Degradability of Fermionic Gaussian Channels

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(Received 12 April 2016; revised manuscript received 17 June 2018; published 13 November 2018)

We study the degradability of fermionic Gaussian channels. Fermionic quantum channels are a central building block of quantum information processing with fermions, and the family of Gaussian channels, in particular, is relevant in the emerging field of electron quantum optics and its applications for quantum information. Degradable channels are of particular interest since they have a simple formula that characterizes their quantum capacity. We derive a simple standard form for fermionic Gaussian channels. This allows us to fully characterize all degradable n-mode fermionic Gaussian channels. In particular, we show that the only degradable such channels correspond to the attenuation or amplitude-damping channel for qubits.

DOI: 10.1103/PhysRevLett.121.200501

The transmission of quantum states in space and time is a fundamental physical process, described by quantum channels [1]. Therefore, the properties of quantum channels and their capacity to transmit classical or quantum information [2] is central to quantum information processing. Channel capacities are difficult to compute since, in general, they require an optimization over entangled inputs to many channels in parallel [3,4] and are only known for a few channels.

These complications do not arise for the quantum capacity of degradable channels [5,6], which can be expressed as a simple formula (which also equals their private capacity [7]). Their characteristic property is that the state of the environment can be reproduced from the channel output by applying another quantum channel. The notion has been generalized to weak [8], conjugate [9], and approximate [10] degradability, maintaining some of its useful properties.

The most natural information carrier in solid-state systems are electrons (quantum dot electrons [11,12] or, more recently, Majorana fermions in quantum wires [13–15]), i.e., fermions, whose anticommutation and superselection rules necessitate the refinement of central concepts of quantum information theory such as entanglement [16–22]. Impressive experimental advances (e.g., edge channels [23,24], moving quantum dots [25–28], quantum dot arrays [29]) demonstrate that electrons can be cleanly and individually transported in well-controlled semiconductor systems, providing fermionic quantum channels over sample-scale distances. These may serve for on-chip information transfer, e.g., between different registers of a quantum processor. This progress motivates the study of fermionic quantum channels [30–33], which are also useful to describe storage (transmission in time) of quantum information using fermionic systems.

Fermionic quantum channels have mostly been studied for noninteracting fermions, leading to the notion of fermionic Gaussian channels (FGCs) [34] (also known as "quasifree" channels [35,36] or "fermionic linear optics" [33,34,37–39]). These works emphasize the analogy with the case of Gaussian bosons, which are a very fruitful model for optical quantum information processing [40,41]. Here we exploit fermionic phase-space methods to analyze the degradability of FGCs. We derive a simple standard form that simplifies further analysis. With the phase-space characterization of quantum channels [34] in this form, we give a full characterization of all degradable $n \to n$-mode FGCs and show that there is only one family of such channels, the single-mode attenuation channel (see Theorem 1).

We consider free fermions with n-dimensional one-particle Hilbert space $\mathcal{H}$ ("n modes") [30,42], described by $2n$ Hermitian operators c_k, $k = 1, \ldots, 2n$, satisfying $\{c_k, c_l\} = 2\delta_{kl}$ and associated annihilation $a_j = (c_{2j-1} - ic_{2j})/2$ and creation operators $a_j^\dagger$, $j = 1, \ldots, n$.

Fermionic Gaussian states are those states for which Wick's theorem holds [43] (all cumulants are zero). They are fully described by the $2n \times 2n$ covariance matrix (CM) [34,36] defined as

$$\gamma_{kl} = \frac{i}{2}\mathrm{tr}(\rho[c_k, c_l]). \tag{1}$$

The matrix γ is real and antisymmetric. We frequently use that any such matrix can be brought to the form ([44], p. 18) $\Lambda = \bigoplus_{j=1}^n \lambda_j J$, with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, by a special orthogonal transformation: there exist $\lambda_j \in [-1, 1]$ and $O \in \mathrm{SO}(2n)$ such that $\gamma = O\Lambda O^T$. The CM Λ describes n modes in

Gibbs states for the Hamiltonian $a_j^\dagger a_j$. The Gaussian state is pure if and only if all $\lambda_j = \pm 1$, or, equivalently, if and only if $\gamma^2 = -\mathbb{1}$ holds.

If we consider a bipartite system of $n + m$ modes, then simplification of γ under *local* operations $\mathrm{SO}(2n) \oplus \mathrm{SO}(2m)$ is of interest. Any pure state Γ can be brought into the Schmidt form [17] with CM

$$\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \equiv \begin{pmatrix} J_{2l} \oplus \Lambda & (0_{2m\times 2l} \quad K)^T \\ (0_{2m\times 2l} \quad -K) & \Lambda \end{pmatrix} \tag{2}$$

by such local operations. Here $n = l + m$ and $J_{2s} = \oplus_{j=1}^{s} J$, $\Lambda = \oplus_{j=1}^{m} \lambda_j J$, and $K = \oplus_{j=1}^{m} \kappa_j \sigma_x$, with $\lambda_j^2 + \kappa_j^2 = 1$ (the parameters κ_j specify the amount of entanglement between the two parties).

Now let us turn to Fermionic Gaussian channels. We consider quantum channels (trace-preserving completely positive maps) that act on a finite set of n fermionic modes and map Gaussian states to Gaussian states. As discussed in [34], they are fully characterized by how they transform the $2n \times 2n$ covariance matrix γ of the input state. An $n \to m$-mode FGC $\mathcal{T}$ is defined by a $2m \times 2n$ matrix A and an antisymmetric $2m \times 2m$ matrix B as

$$\mathcal{T} \equiv \mathcal{T}_{(A;B)}: \gamma \mapsto A\gamma A^T + B. \tag{3}$$

Equivalently, a channel $\mathcal{T}_{(A;B)}$ can be characterized via its Choi-Jamiolkowski (CJ) state [6], which is given by the state obtained if the channel acts on the first half of a maximally entangled state. For Gaussian channels, the CJ state is Gaussian with CM $M_{(A;B)} = \left(\begin{smallmatrix} B & A \\ -A^T & 0 \end{smallmatrix}\right)$, since the maximally entangled state of $2n$ fermionic modes can be chosen Gaussian [$l = 0$ and $\lambda_j = 0$, $\kappa_j = 1$, $\forall\, j$ in Eq. (2)]. This yields a practical necessary and sufficient criterion for $(A; B)$ to define a valid quantum channel [34]: $\mathcal{T}_{(A;B)}$ describes a valid FGC if and only if the corresponding CJ-CM is a valid CM, i.e., if and only if $\mathbb{1} + iM_{(A;B)} \geq 0$, which is readily seen (see the Supplemental Material [45], Lemma S1) to be the case if and only if

$$\mathbb{1} - iB - AA^T \geq 0. \tag{4}$$

This implies that B is a valid CM and that $\mathbb{1} - AA^T \geq 0$ for FGCs and that the kernel of B contains that of $\mathbb{1} - AA^T$: $\ker B \supseteq \ker(\mathbb{1} - AA^T)$. Thus, the singular values of A must be ≤ 1 and B must vanish on the unit eigenspace of AA^T (the perfectly transmitted modes). This ensures that $B' = (\mathbb{1} - AA^T)^{-1/2} B (\mathbb{1} - AA^T)^{-1/2}$ is well defined and a CM (the inverse denotes the Moore-Penrose pseudoinverse [51] if $\mathbb{1} - AA^T$ has a kernel).

Every quantum channel $\mathcal{T}$ can be represented as a unitary acting on the system and an initially factorized environment prepared in some state ρ_E (see Fig. 1). Thus,

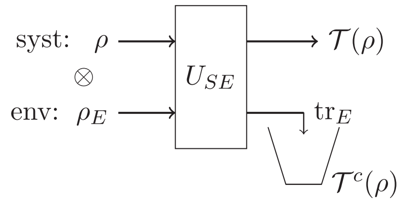

FIG. 1. Channel and complementary channel. For *pure* environmental state ρ_E, the dilation U_E and the complementary channel are unique (up to isometries).

$\mathcal{T}$ comes with a second channel, which describes what is leaked into the environment. For pure ρ_E, this channel is unique up to isometries and is called the complementary channel (of $\mathcal{T}$), denoted by $\mathcal{T}^c$. For our purposes, all these complementary channels are equivalent (see Supplemental Material [45], Lemma S3).

The relation between $\mathcal{T}$ and $\mathcal{T}^c$ has important consequences for certain capacities of $\mathcal{T}$ [5–7]. For example, if it were possible to obtain, for every input ρ, the channel output $\mathcal{T}(\rho)$ by suitably postprocessing the output of $\mathcal{T}^c$, then the channel $\mathcal{T}$ has vanishing quantum capacity, since any nonzero capacity would contradict the no-cloning principle [6]. Such channels for which there exists a completely positive (CP) map $\mathcal{P}$ such that $\mathcal{P}[\mathcal{T}^c(\rho)] = \mathcal{T}(\rho)$ are called "antidegradable" [52].

A more subtle consequence holds if there exists a quantum channel $\mathcal{W}$ such that the concatenation of $\mathcal{W}$ and $\mathcal{T}$ is equal to the complementary channel $\mathcal{T}^c$. Such channels are called "degradable" and have quantum capacity that can be characterized by a simple formula [5]. Degradable and antidegradable channels (and their "conjugate" relatives [9,53]) are at present the only ones for which a good understanding of their quantum capacity can be claimed. As we shall see, the simple structure of fermionic Gaussian channels allows a straightforward answer to the question of which $n \to n$ channels are degradable. This is in contrast to the available characterization for bosons, where the full degradability description is restricted to the one-mode case [52,54] or to the notion of weak degradability [55]. Note that, while degradability can be proven by constructing a degrading channel [6], to show that a channel is not degradable requires to show that *none* of the (possibly many, nonequivalent [56]) degrading maps is CP. In the case of interest here, we can show that the map is effectively unique. Our main result is summarized in the following theorem:

Theorem 1.—All degradable $n \to n$ fermionic Gaussian channels act on the covariance matrix γ as

$$\mathcal{T}_p: \gamma \mapsto (1 - p)\gamma + pJ_{2n}, \tag{5}$$

up to unitary pre- and postprocessing or are a direct sum of such channels. Here, $0 \leq p \leq \frac{1}{2}$ and $J_{2n} = -i \oplus_{j=1}^{n} \sigma_y$.

The proof proceeds in three steps. First, we observe that concatenating a channel with unitary channels does not affect (anti)degradability. This allows us to simplify the further discussion by focusing on FGCs in standard form in which the matrix A is diagonal with descendingly ordered, positive eigenvalues. This form can always be reached by concatenating the channel with Gaussian unitaries that effect the singular value decomposition of $A = O_1 D O_2$ (see Supplemental Material [45], Lemma S6). Additionally, FGCs that act independently on two subsets of modes, i.e., $\mathcal{T}_{(A;B)}$ with $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, are (anti)degradable if and only if both $\mathcal{T}_{(A_i;B_i)}$ are (see Supplemental Material [45], Lemma S7). Furthermore, a degrading map (*CP* or not) can only exist if A is invertible and, in this case, the FGC in question is itself *invertible*. Then the degrading map is unique (see Lemma S4 in the Supplemental Material [45] and Ref. [56] in this Letter) and the FGC is degradable if and only if the degrading map is *CP*. We then show that it is necessary that $D \geq 1/\sqrt{2}$ and that the channel has a small Choi rank [6] (its minimal dilation requires no more than n modes). Finally, we prove that such channels cannot be degradable unless D is a direct sum of (even-dimensional) terms $\propto \mathbb{1}$ ($D = \bigoplus \alpha_k \mathbb{1}_{2n_k}$ for $\alpha_k > 0$) and the environmental state CM γ_p a direct sum of pure CMs $\bigoplus_k \gamma_{p,k}$ (same partition as D).

Let us now construct the standard form and the corresponding degrading map. Without loss of generality, we consider an $n \to m$ FGC $\mathcal{T}_{(A;B)}$ with

$$A = (D \quad 0_{2(n-m)}) \quad \text{or} \quad A = \begin{pmatrix} D \\ 0_{2(m-n)} \end{pmatrix}, \tag{6}$$

depending on whether $m \leq n$ or $m \geq n$. Here $1 \geq D \geq 0$ is a square matrix with dimension $2 \times \min\{n, m\}$.

If D has $L \geq 2$ eigenvalues equal to 1, it implies that $\lfloor L/2 \rfloor$ modes are transmitted perfectly. As B then vanishes on all those modes, the channel is a particular case of a Gaussian product channel $(A; B) = (\mathbb{1} \oplus A_2; 0 \oplus B_2)$ and thus it is degradable if and only if $\mathcal{T}_{(A_2;B_2)}$ is, where A_2 now has at most one singular value equal to 1 and it suffices to consider such channels in our proof.

First, we need the complementary channel to $\mathcal{T}_{A;B}$ in order to express degradability in terms of A and B. To this end, it is useful to find a unitary dilation of $\mathcal{T}_{A;B}$. From Eq. (4), we see that for FGCs $AA^T \leq 1$ and $\ker(\mathbb{1} - AA^T) \subseteq \ker(B)$, and $B' := (\mathbb{1} - AA^T)^{-1/2} B (\mathbb{1} - AA^T)^{-1/2}$ is a valid CM. Then it is easy to check that the $n \to m$ FGC $\mathcal{T}_{A;B}$ can be obtained by a fermionic Gaussian unitary represented by $O_{SE'} \in \mathrm{SO}(2n + 2m)$,

$$O_{SE'} = \begin{pmatrix} A & \sqrt{\mathbb{1}_{2m} - AA^T} \\ -\sqrt{\mathbb{1}_{2n} - A^T A} & A^T \end{pmatrix}, \tag{7}$$

acting on the system and an m-mode environment in the Gaussian state with CM B'. To obtain the complementary channel, however, the environment should be *pure*. Let $l \leq n$ denote the number of pure modes of B', i.e., $B' = O(J_{2l} \oplus L)O^T$ for $0 \leq \lambda_j < 1$, $L = \bigoplus_{j=1}^{m-l} \lambda_j J_2$, and $O \in \mathrm{SO}(2m)$. Then

$$\gamma_E = [O \oplus \mathbb{1}_{2(m-l)}] \left(\begin{array}{c|c|c} J_{2l} & & \\ & L & K \\ \hline & -K & L \end{array} \right) [O^T \oplus \mathbb{1}_{2(m-l)}], \tag{8}$$

where $K = \bigoplus_{j=1}^{m-l} \kappa_j \sigma_x$ and $\lambda_j^2 + \kappa_j^2 = 1$ is a purification of B' and $\mathcal{T}_{A;B}$ can be obtained by coupling with

$$O_{SE} = O_{SE'} \oplus \mathbb{1}_{2(m-l)} \tag{9}$$

to the $2m - l$-mode pure environment in state γ_E.

There are other physical representations of $\mathcal{T}_{A;B}$ with pure environment γ_E' but they are all related isometrically to each other [1] and are all equivalent for our purposes (see Supplemental Material [45], Lemma S3).

Using this representation of $\mathcal{T}_{A;B}$, we can read off its complementary channel $\mathcal{T}^c_{A;B}$. It is the $n \to n + m - l$ map given by

$$\mathcal{T}^c_{(A;B)} \equiv \mathcal{T}_{(A_c;B_c)} \colon \gamma \mapsto A_c \gamma A_c^T + B_c, \tag{10}$$

where

$$A_c = \begin{pmatrix} \sqrt{\mathbb{1} - A^T A} \\ 0 \end{pmatrix}; \quad B_c = (A^T \oplus \mathbb{1}) \gamma_E (A \oplus \mathbb{1}),$$

with γ_E as in Eq. (8).

The question of the degradability of the FGC $\mathcal{T}_{A;B}$ is then, simply, if there exists an $m \to n + m - l$ FGC $\mathcal{T}_{\tilde{A};\tilde{B}}$ such that $\mathcal{T}_{\tilde{A};\tilde{B}} \circ \mathcal{T}_{A;B} = \mathcal{T}_{A_c;B_c}$. The degrading map follows directly from $\mathcal{T}_{A;B}$ and $\mathcal{T}_{A_c;B_c}$. The map only exists if A has no kernel and is then given by $\gamma \mapsto \tilde{A} \gamma \tilde{A}^T + \tilde{B}$ with

$$\tilde{A} = \begin{pmatrix} \sqrt{\mathbb{1} - A^T A} \\ 0 \end{pmatrix} A^{-1} = \begin{pmatrix} A^{-1} \sqrt{\mathbb{1} - AA^T} \\ 0 \end{pmatrix}, \tag{11}$$

$$\begin{aligned} \tilde{B} = {} & (A^T \oplus \mathbb{1}) \gamma_E (A \oplus \mathbb{1}) \\ & - [(A^{-1} - A^T) \oplus 0] \gamma_E [(A^{-T} - A) \oplus 0]. \end{aligned} \tag{12}$$

Using Eq. (4), we see that $\mathcal{T}_{(\tilde{A};\tilde{B})}$ is *CP* if and only if

$$\tilde{M} \equiv \mathbb{1} - \tilde{A}\tilde{A}^T - i\tilde{B} \geq 0. \tag{13}$$

We have thus constructed a fermionic Gaussian degrading map for a given FGC whenever it exists. It is then straightforward to check if it is *CP* via Eq. (13) and we now characterize all FGCs $(A;B)$ for which this is the case. Note that it is sufficient to check the properties of the map $\mathcal{T}_{(\tilde{A};\tilde{B})}$ since the channel $\mathcal{T}_{(A;B)}$ is invertible for invertible A (cf. Supplemental Material [45]), and the degrading map is unique in this case [56]. One may wonder whether our purely Gaussian discussion allows for possible non-Gaussian degrading maps. But since the Gaussian states span the space of all fermionic density matrices (see [45], Lemma S5), $\mathcal{T}_{(A;B)}$ is indeed invertible as a linear map on the space of fermionic density matrices and the Gaussian degrading map (11) and (12) is unique.

First, we claim that for an $n \to n$ FGC to be degradable it is necessary that its Choi rank is $\leq n$ modes. Assuming standard form $A = D$, reexpressing $\tilde{M}$ in Eq. (13) in terms of D and the pure $2n - l$-mode environmental state γ_E of a minimal dilation, we obtain, after repeated application of the Schur complement to check positivity of a block matrix (see Supplemental Material [45], Lemma S1), the inequality

$$2D^{-2} - D^{-4} - [O(0_{2l} \oplus \mathbb{1}_{2(n-l)})O^T] \geq 0 \qquad (14)$$

as a necessary condition for degradability, where $O \in \mathrm{SO}(2n)$ depends on γ_E (for details, see the Supplemental Material [45]). This inequality cannot be fulfilled unless $l = n$; i.e., the environment is no larger than the system. To see this, we use a condition on the eigenvalues of two Hermitian matrices and their sum implied by Horn's conjecture, [57,58]: let λ_i, μ_j, ν_k denote the descendingly ordered eigenvalues of the Hermitian matrices $X, Y, X+Y$, respectively. Then we have [58]

$$\nu_k \leq \lambda_i + \mu_j \quad \forall\, i + j = k + 1. \qquad (15)$$

We take $X = 2D^{-2} - D^{-4}$ and $Y = -O(0_{2l} \oplus \mathbb{1}_{2(n-l)})O^T$ and pick $j = 2l + 1$. Then $\mu_j = -1$ and for all $i > 1$ we have

$$\nu_{2l+i} \leq \lambda_i + \mu_{2l+1} = \frac{2}{d_i^2} - \frac{1}{d_i^4} - 1 \quad \forall\, i = 1, \ldots, 2(n-l).$$

Unless $n - l = 0$ (pure environment), we can take $i = 2$, which means that $d_i < 1$ (since in standard form we have, at most, one singular value of 1), in which case the expression on the rhs is negative.

Let us now focus on an $n \to n$ FGC with an n-mode environment. To complete the proof of Theorem 1, we show the following Lemma.

Lemma 1.—(Only constant-loss channels are degradable.) An $n \to n$ FGC $\mathcal{T}_{(D;B)}$ in standard form with Choi rank of $\leq n$ modes is degradable if and only if $D = \oplus_j (d_j \mathbb{1}_{2n_j})$, $d_j \geq 1/\sqrt{2}$, and $B = \oplus_j B_j$.

Proof.—Following analogous arguments to the construction above (in particular, using that $B = \sqrt{\mathbb{1} - D^{-2}}\gamma_p\sqrt{\mathbb{1} - D^{-2}}$ for an n-mode pure state CM γ_p), the degradability condition (13) becomes

$$2\mathbb{1} - D^{-2} - i\left[D\gamma_p D - \left(\frac{1}{D} - D\right)\gamma_p\left(\frac{1}{D} - D\right)\right] \geq 0. \qquad (16)$$

We show now that this only holds if $D \geq 1/\sqrt{2}$ and if $(\gamma_p)_{kl} = 0$ whenever $d_k \neq d_l$, i.e., if γ_p is a direct sum of pure CMs $\oplus_m \gamma_{p,m}$ and D is a corresponding direct sum of terms proportional to $\mathbb{1}$.

We already saw that $D \geq 1/\sqrt{2}$ is necessary for degradability. If there are one or more eigenvalues $d_i = 1/\sqrt{2}$, then the real part of (16) has a kernel and the inequality can only hold if $\gamma_{ij} = \gamma_{ji} = 0$ for all j such that $d_j \neq 1/\sqrt{2}$. That is, a channel with some $d_j = 1/\sqrt{2}$ can only be degradable if $D = D' \oplus (1/\sqrt{2})\mathbb{1}$ and $\gamma = \gamma_1 \oplus \gamma_2$ in accordance with Theorem 1 (by purity and antisymmetry, both blocks have to have even dimension).

We now assume $D > 1/\sqrt{2}$. Multiplying Eq. (16) by $1/\sqrt{2 - D^{-2}}$ from left and right, the imaginary part becomes $\gamma_p + R$, where

$$R = -\gamma_p + \frac{1}{W}[\gamma_p - D^2\gamma_p - \gamma_p D^2]\frac{1}{W}, \qquad (17)$$

with $W = \sqrt{2D^2 - 1}$.

Since γ_p is pure, $i\gamma_p$ has spectrum $\{\pm 1\}$ with eigenprojectors $P_\pm$. Thus, inequality (16) becomes

$$2P_+ + iR \geq 0, \qquad (18)$$

which shows that the overlap $\mathrm{tr}(P\_R) = -i\mathrm{tr}(\gamma_p R)$ must vanish. As detailed in the Supplemental Material [45], the matrix R is the pointwise (Hadamard) product of γ_p with a symmetric matrix r: $R_{kl} = r_{kl}(\gamma_p)_{kl}$, and the r_{kl} are strictly negative whenever $d_k \neq d_l$. Using the symmetry of r and antisymmetry and purity of γ_p then shows (see [45]) that this imposes $(\gamma_p)_{kl} = 0$ whenever $d_k \neq d_l$, i.e., the direct-sum decomposition of D and γ_p into blocks of even dimension corresponding to constant d_k. □

A final simplification used in Theorem 1 is that the pure state of the environment can be taken to be the vacuum state (with CM $\gamma_E = J$). This is the case since for $A = \sqrt{1-p}\mathbb{1}$ the FGCs $\mathcal{T}_{A;p\gamma_E}$ and $\mathcal{T}_{A;pJ}$ differ only by unitary pre- and postprocessing.

In summary, we have shown that there is only one family of degradable fermionic Gaussian channels, namely, the attenuation channel $\mathcal{T}_p : \gamma \mapsto (1-p)\gamma + pJ_{2n}$ (with losses $p \in [0, 1/2]$). Hence FGCs have a much simpler degradability structure than their qubit or bosonic Gaussian counterparts [6,55]. In contrast to the case of qubits, there are no degradable n-mode FGCs with large environment, nor any (nontrivial) Hadamard channels ([2], p. 196f). Note that even channels very close to the ideal one such as $T_{A;B}$

with $A = \mathrm{diag}(\sqrt{1-x^2}, \sqrt{1-y^2})$, $B = xyJ$, or $A = \alpha\mathbb{1}$, $B = 0$ are not degradable (unless $x = y$ or $\alpha = 1$, respectively).

We can exploit the degradability of $\mathcal{T}_p$ to compute its quantum capacity $Q(\mathcal{T}_p)$ given by the channel's coherent information [2]: $Q(\mathcal{T}_p) = \max_\gamma \{S[\mathcal{T}_p(\gamma)] - S[(\mathcal{T}_p \oplus \mathbb{1})(\Gamma)]\}$, where Γ is a purification of γ. That we can restrict to Gaussian input is a consequence of the extremality of Gaussian states as shown in [54,59] (for bosons) and generalized to fermions in [60] (see also [45]). With $\gamma = \lambda J$ (general one-mode CM), $\mathcal{T}_p(\gamma)$ has eigenvalues $\pm i[p + (1-p)\lambda]$; we can take $\Gamma = \begin{pmatrix} \lambda J & \sqrt{1-\lambda^2}X \\ -\sqrt{1-\lambda^2}X & \lambda J \end{pmatrix}$ and find that $(\mathcal{T} \oplus \mathbb{1})(\Gamma)$ has eigenvalues $\pm i$ (one pure mode) and $\pm i(1 - p + p\lambda)$. This reduces the computation of Q to a simple one-parameter optimization: $Q(p) = \max_{-1\le\lambda\le1}\{H[(1-p)(1-\lambda)/2] - H[p(1-\lambda)/2]\}$, where $H(p) = -p\log p - (1-p)\log(1-p)$ is the binary entropy. The channel $\mathcal{T}_p$ is equivalent to the qubit amplitude-damping channel with Kraus operators $K_1 = |0\rangle\langle 0| + \sqrt{(1-p)}|1\rangle\langle 1|$ and $K_2 = \sqrt{p}|0\rangle\langle 1|$, whose quantum capacity was computed in [61]. Notably, the *classical* capacity [2] of the qubit amplitude damping channel remains unknown to date, although lower [61] and upper bounds [62] have been obtained.

There are several interesting directions for further research. (1) The generalization of the above result to the case of $n \to m$ channels is important, in particular, for the antidegradability even of $n \to n$ channels, since, in general, the complementary channel is a map between systems of different numbers of modes. While it is clear that $n \to m$ channels with $m < n$ are never degradable (since A has a kernel), in the case $m > n$, the positivity of the real part of Eq. (14) is no longer easy to decide since it may depend on the details of γ_E and examples of degradable channels with Choi rank larger than $\max\{n, m\}$ exist (see [45]). Moreover, the uniqueness of the degrading map is no longer ensured and all such maps (both Gaussian and, possibly, non-Gaussian) would need to be examined. (2) FGCs may provide a simple setting to search for exclusively conjugate-degradable channels [9]. (3) Our result on the degradability of FGCs may be of use in bounding the private and quantum capacity of some non-degradable channels or non-Gaussian channels by exploiting the notion of approximate degradability [10] or following [54] via a fermionic Gaussian teleportation channel.

We thank an anonymous referee for drawing our attention to the matter of uniqueness of the degrading map. We acknowledge the Centro de Ciencias de Benasque Pedro Pascual, where the central part of this work was completed. G. G. thanks the Institute for Mathematical Sciences, National University of Singapore for the invitation to the 2013 Mathematical Horizons of Quantum Physics workshop, during which early parts of this work were done. E. G. acknowledges financial support of the Villum Fonden and the Bakala Foundation.

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