Universal Properties of Many-Body Localization Transitions in Quasiperiodic Systems

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The precise nature of many-body localization (MBL) transitions in both random and quasiperiodic (QP) systems remains elusive so far. In particular, whether MBL transitions in QP and random systems belong to the same universality class or two distinct ones has not been decisively resolved. Here, we investigate MBL transitions in one-dimensional ($d = 1$) QP systems as well as in random systems by state-of-the-art real-space renormalization group (RG) calculation. Our real-space RG shows that MBL transitions in 1D QP systems are characterized by the critical exponent $\nu \approx 2.4$, which respects the Harris-Luck bound ($\nu > 1/d$) for QP systems. Note that $\nu \approx 2.4$ for QP systems also satisfies the Harris-Chayes-Chayes-Fisher-Spencer bound ($\nu > 2/d$) for random systems, which implies that MBL transitions in 1D QP systems are *stable* against weak quenched disorder since randomness is Harris *irrelevant* at the transition. We shall briefly discuss experimental means to measure ν of QP-induced MBL transitions.

About one decade ago, Anderson localization in non-interacting systems [1–3] was generalized to many-body localization (MBL) in quantum many-body systems with interactions [4–10]. Isolated systems in MBL phases cannot thermalize where eigenstates thermalization hypothesis [11–13] does not apply. In contrast to thermal systems, entanglement entropy of highly excited eigenstates of MBL systems obey the area law rather than volume law [14], which renders various exotic properties such as extensive emergent integrable operators [15,16] and protected quantum order in excited states [17–19]. More intriguingly, MBL transitions separating ergodic and MBL phases [20–28] are so-called eigenstate phase transitions which can occur for each highly excited eigenstate with finite energy density. For highly excited eigenstate across a MBL transition, entanglement entropy shifts from the volume law in thermal phases to the area law in MBL phases. This novel behavior cannot fit into the framework of conventional equilibrium phase transitions where entanglement entropy at finite temperature satisfies the volume law in both sides of transitions.

Currently, there are two known mechanisms for MBL: one by random disorder and the other by quasiperiodic (QP) potential. The former has been extensively studied for years, while the latter attracted increasing attention very recently [29–37] partly due to its accessibility in cold-atom experiments [38–41]. Recently, an exact diagonalization (ED) study [42] indicated that MBL transitions in random systems and in QP systems belongs to two distinct universality classes. Nonetheless, as ED studies are limited to models with relatively small size (L up to about 20), the finite-size effect could be severe enough preventing us from drawing decisive conclusions. So far, a reliable study of the universal properties of MBL transitions in QP models with a much larger system size is still lacking. Therefore, it is highly desired to compute critical exponents by investigating systems with a sufficiently large size to reliably address important questions such as (i) whether QP MBL transitions are stable or not against weak randomness, and (ii) whether MBL transitions in QP and random systems belong to the same universality class.

A novel real-space renormalization group (RSRG) approach was recently developed [43–45] to investigate critical behaviors of MBL transitions in random systems with much larger size compared to ED. It is probably the only numerical approach so far that obtained critical exponents satisfying the Harris-Chayes-Chayes-Fisher-Spencer (Harris-CCFS) bound ($\nu > 2/d$) [46–50] for randomness-driven MBL transitions. However, directly employing this approach to study MBL transitions in QP
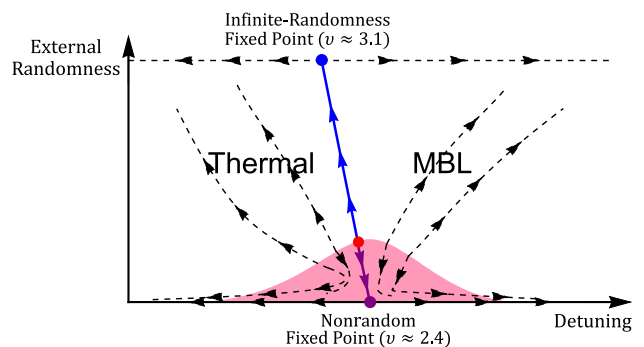

FIG. 1. The schematic RG flow of 1D interacting systems with both QP potential and quenched randomness. Our real-space RG analysis shows that MBL transitions induced by QP potential is Harris stable against weak randomness.

systems encounters serious problems. Here, we propose a different RSRG approach by taking important microscopic details into account, which is more suitable to study MBL transitions in QP systems. By utilizing this improved approach, we can access 1D QP systems with a size of the order of 1000, which is sufficiently large to investigate critical properties of MBL transitions. We systematically analyze different scenarios of MBL transitions using our RSRG approach, and come to conclusions that the critical exponent $\nu \approx 2.4$ for QP-driven MBL transitions. This value of ν is in agreement with the Harris-CCFS bound ($\nu > 2/d$), indicating that QP-induced MBL criticality can survive against weak randomness, as shown in the schematic RG flow in Fig. 1.

Model.—We consider the following one-dimensional spinless fermion model with interactions:

$$H = -\sum_{ij}(t_{ij}c_i^\dagger c_j + \mathrm{H.c.}) + \sum_i W_i n_i + V\sum_{\langle ij\rangle} n_i n_j, \quad (1)$$

where $c_i^\dagger$ creates a fermion at site i, $n_i = c_i^\dagger c_i$ is the fermion density operator, t_{ij} labels the hopping amplitude between sites i and j, V is the interaction between nearest-neighboring (NN) sites, and W_i represents the on-site potential which can vary from site to site. Note that W_i can be a random potential with uniform distribution $W_i \in [0, W]$ for the random case or a cosine potential $W_i = W\cos(2\pi\alpha i + \phi)$ with quasiperiodicity α and phase ϕ (for simplicity, we set the irrational number $\alpha = (\sqrt{5}-1)/2$ as the golden ratio hereafter) for the QP case. Note that the QP potential has long-range correlation as $\langle W_i W_{i+j}\rangle = \frac{1}{2}W^2\cos(2\pi\alpha j)$ which is in contrast to random potential without long-range correlation. In the limit of vanishing interaction $V = 0$, the system with random potential is in the Anderson localization phase for any finite randomness $W > 0$. For finite interaction V, the system with random potential would stay in the localized phase when the interaction V is sufficiently weak but can go across the MBL phase transitions into an ergodic phase when V exceeds a critical value. The nature of this MBL transition has been extensively studied [20–23,25].

Here, we shall focus on the case of QP potential. In the limit of vanishing interaction V, the Hamiltonian with QP potential W_i is the so-called Aubry-André (AA) model [51,52]. The AA model has been extensively studied; it provides an example of single-particle localization in 1D with finite critical potential strength but without single-particle mobility edge. There have been various generalizations of the AA model [53–57]. It turns out that generalized AA models usually possess single-particle mobility edge which indicates that the original AA model with only NN hopping is not generic in the family of QP models. Henceforth, we consider both NN hopping t and next-nearest-neighborhood (NNN) hopping t' such that this extended AA model could describe more realistic and generic systems [38–41]. For this extended AA model with a sufficiently strong QP potential W, all single-particle

states are localized. Then, increasing the interaction V will presumably induce a MBL transition when V exceeds a critical value V_c. We shall employ the real-space RG to study this putative MBL transition to address intriguing questions such as whether the MBL transition induced by QP potential is robust against weak random quenched disorder.

Real-space RG approach.—In order to study the MBL transition in QP systems, we develop a RSRG approach by taking microscopic details into account, which goes beyond the previous RSRG approaches. We first briefly recapitulate the RSRG approach employed in studying the MBL transition in random systems [45]. The RSRG method was constructed by mainly employing the general nature of criticality, namely assumptions of scaling invariance near the MBL transition and the hierarchy of resonance clusters which can be implemented through iterations. To determine the structure of resonance clusters, one needs to track the RG flow of two basic sets of simplified parameters between every two clusters: tunneling amplitude Γ_{ij} (namely, matrix element for transitions between different clusters) and typical energy mismatch ΔE_{ij}.

For the initial condition or data input for Γ and ΔE, the PVP RSRG assumes $\Delta E_{ij} = |\mu_i - \mu_j|$ (μ_i is the on-site potential at i and it takes box distribution for the case of random disorder) and $\Gamma_{ij} = V\exp(-|i - j|/x_0)$ (x_0 is the localization length and single-particle localization length is usually used as an approximation). Simply using chemical potential and sites positions as input data works well for studying MBL transitions in random systems, but encounters serious problems for studying MBL transitions in QP systems. In fact, if insisting to employ the original algorithm to study QP systems, one would obtain physical observables whose dependence with W is not smooth. For reasons why the adiabatic approximation fails in QP systems, see Supplemental Material [58].

To be capable of studying MBL transitions in QP systems, we improved the RSRG algorithm by making full use of the microscopic information of QP Hamiltonians. Specifically, we employ true spectrums of the single-particle Hamiltonian and localization centers of single-particle wave functions as input data, rather than using the simplified ones. By doing this, we avoid the approximation caused by oversimplifications in the original approach. For technical details of the RSRG approach developed in the present work, see Supplemental Material [58]. To the best of our knowledge, it is the first time that critical exponents for MBL transitions in QP systems can be obtained from RG analysis of models with sufficiently large system size (L order of 1000).

Results.—In performing real-space RG calculations of the interacting QP systems, we employ normalized entanglement entropy (EE) s to analyze critical behaviors of the MBL transitions for the following two reasons. First, entanglement entropy can be derived simply from the

configurations and distributions of final resonance cluster structures; second, EE satisfies the scaling form as $s = f[(W - W_c)L^{1/\nu}]$ around MBL critical point W_c, where f is some unspecified function, L is the linear size of the system, and ν is the correlation (localization) length critical exponent.

Our RSRG results of normalized EE for various QP strength W and various system size L are shown in Fig. 2, where we choose $t' = 0.1t$ and $V = 0.5t$. As mentioned before, the noninteracting extended AA model with finite NNN hopping t' can have mobility edge in single-particle spectrum for $1.5 \lesssim W \lesssim 2.6$. Note that many-body mobility edges or exotic nonergodic metal phase may occur between fully thermal and fully localized phases [59–64] when weak interactions are added into noninteracting QP models with single-particle mobility edge (SPME). To understand the precise nature of such putative phases as well as the role SPME plays in such phases, more future work is desired. Nonetheless, for sufficiently strong QP potential W that can localize all single-particle states, a full MBL phase should naturally emerge when weak interactions are added. Indeed, as shown in Fig. 2, the EE for all L cross reasonably at the putative critical point $W = W_c \approx 3.36$ where all single-particle states are fully localized without single-particle mobility edge.

As mentioned above, the EE around the transition should satisfy the scaling form. Thus, we perform the scaling collapse, as shown in the inset of Fig. 2, to extract the value of the critical exponent ν [65]. The obtained result $\nu = 2.4 \pm 0.3$ for the MBL transition in QP systems without quenched randomness is consistent with the Harris-Luck criterion ($\nu > 1/d$) of QP systems [49]. To show the robustness of the obtained results, we also performed the same type of calculations for various settings of the model,
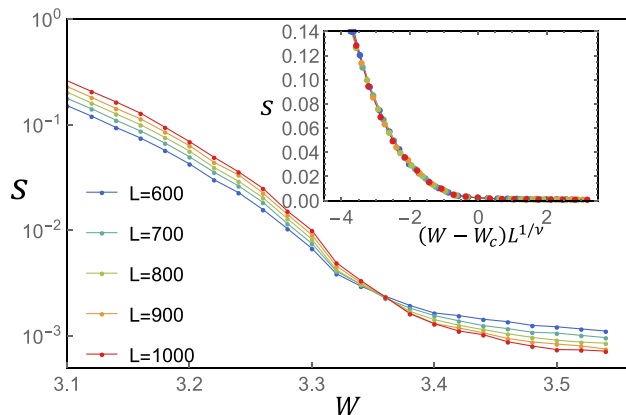

FIG. 2. Finite-size scaling analysis of entanglement entropy of 1D systems with quasiperiodic potentials $W_i = W \cos(2\pi\alpha i + \phi)$. The MBL transition is identified as the crossing point of entanglement entropy s for different sizes $W_c = 3.36 \pm 0.01$. The data collapse shown in the inset gives rise to $\nu = 2.4 \pm 0.3$. Results are obtained by averaging over 10^5–10^6 QP configurations (namely 10^5–10^6 choices of ϕ).

including different interaction strength and different wave vectors. It is remarkable that critical exponents obtained for various settings are consistent with one another within the error bar. This is not only a strong support to the numerical results we obtained, but also a clear demonstration of RSRG's effectiveness of capturing universal properties of MBL transitions. Moreover, this value of ν is also in agreement with the Harris-CCFS bound ($\nu > 2/d$), implying that the MBL transition in QP systems is Harris stable against sufficiently weak randomness since the quenched randomness is Harris irrelevant at the QP MBL transition.

Note that the result of $\nu \approx 2.4$ for QP MBL transitions in the present work is not in accord with the one in Ref. [42], where $\nu \sim 1$ was obtained through exact diagonalization (ED) calculations of models with relatively small system size ($L \leq 18$) and where the randomness was considered as Harris relevant due to the value of ν obtained by their ED calculations does not respect the Harris-CCFS bound ($\nu > 2/d$). The result of ν of the present paper derived from real-space RG calculations of models with system size up to $L = 1000$ should have significantly less finite-size effect than ED calculations of models with system size (L up to 20–30). To directly verify that the quenched randomness is Harris irrelevant at the QP-induced MBL transitions, we further carried out RSRG calculations on QP systems in the presence of a weak on-site randomness potential ($W' = 0.1t$). With this weak quenched randomness, we obtained the critical value of QP potential $W_c = 3.32 \pm 0.02$ as well as the critical exponent $\nu = 2.5 \pm 0.3$. It is remarkable that the critical exponent of the QP-induced MBL transition in the presence of weak quenched randomness is consistent with the one without any quenched randomness, which confirms that the QP-MBL criticality is stable against weak randomness. Consequently, we reasonably conclude that weak randomness should be Harris irrelevant, rather than relevant, at the MBL transition of QP systems without quenched randomness, as shown in the schematic RG flow in Fig. 1.

If MBL transitions induced by QP and by randomness belong to the same university class, they should feature the same ν. Thus, we further performed RSRG calculations of models with quenched randomness; the results of normalized EE for different randomness W and different system size L are shown in Fig. 3. From the scaling collapse shown in the inset of Fig. 3, we obtained the critical exponent $\nu = 3.1 \pm 0.3$ for the MBL transitions induced by quenched randomness. This result is quite consistent with the one ($\nu = 3.2 \pm 0.3$) obtained by previous RSRG studies [43–45], which indicates that our RSRG algorithm works well not only for QP systems but also for systems with quenched randomness. Since the result of $\nu \approx 3.1$ for randomness-induced MBL transitions is significantly different from the result of $\nu \approx 2.4$ of QP-induced MBL transitions, we believe that it is quite likely that they belong to two distinct universality classes. The results on critical
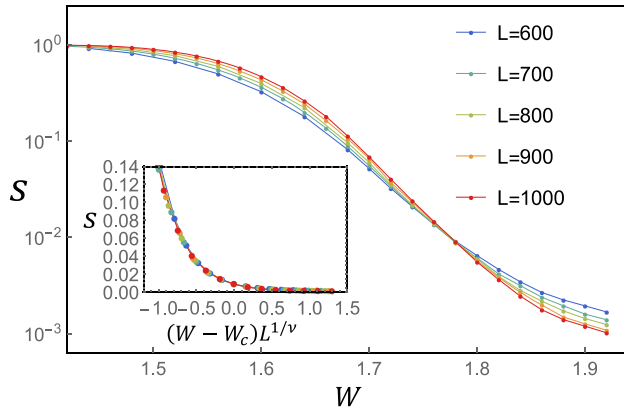
FIG. 3. The finite-size scaling for entanglement entropy s of 1D system with random potentials $W_i \in [0, W]$. The MBL transition occurs at $W_c = 1.78 \pm 0.01$. The data collapse shown in the inset gives rise to $\nu = 3.1 \pm 0.3$. Results are obtained by averaging over 10^5–10^6 disorder configurations.

TABLE I. The results of critical exponents obtained from our improved algorithms of real-space RG for both randomness- and QP-induced MBL transitions.

	QP-induced MBL	Randomness-induced MBL
Potential W_i	$W \cos(2\pi\alpha i + \phi)$	$[0, W]$
ν from RSRG	2.4 ± 0.3	3.1 ± 0.3
Lower bound of ν	1	2

behavior of both cases are summarized in Table I. Nonetheless, due to the difference between the values of ν for two cases are not that large after taking the error bar into consideration, the RSRG results obtained in the present work could not fully rule out the possibility that MBL transitions driven by randomness and by QP potentials belong to the same universality class.

Discussions and concluding remarks.—Note that the normalized EE curves of the QP systems shown in Fig. 2 are less smooth than the ones of random systems shown in Fig. 3. This may provide some insight into understanding the difference between two seemingly distinct universality classes. We think that the less smooth behaviors of the QP case should be related to the multifractal properties of single-particle spectrum and wave functions of QP systems. In contrast to the random case where distribution of energy mismatch between neighboring localized single-particle eigenstates is continuous and smooth, the single-particle level mismatch between neighboring localized wave functions in QP systems possesses a self-similar feature with several gaps between sub-bands. Consequently, due to the fractal structure and gaps in the distribution of energy mismatch in QP systems, initial input data as well as final results of RSRG may show less smooth behaviors (even cusps) compared to the random case.

We now discuss experimental ways to measure the critical exponent ν associated with the QP-induced MBL transitions, which could ultimately answer the question whether the QP MBL transition is stable or not against weak quenched randomness. One way of extracting the critical exponent ν is to experimentally measure transport properties in the Griffith region around MBL transitions. Specifically, the divergent behavior of dynamical exponent z in the Griffith region around the MBL critical point W_c is related with ν: $z(W) \sim [1/(W - W_c)^\nu]$. $z > 2$ for

subdiffusive transport in Griffith region while $z = 2$ for diffusive transport. Despite there being no randomness-induced rare regions in QP systems, the Griffith region still appears around the QP-induced MBL transition [40,66,67] since rare regions can appear from preparing initial states. In the Griffith region of 1D QP systems, subdiffusive behaviors should emerge. To probe subdiffusive transport experimentally, one can measure time dependence of the density imbalance $I(t)$, which is defined as $I = N_e - N_o/N_e + N_o$ with $N_{e/o}$ the number of particles on even (odd) sites of 1D lattices. If the system is prepared with all atoms residing on even sites, the decaying behaviors of imbalance I after a long time can be used as a sign to distinguish between ergodic and MBL phases. In the putative Griffith region, the imbalance I should decay in a power law [40,44] $I(t) \sim t^{-1/z}$. Thus, by measuring time dependence of imbalance I, the Griffith region and the dynamic exponent $z(W)$ as a function of W around MBL transition W_c can be determined. From the behaviors of $z(W)$, one can extract ν which could ultimately help to determine the universality class of the MBL transitions in QP systems.

In conclusion, we improved the RSRG approach to make it more versatile to study models with qualitatively different disorders, especially suitable for studying MBL transitions in systems with QP potentials. Moreover, it paves one way for further investigations of MBL criticality in various types of models, such as the ones with longer-range hoppings, different dimensions, or different interactions. In the present Letter, the critical exponent ν of MBL transitions in 1D QP systems obtained from our RSRG calculations satisfies the Harris-CCFS bound ($\nu > 2/d$) for random systems, which suggests that the MBL transition in 1D QP systems is Harris stable against weak quenched randomness. It would be interesting to study MBL transitions and their universal properties in higher-dimensional QP systems by employing this RSRG approach in the future.

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