Small Magnetic Charges and Monopoles in Nonassociative Quantum Mechanics

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Weak magnetic monopoles with a continuum of charges less than the minimum implied by Dirac's quantization condition may be possible in nonassociative quantum mechanics. If a weakly magnetically charged proton in a hydrogen atom perturbs the standard energy spectrum only slightly, magnetic charges could have escaped detection. Testing this hypothesis requires entirely new methods to compute energy spectra in nonassociative quantum mechanics. Such methods are presented here, and evaluated for upper bounds on the magnetic charge of elementary particles.

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In 1931, Dirac [1] showed that magnetic monopoles with charge g can be consistently described by wave functions provided the quantization condition $eg = N\hbar$ holds with half-integer N. Since the elementary electric charge e (or, rather, the fine structure constant) is small, the elementary magnetic charge is large. Therefore, there are strict limits on the possible magnetic charge of, say, a proton in a hydrogen nucleus because the strong magnetic charge would significantly alter the energy spectrum [2].

The aim of this Letter is to point out and analyze the fact that Dirac's argument relies on properties of wave functions in a Hilbert space, and therefore implicitly assumes that quantum mechanics is associative. If the assumption of associativity is dropped, there is no Hilbert-space representation of the algebra of observables (which by necessity would always be associative), but quantum mechanics may still be meaningful [3–6]. Indeed, the existence of consistent nonassociative algebras for magnetic charge densities has recently been demonstrated [7–11]. Nonassociative quantum mechanics can therefore be defined by replacing the operator product of observables with an abstract product, such that $\hat{a}_1(\hat{a}_2\hat{a}_3) \neq (\hat{a}_1\hat{a}_2)\hat{a}_3$ in general. States are defined as expectation-value functionals that assign complex numbers $\langle\hat{a}\rangle$ to algebra elements $\hat{a}$, subject to certain consistency conditions which make sure that uncertainty relations are respected. No wave functions appear in this formalism, and there is no analog of "single valuedness" used crucially by Dirac. Without wave functions, Dirac's argument therefore loses its footing. Magnetic monopoles are then possible with small charges much less than the smallest nonzero value, $g_0 = \frac{1}{2}\hbar/e$, allowed by Dirac. It is conceivable that a small magnetic charge of the proton could have escaped detection in precision spectroscopy such as [12].

Here, we show that even a small magnetic charge of the nucleus would significantly shift the ground-state energy of a hydrogen atom. To the best of our knowledge, this is the first time that properties of energy spectra have been computed in nonassociative quantum mechanics. We provide new methods to compute spectra in an algebraic manner, which may also be useful in other contexts.

Harmonic oscillator.—We first demonstrate the new methods in an application to the harmonic oscillator in standard, associative quantum mechanics. We have two distinguished observables $\hat{q}$ and $\hat{p}$ with $[\hat{q},\hat{p}] = i\hbar$, and the quantum Hamiltonian $\hat{H} = \frac{1}{2}(\hat{p}^2/m + m\omega^2\hat{q}^2)$.

An eigenstate $|\psi_E\rangle$ of $\hat{H}$ with eigenvalue E obeys the equation $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$, which implies

$$\langle\hat{a}(\hat{H}-E)\rangle_E = 0 \tag{1}$$

for the expectation value $\langle\cdot\rangle_E$ taken in $|\psi_E\rangle$, where $\hat{a}$ can be any polynomial in $\hat{q}$ and $\hat{p}$. We will first show that (1), which amounts to infinitely many equations given the freedom of choosing $\hat{a}$, allows one to compute the spectrum of $\hat{H}$ even if the eigenstates $|\psi_E\rangle$ are not known. In [13,14], it has been shown how observables can be computed using algebraic relations between moments of a state. The methods used here are closely related to these papers but provide a new application to energy spectra. In this way, we will set up a method to compute eigenvalues without using wave functions or boundary conditions. The same method can then be applied to the Coulomb problem in nonassociative quantum mechanics.

The demonstration is based on recurrence with respect to the degree of the polynomial $\hat{a}$ in $\hat{q}$ and $\hat{p}$. The ground-state

energy can be obtained by elementary calculations as follows: First, $\hat{a} = \hat{1}$ (the identity operator) gives $E = \frac{1}{2}(\langle \hat{p}^2 \rangle_E / m + m\omega^2 \langle \hat{q}^2 \rangle_E)$. For $\hat{a}$ not the identity, it is useful to refer to the equation

$$\langle [\hat{a}, \hat{H}] \rangle_E = \langle \hat{a}\hat{H} \rangle_E - \overline{\langle \hat{a}^\dagger \hat{H} \rangle_E} = E(\langle \hat{a} \rangle_E - \overline{\langle \hat{a}^\dagger \rangle_E}) = 0, \quad (2)$$

with the complex conjugate $\bar{z}$ of a complex number z. In particular, $\langle [\hat{q}, \hat{H}] \rangle_E = i\hbar \langle \hat{p} \rangle_E / m = 0$ from $\hat{a} = \hat{q}$ and $\langle [\hat{p}, \hat{H}] \rangle_E = -i\hbar m\omega^2 \langle \hat{q} \rangle_E = 0$ from $\hat{a} = \hat{p}$. From quadratic monomials, we obtain $\langle [\hat{q}^2, \hat{H}] \rangle_E = \frac{1}{2} i\hbar \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle_E / m = 0$ and $\langle [\hat{q}\hat{p}, \hat{H}] \rangle_E = i\hbar(\langle \hat{p}^2 \rangle_E / m - m\omega^2 \langle \hat{q}^2 \rangle_E) = 0$. Therefore, any eigenstate has fluctuations obeying $\Delta_E p = m\omega \Delta_E q$, and zero covariance $0 = C^E_{qp} = \frac{1}{2} \langle \hat{q}\,\hat{p} + \hat{p}\,\hat{q} \rangle_E - \langle \hat{q} \rangle_E \langle \hat{p} \rangle_E$. From the condition for $\hat{a} = \hat{1}$, $(\Delta_E q)^2 = E/(m\omega^2)$ and $(\Delta_E p)^2 = mE$.

So far, we have computed moments of a bound state in terms of its energy value E. We obtain a restriction on E by making sure that the fluctuations we derived obey the uncertainty relation

$$(\Delta_E q)^2 (\Delta_E p)^2 - (C^E_{qp})^2 = \frac{E^2}{\omega^2} \geq \frac{\hbar^2}{4} \quad (3)$$

and therefore $E \geq \frac{1}{2}\hbar\omega$.

In order to evaluate all the conditions imposed on eigenstates by (1), we follow [15,16] and introduce the operators $\hat{T}_{m,n} := (\hat{q}^m \hat{p}^n)_{\text{Weyl}}$ where m and n are nonnegative integers, and the subscript indicates that the product is taken in the totally symmetric ordering. The Hamiltonian is a linear combination $\hat{H} = \frac{1}{2}(\hat{T}_{2,0}/m + m\omega^2 \hat{T}_{0,2})$ of $\hat{T}_{2,0}$ and $\hat{T}_{0,2}$, and therefore (1) contains products of the form $\hat{T}_{m,n}\hat{T}_{m',n'}$. Using the basic commutation relation of $\hat{q}$ and $\hat{p}$, such products can always be rewritten as sums over individual $\hat{T}_{m'',n''}$ of order $m + n + m' + n'$ or less, as derived explicitly in [17]. The condition (1) is therefore equivalent to a recurrence relation for $\langle \hat{T}_{m,n} \rangle_E$ which is shown and discussed in more detail in our Supplemental Material [18]. (This material also uses an algebraic notion of states [19] and makes contact with effective constraints [20,21].)

In addition to higher-order moments $\langle \hat{T}_{m,n} \rangle_E$ of an eigenstate, we have higher-order uncertainty relations. They can be obtained just like Heisenberg's version, by applying the textbook derivation to integer powers of $\hat{q}$ and $\hat{p}$ or their products instead of just $\hat{q}$ and $\hat{p}$. A systematic procedure to organize these higher-order, or generalized, uncertainty relations has been given in [15,16]. For our purposes, a subset of these relations is sufficient, which can be constructed as follows: We define $\hat{\xi}_J$ as the $2J$-dimensional column vector consisting of all $\hat{T}_{m,0}$ and $\hat{T}_{m-1,1}$ up to order $m = 2J$, where J is an integer or half-integer. According to the generalized uncertainty principle, the matrix $M_J = \langle \hat{\xi}_J \hat{\xi}_J^\dagger \rangle$ is positive semidefinite for all J, where the expectation value is taken element by element. For $J = 1/2$, we have Heisenberg's uncertainty principle because a positive semidefinite matrix has a nonnegative determinant.

As outlined in the Supplemental Material [18], positive semidefiniteness of M_J can be reduced to the conditions

$$\prod_{k=1}^{n} (E/\hbar\omega - \alpha_k)(E/\hbar\omega + \alpha_k) \geq 0 \quad (4)$$

for all integer $n \geq 1$, where $\alpha_k = (2k-1)/2$ are the odd half-integer multiples. Considered as functions of E for all n, these expressions have nodes at $\hbar\omega\alpha_k$ up to some maximum k that depends on the particular value of n. Between nodes, the functions are nonzero and alternate in sign. Moreover, sending n to $n+1$ causes the signs at fixed E to alternate. This behavior combined with the nonnegativity of (4) implies that the only allowable values for E occur at the nodes. We can exclude negative values of E because we have already shown that $E \geq \frac{1}{2}\hbar\omega$. Thus, the only possible values for E are such that $E/\hbar\omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ in agreement with the well-known eigenvalues of the harmonic oscillator.

Moreover, the arguments just given show that, for each eigenvalue $E_n = (n - \frac{1}{2})\hbar\omega$, there is a generalized uncertainty relation which restricts higher-order moments and is saturated by the corresponding excited state with energy E_n. This result generalizes the well-known statement that the ground state of the harmonic oscillator saturates Heisenberg's uncertainty relation. Also note that our derivation, based on expectation values, still applies if the state used is mixed, given by a density matrix. Since we obtain the usual energy spectrum of the harmonic oscillator, it follows that mixed states do not to enlarge the spectrum.

As another consequence, we obtain the full energy spectrum of the harmonic oscillator from the unfamiliar condition (1) on energy eigenvalues. This result serves as a proof of concept of the new algebraic method introduced here, which we now apply to the Coulomb problem. We will then be ready to generalize the results to nonassociative quantum mechanics, where the usual methods of computing eigenvalues are not available.

Hydrogen.—The hydrogen atom has the Hamiltonian $\hat{H} = \frac{1}{2}|\hat{p}|^2/m - \alpha\hat{r}^{-1}$, where $|\hat{p}|^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$ and $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$. The position and momentum components are subject to the basic commutation relations $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$. For our purposes a different choice of distinguished observables,

$$\hat{r}, \qquad \hat{P} := \hat{r}|\hat{p}|^2, \qquad \hat{Q} := \hat{x}\hat{p}_x + \hat{y}\hat{p}_y + \hat{z}\hat{p}_z, \quad (5)$$

is more useful. Closely related variables have been used, quite differently, to compute hydrogen spectra in deformation quantization [22–24].

These operators have linear commutation relations

$$[\hat{r},\hat{Q}] = i\hbar\hat{r}, \qquad [\hat{r},\hat{P}] = 2i\hbar\hat{Q}, \qquad [\hat{Q},\hat{P}] = i\hbar\hat{P}, \quad (6)$$

and there is a Casimir operator

$$\hat{K} = \frac{1}{2}(\hat{r}\,\hat{P} + \hat{P}\,\hat{r}) - \hat{Q}^2 \quad (7)$$

that commutes with $\hat{r}$, $\hat{P}$, and $\hat{Q}$. A direct calculation in terms of the position and momentum components in (5) shows that $\hat{K}$ is equal to the total angular momentum squared. We should keep in mind that not all the distinguished observables are self-adjoint. We do have $\hat{r}^\dagger = \hat{r}$, but $\hat{Q}^\dagger = \hat{Q} - 3i\hbar$ and

$$\hat{P}^\dagger = \hat{P} - 2i\hbar\hat{r}^{-1}\hat{Q} = \hat{P} - 2i\hbar\hat{Q}\hat{r}^{-1} - 2\hbar^2\hat{r}^{-1}. \quad (8)$$

As in our demonstration using the harmonic oscillator, we will be interested in expectation values of monomials in $\hat{r}$, $\hat{P}$, and $\hat{Q}$ evaluated in eigenstates that obey (1). We have another useful relationship between certain expectation values given by the virial theorem

$$\alpha\langle\hat{r}^{-1}\rangle_E = 2E = -\frac{1}{m}\langle\hat{p}^2\rangle_E. \quad (9)$$

The procedure used for the harmonic oscillator does not directly apply to the Coulomb problem because the Hamiltonian is no longer quadratic, leading to highly coupled recurrence relations. We therefore reformulate the condition (1) in terms of a constraint linear in $\hat{P}$ and $\hat{r}$, introducing

$$\hat{C}_E = \hat{r}(\hat{H} - E) = \frac{1}{2m}\hat{P} - E\hat{r} - \alpha. \quad (10)$$

The condition on the spectrum of $\hat{H}$ then takes the form $\langle\hat{a}\hat{C}_E\rangle_E = 0$ for all polynomials $\hat{a}$ in $\hat{r}$, $\hat{r}^{-1}$, $\hat{P}$, and $\hat{Q}$. Unlike the Hamiltonian, $\hat{C}_E$ is not self-adjoint. It is still useful to apply commutator identities as in (2), but with a non-self-adjoint $\hat{C}_E$, there are additional terms: In an eigenstate such that $\langle\hat{a}\hat{C}_E\rangle_E = 0$ and $\langle\hat{a}^\dagger\hat{C}_E\rangle_E = 0$,

$$0 = \langle\hat{a}\hat{C}_E\rangle_E - \overline{\langle\hat{a}^\dagger\hat{C}_E\rangle_E} = \langle(\hat{a}\hat{C}_E - \hat{C}_E^\dagger\hat{a}\rangle_E. \quad (11)$$

With

$$\hat{C}_E^\dagger = \hat{C}_E - \frac{i\hbar}{m}\hat{r}^{-1}\hat{Q} = \hat{C}_E - \frac{i\hbar}{m}\hat{Q}\hat{r}^{-1} - \frac{\hbar^2}{m}\hat{r}^{-1} \quad (12)$$

using (8), we have

$$0 = \frac{\langle[\hat{a},\hat{C}_E]\rangle_E}{i\hbar} + \frac{\langle\hat{Q}\hat{r}^{-1}\hat{a}\rangle_E}{m} - \frac{i\hbar\langle\hat{r}^{-1}\hat{a}\rangle_E}{m}. \quad (13)$$

For $\hat{a} = \hat{Q}$,

$$0 = \frac{\langle\hat{P}\rangle_E}{2m} + E\langle\hat{r}\rangle_E + \frac{\langle\hat{Q}^2\hat{r}^{-1}\rangle_E}{m} + \frac{\hbar^2}{m}\langle\hat{r}^{-1}\rangle_E. \quad (14)$$

If we replace $\hat{Q}^2$ using the Casimir operator $\hat{K}$, and $\langle\hat{P}\rangle_E$ using $\langle\hat{C}_E\rangle_E = 0$, we have $0 = 3\alpha + 4E\langle\hat{r}\rangle_E - K_\ell\langle\hat{r}^{-1}\rangle_E/m$. The eigenvalues $K_\ell = \ell(\ell+1)\hbar^2$ of $\hat{K}$ follow from angular-momentum quantization, and $\langle\hat{r}^{-1}\rangle_E$ is related to E by (9). With these ingredients and similar calculations for $\hat{a} = \hat{r}\,\hat{Q}$, we obtain

$$\langle\hat{r}\rangle_E = \frac{1}{2}\frac{K_\ell}{m\alpha} - \frac{3}{4}\frac{\alpha}{E},$$
$$\langle\hat{r}^2\rangle_E = \frac{3}{4}\frac{K_\ell}{mE} + \frac{5}{8}\frac{\alpha^2}{E^2} - \frac{1}{4}\frac{\hbar^2}{mE}. \quad (15)$$

In order to determine the allowed eigenvalues E, as before, we have to impose uncertainty relations. We are interested here in the ground state, for which we can focus on the lowest-order uncertainty relations, computed for our noncanonical operators $\hat{r}$, $\hat{P}$, and $\hat{Q}$ using the Cauchy-Schwarz inequality. There is only one nontrivial relation,

$$(\Delta_E r)^2 C^E_{\hat{Q}\hat{Q}} \geq \left| C^E_{r\hat{Q}} + \frac{1}{2}i\hbar\langle\hat{r}\rangle_E \right|^2, \quad (16)$$

with two covariances. Again using (13), we compute $\langle\hat{Q}\rangle_E = \frac{1}{2}i\hbar$ using $\hat{a} = \hat{r}$, $\langle\hat{r}\,\hat{Q} + \hat{Q}\,\hat{r}\rangle_E = i\hbar\langle\hat{r}\rangle_E$ using $\hat{a} = \hat{r}^2$. Finally, $\langle\hat{Q}^\dagger\hat{Q}\rangle_E = \langle\hat{Q}^2\rangle_E - 3i\hbar\langle\hat{Q}\rangle_E$ can be obtained using $\hat{K}$.

Inserting all the required moments and factorizing the resulting polynomial in E, (16) gives the condition

$$\ell^2(\ell+1)^2(\ell^2+\ell-1)\frac{1}{E}\left(E + \frac{1}{2}\frac{m\alpha^2}{\hbar^2(\ell+1)^2}\right) \times \left(E + \frac{1}{2}\frac{m\alpha^2}{\hbar^2\ell^2}\right)\left(E - \frac{1}{2}\frac{m\alpha^2}{\hbar^2(\ell^2+\ell-1)}\right) \geq 0. \quad (17)$$

It is saturated for all energy eigenvalues with maximal ℓ, for which

$$E_{\ell+1} = -\frac{m\alpha^2}{2\hbar^2(\ell+1)^2}. \quad (18)$$

Assuming the well-known degeneracy of the hydrogen spectrum, we obtain the full set of bound-state energies. As in the example of the harmonic oscillator, every eigenstate saturates an uncertainty relation, in this case (16).

Nonassociative hydrogen.—We are now in a position to derive our main result. In the presence of a magnetic

central charge, we cannot use canonical momenta because they require a vector potential of the magnetic field $\vec{B}$. Instead, we generate an algebra using kinematical electron momenta, quantizing $p_i = m\dot{x}_i$. Their commutators are obtained by generalizing the case in which there is a vector potential $\vec{A}$ depending on $\vec{x}$, and canonical momenta are $\pi_i = p_i + eA_i$. Therefore,

$$[\hat{p}_j, \hat{p}_k] = i\hbar e\left(\frac{\partial \hat{A}_k}{\partial x_j} - \frac{\partial \hat{A}_j}{\partial x_k}\right) = i\hbar e \sum_{l=1}^{3} \epsilon_{jkl}\hat{B}^l, \quad (19)$$

while $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}$ is unchanged.

The final result depends only on $\vec{B}$ and therefore can be used to define the commutators $[\hat{p}_j, \hat{p}_k]$ also if $\nabla \cdot \vec{B} \neq 0$ in the presence of magnetic charges. A direct calculation shows that these commutators then no longer obey the Jacobi identity:

$$\begin{aligned}&[[\hat{p}_x, \hat{p}_y], \hat{p}_z] + [[\hat{p}_y, \hat{p}_z], \hat{p}_x] + [[\hat{p}_z, \hat{p}_x], \hat{p}_y] \\ &\quad = i\hbar e \sum_{j=1}^{3} [\hat{B}^j, \hat{p}_j] = -\hbar^2 e \hat{\mathrm{div}}\vec{B} \neq 0. \end{aligned} \quad (20)$$

Even a single pointlike monopole cannot be excised, as in Dirac's construction, if we consider weak charges that do not obey the quantization condition. However, a nonassociative algebra generated by commuting $\hat{x}_i$ and noncommuting $\hat{p}_j$, with standard commutators between $\hat{x}_i$ and $\hat{p}_j$, is still meaningful [3,4].

Another direct calculation shows that the commutators of $(\hat{r}, \hat{Q}, \hat{P})$ remain unchanged provided that $\vec{r} \times \vec{B} = 0$. This result, which relies on unexpected cancellations of the extra terms in commutators implied by (19), is crucial for the new application in this Letter. In this case, $\vec{B} = g(\vec{r})\vec{r}$. For a static magnetic field, we have $\nabla \times \vec{B} = 0$, which implies that $g(r)$ is spherically symmetric. A monopole density $\nabla \cdot \vec{B} \neq 0$ then requires that $g(r) = Q_m(r)/(4\pi r^3)$ with the magnetic charge

$$Q_m(r) = 4\pi \int \nabla \cdot \vec{B}(r) r^2 dr \quad (21)$$

enclosed in a sphere of radius r. For a single monopole at $r = 0$, $g(r) = g$ is constant.

The virial theorem relies only on algebraic properties and remains valid. With monopole commutators for momentum components, however, the modified angular momentum $\hat{\vec{L}}' = \hat{\vec{L}} + eg\hat{\vec{r}}/\hat{r}$, not $\hat{\vec{L}}$ itself, satisfies the usual commutators of angular momentum [25,26]. The Casimir of the algebra generated by $(\hat{r}, \hat{Q}, \hat{P})$ is still equal to $\hat{K} = \hat{\vec{L}}^2$, but in terms of the modified angular momentum it has an extra term:

$$\hat{K} = \hat{\vec{L}}^2 = \hat{\vec{L}}'^2 - e^2g^2. \quad (22)$$

For a single monopole at the center, the spectrum of $\hat{K}$ has a simple shift compared with the standard spectrum of $\hat{L}^2$, which is known to break the ℓ degeneracy of the hydrogen spectrum [2]. Moreover, the allowed values of ℓ are restricted for nonzero g because $\hat{K}$, by definition, is positive, and so must be its eigenvalues. Therefore, $\ell = 0$ is not possible for $g \neq 0$, and larger ℓ may be ruled out as well for strong magnetic charges.

We will focus now on the range of weak magnetic charges given by

$$0 < \frac{eg}{\hbar} = N < \frac{1}{2}. \quad (23)$$

None of these values could be modeled by a Dirac monopole (they would not correspond to single-valued wave functions), but they can be considered if quantum mechanics is nonassociative. Since the algebraic relations used to derive (17) are still applicable, we obtain conditions on the energy spectrum. The only difference is that the eigenvalues of $\hat{K}$ are now given by $K_\ell = \ell(\ell+1)\hbar^2 - e^2g^2$, which can be taken into account by replacing ℓ in (17) with

$$\tilde{\ell} = \sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{e^2g^2}{\hbar^2}} - \frac{1}{2}. \quad (24)$$

For quantized magnetic charges, the corresponding eigenvalues for which the first parenthesis in (17) is zero are indeed included in the spectrum found in [2], but they no longer constitute the full spectrum.

For weak magnetic charges, positivity of $\hat{K}$ requires that the smallest possible ℓ is $\ell = 1/2$, which we use for the ground state. The corresponding $\tilde{\ell}$ is equal to

$$\tilde{\ell} = \sqrt{1 - N^2} - \frac{1}{2} \quad (25)$$

and lies in the range $\frac{1}{2}(\sqrt{3} - 1) < \tilde{\ell} < \frac{1}{2}$. This range does not come close to the integer values 0 or 1 which would amount to standard hydrogen eigenvalues. Therefore, even for weak magnetic monopoles the energy spectrum of hydrogen is strongly modified. The ground-state energy is discontinuous in the central magnetic charge as a consequence of the positivity condition $K \geq 0$, which is the reason why even a small magnetic charge is not a simple perturbation of the usual hydrogen spectrum.

This result would seem to rule out any nonzero magnetic charge of the proton. However, from a purely experimental perspective, the smallest eigenvalue of the total angular momentum, used in our evaluation of $K \geq 0$, is zero only within some uncertainty. The angular momentum spectrum is very basic and hard to modify. For instance,

the conservation law and its role played in parity considerations implies that, for a single component, it has the form of a ladder centered around zero. It is, however, conceivable that its values are washed out to within some δL^2. To estimate this quantity, we are not restricted to hydrogenlike systems because all energy levels depend in some way on the eigenvalues of $\hat{L}^2$. The best relative precision, of about 5×10^{-19}, is obtained for spectral lines used in atomic clocks [27]. In SI units, a nonzero upper bound

$$g \leq \frac{4\pi\epsilon_0\sqrt{\delta L^2}c^2}{e} \approx 4.7 \times 10^{-18}\, A\, m = 1.4 \times 10^{-9} g_{\text{Dirac}} \quad (26)$$

then follows from $K \geq 0$ and (22), where g_{Dirac} is the smallest magnetic charge allowed by Dirac.

For the proton, this bound is not as strong as existing ones [28,29]. However, the bounds in [28,29] are obtained by limiting the total magnetic charge of a macroscopic object, adding the individual charges of all electrons or nucleons. Our bound is obtained directly for a single proton. Moreover, the magnetic charge of the muon is more difficult to bound [29]. Our bound, on the other hand, also applies to a muon as the nucleus of muonium, and to antimatter such as the antiproton in antihydrogen [30,31] or the positron in positronium [32].

If we directly apply hydrogen or muonium spectroscopy, with accuracies of $\Delta E/E \approx 4.5 \times 10^{-15}$ [12] and about 10^{-9} [33], respectively, we obtain weaker bounds: $g_{\text{proton}} \leq 9.5 \times 10^{-8} g_{\text{Dirac}}$ and $g_{\text{muon}} \leq 4.5 \times 10^{-5} g_{\text{Dirac}}$.

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