[48] P. Calabrese, J. Stat. Mech. (2016) 064001.

[49] L. Vidmar and M. Rigol, J. Stat. Mech. (2016) 064007.

[50] F. H. L. Essler and M. Fagotti, J. Stat. Mech. (2016) 064002.

[51] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Phys. Rev. Lett. **115**, 157201 (2015).

[52] E. Ilievski, E. Quinn, and J.-S. Caux, Phys. Rev. B **95**, 115128 (2017).

[53] B. Doyon and T. Yoshimura, SciPost Phys. **2**, 014 (2017).

[54] L. Piroli, J. De Nardis, M. Collura, B. Bertini, and M. Fagotti, Phys. Rev. B **96**, 115124 (2017).

[55] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Phys. Rev. Lett. **119**, 220604 (2017).

[56] B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, Phys. Rev. Lett. **119**, 195301 (2017).

[57] E. Ilievski and J. De Nardis, Phys. Rev. B **96**, 081118 (2017).

[58] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.230602 for the technical background and detailed derivations associated with this Letter.

[59] J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. Lett. **103**, 216602 (2009).

[60] J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. B **83**, 035115 (2011).

[61] C. Karrasch, R. G. Pereira, and J. Sirker, New J. Phys. **17**, 103003 (2015).

[62] B. Altshuler, R. Konik, and A. Tsvelik, Nucl. Phys. **B739**, 311 (2006).

[63] S. Sachdev and K. Damle, Phys. Rev. Lett. **78**, 943 (1997).

[64] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999).

[65] A. De Luca, M. Collura, and J. De Nardis, Phys. Rev. B **96**, 020403 (2017).

[66] H. Spohn, J. Math. Phys. (N.Y.) **59**, 091402 (2018).

[67] At discrete points $\Delta = \cos(\pi/\ell + 1/\nu)$, the number of magnonic bound states is $\ell + \nu$ and, therefore, after subtracting a finite Drude weight, the Drude curvature diverges as $\nu \to \infty$, similarly to the case $\Delta \to 1^-$. This suggests that the spin diffusion constant, similarly to the Drude weight, is not a smooth function of Δ, and that it diverges almost everywhere for $\Delta \in [-1, 1]$.

[68] M. Žnidaric, Phys. Rev. Lett. **106**, 220601 (2011).

[69] A. Zamolodchikov, Phys. Lett. B **253**, 391 (1991).

[70] F. H. L. Essler, V. E. Korepin, and K. Schoutens, Phys. Rev. Lett. **68**, 2960 (1992).

[71] H. Frahm, Nucl. Phys. **B559**, 613 (1999).

[72] N. Gromov and V. Kazakov, Lett. Math. Phys. **99**, 321 (2012).

[73] S. Frolov and E. Quinn, J. Phys. A **45**, 095004 (2012).

[74] D. Volin, Lett. Math. Phys. **102**, 1 (2012).

[75] E. Ilievski, E. Quinn, J. D. Nardis, and M. Brockmann, J. Stat. Mech. (2016) 063101.

[76] V. Kazakov, S. Leurent, and D. Volin, J. High Energy Phys. 02 (2016) 044.

[77] T. Prosen and B. Žunkovič, Phys. Rev. Lett. **111**, 040602 (2013).

[78] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[79] A. Nahum, S. Vijay, and J. Haah, Phys. Rev. X **8**, 021014 (2018).

[80] M. Bauer, D. Bernard, and T. Jin, SciPost Phys. **3**, 033 (2017).