Detailed Fluctuation Relation for Arbitrary Measurement and Feedback Schemes

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Fluctuation relations are powerful equalities that hold far from equilibrium. However, the standard approach to include measurement and feedback schemes may become inapplicable in certain situations, including continuous measurements, precise measurements of continuous variables, and feedback induced irreversibility. Here we overcome these shortcomings by providing a recipe for producing detailed fluctuation relations. Based on this recipe, we derive a fluctuation relation which holds for arbitrary measurement and feedback control. The key insight is that fluctuations inferable from the measurement outcomes may be suppressed by postselection. Our detailed fluctuation relation results in a stringent and experimentally accessible inequality on the extractable work, which is saturated when the full entropy production is inferable from the data.

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Introduction.—Most devices that simplify our daily lives are far from equilibrium, consuming and dissipating energy. A thorough understanding of nonequilibrium physics is therefore of pivotal importance for the development of novel technologies. However, systems that are far from equilibrium are notoriously difficult to describe. This holds especially true for small systems, where fluctuations cannot be neglected. During the last 25 years, a number of powerful thermodynamic equalities that hold far from equilibrium have been developed (for recent reviews, see Refs. [1–7]). The most prominent of these are the Jarzynski relation [8,9] and the Crooks fluctuation theorem [10–14] (see also Refs. [15,16]). These equalities involve the probability distributions of work or entropy production along trajectories through phase space and constitute important results in the field of stochastic thermodynamics.

Recent experimental advances in observing and controlling small systems opened up the possibility of optimizing the process at hand using feedback control [17]. Promising platforms for such experiments include electronic systems [18–22], DNA molecules [23,24], photons [25], Brownian particles [26], and superconducting circuits in the quantum regime [27–29]. These experiments probe the thermodynamics of information [30–33], a field which goes back to the thought experiments of Maxwell and Szilard [34–36], where microscopic information is used to seemingly violate the second law and to produce useful work. Under measurement and feedback schemes, fluctuation relations and second-law-like inequalities can still be derived by including a term that represents the obtained information [37–57]. For the Jarzynski relation, the most prominent generalizations read [39,43]

$$\langle e^{-\sigma-I}\rangle = 1 \Rightarrow \langle\sigma\rangle \geq -\langle I\rangle, \tag{1}$$

$$\langle e^{-\sigma}\rangle = \gamma \Rightarrow \langle\sigma\rangle \geq -\ln\gamma, \tag{2}$$

where I denotes the transfer entropy (the average of which reduces to the mutual information for a single measurement), γ the efficacy parameter, and σ the entropy production.

While existing fluctuation relations constitute powerful results, they are unfortunately not always applicable and a detailed fluctuation relation for arbitrary measurement and feedback scenarios is still lacking. The problems that can arise can be exemplified with the help of Eqs. (1) and (2), where we identified three key shortcomings. (i) The quantities I, $\langle I\rangle$, and γ can diverge, rendering Eqs. (1) and (2) inapplicable. In particular, I diverges when the feedback introduces absolute irreversibility. A naive evaluation of the Jarzynski relation in Eq. (1) then yields the wrong result [40,58]. The average of the transfer entropy $\langle I\rangle$ can diverge, e.g., for continuous measurements, when the amount of information extracted from the system diverges [50]. Moreover, the efficacy parameter γ can diverge for feedback schemes that include a large number of control protocols to choose from (see below). (ii) The transfer entropy I is not directly measurable as it contains information on the correlations between system and measurement apparatus [44,45]. This limits the practical relevance of Eq. (1). (iii) For Eq. (2), there is to date no corresponding detailed fluctuation relation which relates probabilities in a *forward* experiment to probabilities in a *backward* experiment. Given these shortcomings, it is highly desirable to obtain refined detailed fluctuation relations which hold for any measurement and feedback scheme. For error-free measurements, an effort in this direction has been made in Ref. [49].

In this Letter, we overcome the shortcomings of fluctuation relations in the presence of measurement and feedback with two interrelated contributions. First, we provide a novel recipe for obtaining fluctuation relations. Upon defining a backward experiment our recipe provides the associated fluctuation relation, including the corresponding information terms. This allows one to tailor useful fluctuation relations, Jarzynski relations, and second-law-like inequalities for the problem at hand. Second, we use this recipe to find a detailed fluctuation relation that circumvents the problems (i)–(iii) listed above. In the case of error-free measurements, our fluctuation relation reduces to the one found in Ref. [49].

A recipe for fluctuation relations.—Our starting point is the detailed fluctuation relation for a fixed control protocol, a fundamental relation which generalizes the second law for stochastic systems [10–12,43,59–63]. In the notation of Ref. [43], largely followed throughout this Letter, we have [64]

$$\frac{P[X^\dagger|\Lambda^\dagger]}{P[X|\Lambda]} = e^{-\sigma[X,\Lambda]}. \tag{3}$$

Here the vector $X = (x_1, \ldots, x_N)$ denotes a system trajectory through phase space, where time is discretized and x_j denotes the point in phase space the system occupies at time t_j. The time step $t_{j+1} - t_j = \delta t$ is assumed to be infinitesimally small. Similarly, $\Lambda = (\lambda_1, \ldots, \lambda_N)$ denotes a trajectory of the control parameter (sometimes called protocol). For instance, λ_j can be the value of an electric field at time t_j. The daggered quantities denote the time-reverse of the undaggered ones, e.g., $X^\dagger = (x_N^*, \ldots, x_1^*)$, where x_j^* is the time reverse of x_j and similarly for Λ. Note that the daggered quantities are uniquely defined by the undaggered ones.

Equation (3) can be understood as follows: $P[X|\Lambda]$ denotes the probability that the system takes trajectory X when the control parameter is determined by Λ. The probability $P[X^\dagger|\Lambda^\dagger]$ of realizing the time-reversed trajectory when applying the time-reversed control parameter is related to $P[X|\Lambda]$ by the exponentiated entropy production [63] (see the Supplemental Material for a general definition [65]). For experiments that start in thermal equilibrium, and systems coupled to a single bath at temperature T, the entropy production can be written as

$$k_B T\sigma[X,\Lambda] = \Delta F[\Lambda] - W[X,\Lambda], \tag{4}$$

where $\Delta F[\Lambda]$ corresponds to the free energy difference of the equilibrium states at the beginning and at the end of the experimental run and $W[X,\Lambda]$ denotes the work *extracted* from the system.

To include measurement and feedback, we denote by $Y = (y_1, \ldots, y_N)$ a trajectory of measurement outcomes, encoding information on X. Discrete measurements can be obtained by taking most y_j independent of the system trajectory. Feedback is included by determining the control parameter based on the measurement outcomes, i.e., $\Lambda(Y)$. We stress that Eq. (3) is still valid since it only involves probabilities which are conditioned on the control parameter. For ease of notation, we omit the Y dependence of Λ whenever there is no explicit Y dependence.

In the presence of measurement and feedback, the forward experiment is described by a joint probability distribution for system trajectory X and measurement outcome Y [43]

$$P[X,Y] = P_m[Y|X]P[X|\Lambda(Y)], \tag{5}$$

where $P_m[Y|X]$ denotes the probability that a fixed trajectory X results in the measurement outcomes Y. For more details, see Ref. [43]. For our purposes, the last equation can be seen as the definition of $P_m[Y|X]$. Equation (5) illustrates that a feedback experiment includes two ingredients. (1) A set of possible trajectories for the control parameter, and (2) a decision procedure to determine which trajectory is applied. Throughout this Letter, an experiment is defined by these two ingredients as well as a possible third one: (3) postprocessing of the measured data.

In the absence of measurement and feedback, there is usually only a single trajectory for the control parameter and ingredients (2) and (3) are unnecessary. The detailed fluctuation relation in Eq. (3) then relates the forward experiment to the backward experiment, which is provided by applying the time-reverse of the control parameter trajectory. In the presence of measurement and feedback, defining a backward experiment is much less trivial. While the control parameter trajectories can simply be time reversed, it is not *a priori* clear how to fix ingredients (2) and (3). As we will now discuss in detail, this freedom in choosing the backward experiment results in many different fluctuation relations.

Rewriting Eq. (3), we arrive at our first main contribution, a general detailed fluctuation relation for joint probabilities

$$\frac{P_B[X^\dagger, Y^\dagger]}{P[X,Y]} = e^{-\sigma[X,\Lambda(Y)]-(I[X:Y]-I^\dagger[X^\dagger:Y^\dagger])}, \tag{6}$$

where $P_B[X^\dagger, Y^\dagger]$ denotes the probability distribution for the backward experiment; unspecified thus far. Here we introduced the transfer entropy in the forward experiment

$$I[X:Y] = \ln\frac{P[X,Y]}{P[X|\Lambda(Y)]P[Y]} = \ln\frac{P_m[Y|X]}{P[Y]}, \tag{7}$$

and in the backward experiment

$$I^\dagger[X^\dagger:Y^\dagger] = \ln\frac{P_B[X^\dagger,Y^\dagger]}{P[X^\dagger|\Lambda(Y)^\dagger]P[Y]}, \tag{8}$$

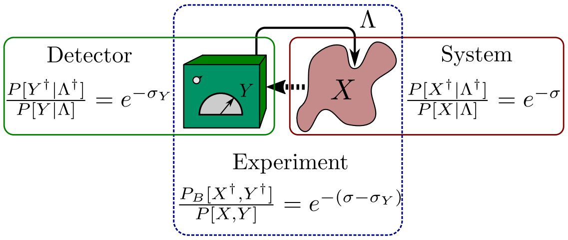

FIG. 1. Illustration of the fluctuation relation for measurement and feedback. Both the system as well as the detector output fulfill a detailed fluctuation relation. Here X (Y) denotes a trajectory of the system state (detector output) and Λ a trajectory of the control parameter. A detailed fluctuation relation for the full experiment can be obtained, where the total entropy production σ is reduced by the inferable entropy production σ_Y. Probability distributions are defined in the text.

and $P[Y] = \int dX P[X,Y]$. To illustrate the usefulness of Eq. (6) as a recipe for fluctuation relations, we consider the following scenario: An experiment using measurement and feedback has been designed and it is desired to investigate the physics of the experiment with fluctuation relations. While the forward experiment is fixed by the designed experiment, there is a freedom in choosing the backward experiment. For any chosen backward experiment, Eq. (6) provides a fluctuation relation and allows for identifying the corresponding information terms.

It is instructive to see how previous results can be recovered from Eq. (6). To this end, we consider a backward experiment where no feedback is performed. Instead, the fixed control parameter $\Lambda^\dagger$ is performed with the same probability as Λ is applied in the forward experiment (where it arises from feedback). This corresponds to the backward probability $P_B[X^\dagger, Y^\dagger] = P[X^\dagger|\Lambda(Y)^\dagger]P[Y]$. Equation (6) then results in the fluctuation relation associated to Eq. (1) [39,43]. Here we mainly focus on scenarios where P_B describes an actual experiment and is thus a normalized probability distribution. However, for any function P_B, Eq. (6) can be used to derive integral fluctuation relations. For instance, we can recover the integral fluctuation relation in Eq. (2) by choosing $P_B[X^\dagger, Y^\dagger] = P_m[Y|X]P[X^\dagger|\Lambda(Y)^\dagger]$, which is not a normalized probability distribution. Indeed, when Eq. (11) below holds, this distribution is normalized to the efficacy parameter γ.

Other definitions of P_B will result in different fluctuation relations. More generally, one can demand conditions on the backward experiment and/or the information terms in Eq. (6) to find novel fluctuation relations. Generalized Jarzynski relations and second-law-like inequalities can then be derived in a straightforward manner.

A versatile fluctuation relation.—We now apply our recipe to find a fluctuation relation which circumvents the shortcomings (i)–(iii) listed in the introduction. To this end, we impose two conditions: (I) The quantity $\Delta I[Y] \equiv I[X:Y] - I^\dagger[X^\dagger : Y^\dagger]$ shall be fully determined by the measurement outcomes; (II) The Y marginals of the forward and backward probabilities shall be the same $\int dX P_B[X^\dagger, Y^\dagger] = P[Y]$. The first condition ensures that the information term ΔI is experimentally accessible, overcoming shortcoming (ii). The second condition demands that a given set of measurement outcomes Y is equally likely in the forward and in the backward experiment.

These two conditions uniquely fix P_B in Eq. (6), resulting in our second main contribution, a detailed fluctuation relation applicable for arbitrary measurement and feedback scenarios. We now discuss both the backward probability distribution as well as the information term derived from our conditions (see the Supplemental Material for detailed derivations [65]). First, we have $\Delta I[Y] = -\sigma_{\rm cg}$, where we introduced the coarse-grained entropy production [43,66]

$$e^{-\sigma_{\rm cg}[Y]} \equiv \int dX e^{-\sigma[X,\Lambda(Y)]} P[X|Y], \tag{9}$$

where $P[X|Y] = P[X,Y]/P[Y]$. We note that as long as the total entropy production remains finite, $\sigma_{\rm cg}$ remains finite as well, preventing the divergences related to shortcoming (i). We find a generalized Jarzynski relation including the coarse-grained entropy production

$$\langle e^{-(\sigma - \sigma_{\rm cg}[Y])} \rangle = 1 \Rightarrow \langle \sigma \rangle \geq \langle \sigma_{\rm cg}[Y] \rangle, \tag{10}$$

where $\langle \cdots \rangle$ denotes an average over the forward probability distribution and the second-law-like inequality follows from Jensen's inequality.

Of key importance are scenarios which fulfill the measurement time-reversal symmetry

$$P_m[Y|X] = P_m[Y^\dagger|X^\dagger]. \tag{11}$$

As we will see below, this condition leads to a particularly illuminating physical interpretation of our fluctuation relation and ensures that the backward probability distribution has an operational meaning. We also note that this condition underlies Eq. (2). Given Eq. (11), it can be shown that a detailed fluctuation relation for the detector output holds [43]

$$e^{-\sigma_{\rm cg}[Y]} = e^{-\sigma_Y} \equiv \frac{P[Y^\dagger|\Lambda(Y)^\dagger]}{P[Y|\Lambda(Y)]}, \tag{12}$$

where $P[Y|\Lambda] = \int dX P_m[Y|X]P[X|\Lambda]$ denotes the probability of obtaining the outcomes Y given the control parameter Λ. From Eq. (5), we thus find $P[Y|\Lambda(Y)] = P[Y]$. Comparing Eq. (12) with the detailed fluctuation relation in Eq. (3), we conclude that σ_Y is the entropy production that we infer from observing only the measurement outcomes (see also Fig. 1). We thus call it the *inferable entropy production*. We note that the coarse-grained entropy production is only equal to the inferable entropy production when Eq. (11) holds. In the following, we thus identify $\sigma_Y = \sigma_{\mathrm{cg}}$, deferring a discussion on scenarios where this is not the case to the Supplemental Material [65]. Equation (12) implies $\langle \exp(-\sigma_Y) \rangle = \gamma$. From Jensen's inequality we then find $\langle \sigma_Y \rangle \geq -\ln \gamma$. The inequality in Eq. (10) is thus strictly more stringent than the inequality based on the efficacy parameter given in Eq. (2).

The backward probability obtained from our conditions (I), (II), and Eq. (11) reads

$$P_B[X^\dagger, Y^\dagger] = \frac{P[X^\dagger | \Lambda(Y)^\dagger]}{P[Y^\dagger | \Lambda(Y)^\dagger]} P_m[Y^\dagger | X^\dagger] P[Y]. \tag{13}$$

This distribution has an operational meaning [overcoming shortcoming (iii)] and can be obtained as follows: In a backward experiment, the control parameter $\Lambda(Y)^\dagger$ is applied with probability $P[Y]$. Just as in Ref. [43], $Y^\dagger$ is thus determined probabilistically at the beginning of each experimental run. The same measurements as in the forward experiment are then carried out but in time-reversed order. Importantly, the measurement outcomes are not used to update the control parameter. The data are then postselected, discarding all experimental runs where the measurement outcomes are not equal to $Y^\dagger$ when applying $\Lambda(Y)^\dagger$. The distribution $P_B[X^\dagger, Y^\dagger]$ is the joint probability for realizing $X^\dagger$ and $Y^\dagger$ in this backward experiment. It is the postselection which results in the reduction of the entropy production by the inferable entropy production σ_Y. Intuitively, having access to the measurement outcomes, their fluctuations can be suppressed. This is illustrated in Fig. 1. In case the full entropy production is inferable from the measurement outcomes, i.e., $\sigma_Y = \sigma$, our fluctuation relation reduces to the trivial equality $1 = 1$ reflecting the fact that the full entropy production is accessible. Finding deviations from this trivial identity then reflects the fact that not all entropy producing degrees of freedom are perfectly measured. To verify this, the entropy production must be measurable independently from Y.

Under our conditions, one can integrate Eq. (6) over all X which result in the same σ to obtain a fluctuation relation for entropy production [65]. We note that this is not generally possible for previous fluctuation relations. For an entropy production given by Eq. (4), this results in a fluctuation relation for the extracted work W

$$\frac{P[W, Y]}{P_B[-W, Y^\dagger]} = e^{-\beta(W - \Delta F[\Lambda(Y)]) - \sigma_Y}, \tag{14}$$

$$\Rightarrow \langle W \rangle \leq \langle \Delta F[\Lambda(Y)] \rangle - k_B T \langle \sigma_Y \rangle, \tag{15}$$

where $P[W, Y]$ is the joint probability of obtaining a value W for the work and a measurement outcome equal to Y in the forward experiment (and similarly for the backward experiment). We note that in the absence of feedback, the probability distributions factorize and Eq. (14) reduces to a simple product between the Crooks fluctuation relation and Eq. (12). To illustrate our results, we consider two well-studied examples, the Szilard engine and a Brownian particle in a harmonic trap. We note that Eq. (11) holds for both examples.

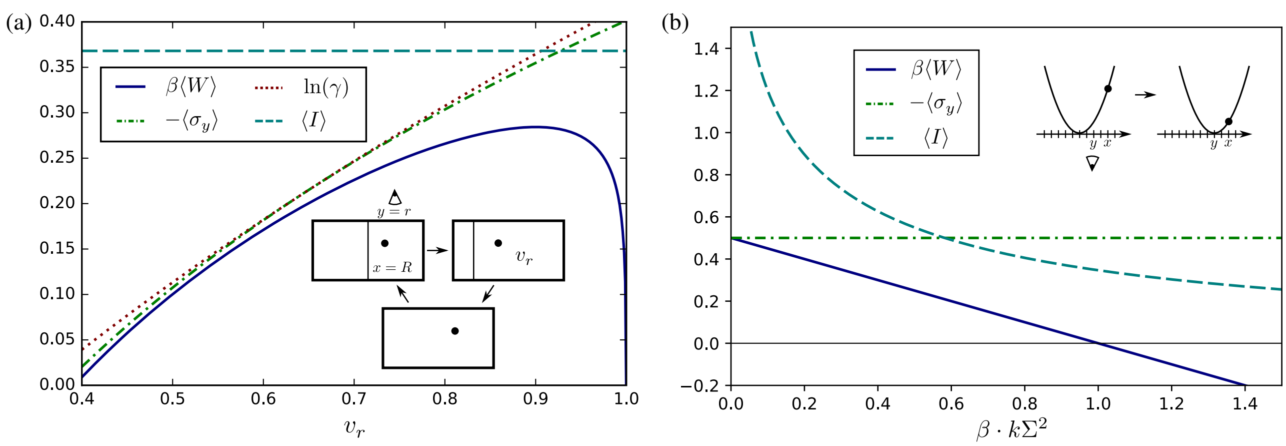

FIG. 2. Second-law-like bounds for the extracted work. The extracted work (blue, solid line) is compared to the inferable entropy production (green, dash-dotted line), the logarithm of the efficacy parameter (red, dotted line), and the transfer entropy (cyan, dashed line). (a) Szilard engine. On the horizontal axis, the final volume for measurement outcome $y = r$ is varied. For a broad range of parameters, the inferable entropy provides the tightest bound on the extracted work. Here the measurement error probability is $\varepsilon = 0.1$ and the final volume for measurement outcome $y = l$ is $v_l = 0.65$. (b) Brownian particle in a harmonic trap. On the horizontal axis, the measurement error Σ^2 divided by $k_B T/k$ is varied, where k denotes the spring constant of the trap. The transfer entropy diverges as the measurement error goes to zero and the efficacy parameter diverges for all parameters. The inferable entropy provides a bound that becomes tighter as the measurement becomes more precise. We note that in both examples, the transfer entropy equals the mutual information since there is only a single measurement.

The Szilard engine.—We consider a particle in a box of volume $v = 1$. A separation in the middle of the box is introduced and the particle will be found to the left $x = L$ or to the right $x = R$ of the separation with equal probabilities. Subsequently, the location of the particle is measured with an error ε resulting in a measurement outcome $y \in \{l, r\}$. The separation is then slowly moved with the aim of increasing the volume available to the particle to v_y, depending on the outcome of the measurement. Finally, the separation is removed and the system returns to its initial state.

Detailed calculations are given in the Supplemental Material, where we verify the detailed fluctuation relation given in Eq. (14) [65]. In Fig. 2(a), we show the extracted work and compare it to the bounds given in Eqs. (1), (2), and (15). We find that the inequality involving the inferable entropy production gives a tighter bound than the established inequalities for a range of parameters.

Brownian particle in a harmonic trap.—Our second example consists of a Brownian particle in a harmonic trap potential with spring constant k. After a position measurement is performed, the trap potential is shifted, such that the new minimum coincides with the measurement outcome. As long as the thermal spread, $k_B T/k$ is larger than the measurement error, denoted by Σ^2, a positive amount of work is extracted from the particle on average. As for the Szilard engine, detailed calculations are given in the Supplemental Material where Eq. (14) is explicitly verified [65]. In Fig. 2(b), the extracted work is compared to the transfer entropy and the inferable entropy production. The efficacy parameter diverges in this scenario since the position measurement has infinitely many outcomes, resulting in infinitely many control parameter trajectories. The transfer entropy diverges as the measurement error goes to zero. The inferable entropy production provides a useful bound for all parameters. We note that Ref. [49] discussed the same example in the limit $\Sigma \to 0$, where the bound provided by the inferable entropy becomes tight.

As an additional example published elsewhere, our results are applied to continuous measurements in single molecule force spectroscopy experiments [67].

Conclusions.—We provided a recipe for obtaining fluctuation relations in the presence of measurement and feedback. This recipe relies on the freedom of choosing a backward experiment and can be employed to develop useful and experimentally relevant fluctuation relations. This is illustrated with a detailed fluctuation relation which overcomes the shortcomings identified in previous works. The resulting relation allows for an intuitive explanation and provides a second-law like inequality in situations where previous fluctuation relations break down.

The freedom of choosing a backward experiment indicates that there is no single fluctuation relation which is universally optimal, but that each class of problems might be best described by a tailor-made fluctuation relation. The general validity of our recipe allows for the construction of relevant fluctuation relations for any given problem including measurement and feedback. The approach outlined here has thus great potential for obtaining a better understanding of nonequilibrium processes and will likely result in additional practically useful equalities and inequalities.

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[64] If not specifically stated otherwise, our results only require Eq. (3) to hold and do not depend on the specifics of the entropy production. For non-equilibrium initial states, there are cases when Eq. (3) becomes inapplicable [52,58]. This problem can be circumvented by including the preparation of the initial state in the process.

[65] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.210603 for details on the entropy production, a summary of the employed probability distributions, a derivation of the fluctuation relation, a discussion on measurements without time-reversal symmetry, as well as details on the examples.

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