Quantum Error Correction Decoheres Noise

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Typical studies of quantum error correction assume probabilistic Pauli noise, largely because it is relatively easy to analyze and simulate. Consequently, the effective logical noise due to physically realistic coherent errors is relatively unknown. Here, we prove that encoding a system in a stabilizer code and measuring error syndromes decoheres errors, that is, causes coherent errors to converge toward probabilistic Pauli errors, even when no recovery operations are applied. Two practical consequences are that the error rate in a logical circuit is well quantified by the average gate fidelity at the logical level and that essentially optimal recovery operators can be determined by independently optimizing the logical fidelity of the effective noise per syndrome.

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Introduction.—Quantum computers are likely to dramatically outperform classical computers, provided that errors can be corrected enough to make the output reliable. Errors in a quantum computer can take many forms with differing impacts on an error-correction procedure. Most studies of the performance of quantum error-correcting codes only consider probabilistic Pauli errors because they are easy to simulate via the Gottesman-Knill theorem [1]. However, in real systems, it is likely that other noise will also be present.

Determining the performance of an error-correcting code at the logical level under general noise is complicated because such noise is harder to simulate. Previous approaches have expanded the class of errors to some larger class that can still be efficiently simulated [2], performed full density-matrix simulations [3], used tensor network descriptions of specific codes [4,5] or effective logical process matrices [6–8]. These methods are suboptimal because they either require a huge amount of resources to simulate or are indirect approximations. They also do not easily give structural insight because extrapolating the effective logical noise from the description of the encoded state is difficult and determining the scaling with parameters of interest typically requires extensive recalculations.

Optimistically, one may hope that a (numerical or analytical) estimate of the infidelity of the logical noise under a probabilistic Pauli channel generalizes directly to general logical noise. However, even quantifying the error becomes more complicated for more general noise. The "error rate" due to a noise process $\mathcal{N}$ acting on a m-level system is often experimentally quantified via the average gate infidelity to the identity (hereafter the infidelity)

$$r(\mathcal{N}) = 1 - \int d\psi \langle\psi|\mathcal{N}(|\psi\rangle\langle\psi|)|\psi\rangle \tag{1}$$

because it can be efficiently estimated via randomized benchmarking [9–13]. However, theoreticians often report rigorous bounds on the performance of a quantum error-correcting code or a circuit in terms of the diamond distance to the identity (hereafter the diamond distance) [14]

$$\epsilon(\mathcal{N}) = \sup_{\psi} \frac{1}{2} \|[\mathcal{N} \otimes \mathcal{I}_m - \mathcal{I}_{m^2}](\psi)\|_1, \tag{2}$$

where $\|A\|_1 = \sqrt{\mathrm{Tr}A^\dagger A}$ and the maximization is over all m^2-dimensional pure states (to account for the error introduced when acting on entangled states).

The infidelity and diamond distance are related via the bounds [15,16]

$$r(\mathcal{N})(1 + m^{-1}) \leq \epsilon(\mathcal{N}) \leq \sqrt{m(m+1)r(\mathcal{N})}. \tag{3}$$

which scale optimally with respect to r and m [17]. For unitary noise, $\epsilon(\mathcal{N})$ scales as $\sqrt{r(\mathcal{N})}$, though it does not necessarily saturate the upper bound of Eq. (3); this scaling follows from the magnitude of the coherent (non-Pauli) part of the noise [18]. Pauli noise saturates the lower bound of

Eq. (3) and the effect of coherent noise is often assumed to be negligible, so that experimental infidelities are often compared to diamond distance targets to determine whether fault tolerance is possible [17]. However, even if coherent errors make a negligible contribution to the infidelity, they can dominate the diamond norm [19]. Because of this uncertainty about how to quantify errors effectively, it is unclear what figure of merit recovery operations should optimize and how to quantify the logical error rate [3,8,20].

Previous studies have shown that the contribution to the logical noise from the coherent part of the physical noise decays exponentially as a function of code distance [7], although the decay rate was only given as an abstract property of the noise map. Recently, the decay rate was analyzed for specific noise models in the repetition code [21].

In this Letter, we directly relate the decay rate of coherent terms at the logical level of a general stabilizer code to the infidelity of the physical noise of a general local noise process, which can be estimated by randomized benchmarking. Further, we give physical motivation for the decoherence of errors with increasing code distance by relating the scaling of errors to projective syndrome measurements. We demonstrate that—even without applying recovery operations—encoding a system in a quantum error-correcting code and measuring error syndromes decoheres errors, that is, causes rapid convergence toward probabilistic Pauli errors. To isolate the contribution from local noise, we assume that there is no other contributing noise. That is, encoding, syndrome measurements, recovery operations, and decoding are all assumed to be noiseless.

Our results show that the effective logical noise is well characterized by the logical infidelity. This provides a rigorous justification for choosing recovery maps to independently optimize the logical fidelity per syndrome (instead of, e.g., optimizing the diamond norm of the logical noise averaged over all syndromes). Complementary results on the scaling of the diamond distance with quantum error-correction protocols were independently obtained in Ref. [22].

The Letter is structured as follows. We first introduce Markovian noise processes and review the process matrix formalism, a convenient representation of quantum channels (not to be confused with the χ matrix representation). We then give an expression for the infidelity in terms of this representation and discuss the implications and bounds on the entries of a process matrix in terms of its infidelity. Next, we introduce stabilizer codes and, using the aforementioned bounds, discuss the behavior of the effective logical noise of an encoded state after syndrome measurements with and without the application of recovery operations in terms of the physical infidelity of the qubits. We conclude by discussing some implications of our work and discuss how our results relate to existing results showing coherent errors at the logical level.

Markovian noise processes.—We represent quantum states and measurements of a m-dimensional system by vectors as follows. Let $\{e_j : j \in \mathbb{Z}_m\}$ be the canonical basis of $\mathbb{C}^{m^2}$ and $\mathbb{B}$ be an arbitrary trace-orthonormal basis of $\mathbb{C}^{m\times m}$ respectively, that is, $\mathrm{Tr}(B_j^\dagger B_k) = \delta_{j,k}$ for all B_j, $B_k \in \mathbb{B}$. We will generally choose $\mathbb{B}$ to be the set of normalized (physical or logical) Pauli operators, $\mathbb{P} = \{I_2, X, Y, Z\}/\sqrt{2}$, or tensor products thereof. We define a map $|\cdot\rangle\rangle$: $\mathbb{C}^{m\times m} \to \mathbb{C}^{m^2}$ by setting $|B_j\rangle\rangle \to e_j$ for all $B_j \in \mathbb{B}$ and extending to a linear map, so that

$$|M\rangle\rangle = \sum_j \mathrm{Tr}(B_j^\dagger M) e_j. \tag{4}$$

Defining $\langle\langle M| = |M\rangle\rangle^\dagger$, we have

$$\langle\langle M|N\rangle\rangle = \mathrm{Tr}(M^\dagger N). \tag{5}$$

A Markovian noise process is a linear map $\mathcal{N}$ that maps valid quantum states of one system to valid quantum states of another system, and so is completely positive and trace preserving. Let $\mathbb{B}_{\mathrm{in}}$ and $\mathbb{B}_{\mathrm{out}}$ be trace-orthonormal bases for the input and output systems, respectively. Then

$$|\mathcal{N}(M)\rangle\rangle = \sum_{B\in\mathbb{B}_{\mathrm{in}}} |\mathcal{N}(B)\rangle\rangle\langle\langle B|M\rangle\rangle = \mathcal{N}|M\rangle\rangle, \tag{6}$$

where we abuse notation slightly by using $\mathcal{N}$ to denote both an abstract map and its matrix representation $\sum_{B\in\mathbb{B}_{\mathrm{in}}} |\mathcal{N}(B)\rangle\rangle\langle\langle B|$. Note that $|\mathcal{N}(B)\rangle\rangle$ is a state of the output system and so is expanded relative to $\mathbb{B}_{\mathrm{out}}$ via Eq. (4). The composition of two channels is then given by the standard matrix product of the process matrices.

The average infidelity of a single-qubit noise process $\mathcal{N}$ with the identity in terms of process matrices is [23]

$$r = \frac{\mathrm{Tr}[\mathcal{I} - \mathcal{N}]}{6}. \tag{7}$$

The infidelity only captures the effects of the Pauli part of the noise, that is, the diagonal part, whereas the disconnect between the infidelity and the diamond norm in Eq. (3) for non-Pauli noise is due to the off-diagonal terms, which we call the coherent part of the noise.

Setting $B_0 = I_2/\sqrt{2}$ and defining the single-qubit error matrix $E \equiv |I_4 - \mathcal{N}|$, we have the following bounds on the matrix entries $E_{\sigma,\tau} = \langle\langle\sigma|E|\tau\rangle\rangle$ of E in terms of the infidelity.

Lemma 1.—For any single-qubit Markovian noise process with infidelity r,

$$E_{\sigma_0,\sigma} = 0, \tag{8a}$$

$$E_{\sigma,\sigma_0} \le 3r, \tag{8b}$$

$$E_{\sigma,\sigma} \le 3r, \tag{8c}$$

$$E_{\sigma,\tau} \le \sqrt{6r}, \tag{8d}$$

for all $\sigma, \tau \in \vec{\sigma} = I, X, Y, Z/\sqrt{2}$.

Proof.—Equation (8a) follows directly from the trace-preserving condition. Equation (8b) was proven in [16], Prop. 12. To prove Eq. (8c), note that the Pauli twirl of $\mathcal{N}$,

$$\frac{1}{4}\sum_{P\in\{I,X,Y,Z\}}\mathcal{P}\mathcal{N}\mathcal{P} \tag{9}$$

where $\mathcal{P}$ denotes the channel that acts via conjugation by P, is a valid channel whose process matrix is the diagonal part of $\mathcal{N}$ whose singular values are consequently the diagonal entries. We can then write $E_{\sigma,\sigma} = a_\sigma r$ [24] where the a_σ must satisfy

$$(a_\sigma - a_\tau)^2 \le a_\nu^2 \tag{10}$$

for all permutations $\{\sigma,\tau,\nu\}$ of $\sigma\backslash\{\sigma_0\}$ in order for the map to be completely positive and trace preserving [Eq. (63) in [16]] and must add to 6, by Eq. (7), as $\mathcal{N}$ has infidelity r.

Equation (8d) holds as the Euclidean norm of any column of $\mathcal{N}_u$ is upper bounded by 1 where $\mathcal{N}_u$ is the unital block obtained by deleting the first row and column of $\mathcal{N}$ [24]. Note that the term in the square root was only kept to $\mathcal{O}(r)$; an r^2 term was dropped, reducing the inequality from $E_{\sigma,\tau} \le \sqrt{6r - 9r^2}$. This convention will be followed for the remainder of the Letter. This bound can be tightened further by considering unitarity [25].

Stabilizer codes.—We now review stabilizer codes; for more details, see, e.g., Ref. [26]. Let $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$. An n-qubit Pauli operator P is the tensor product of n single-qubit Pauli operators, and the weight $w(P)$ of a Pauli operator P is the number of qubits P acts on nontrivially. An $[\![n,k,d]\!]$ stabilizer code encodes k logical qubits in n physical qubits and is distance d; it is defined by an Abelian group $\mathbb{S}\not\ni -I$ of 2^{n-k} n-qubit Pauli operators, which can be described by a set of generators $g_1,\ldots,g_{n-k}$. We can define a set of 2^{n-k} mutually orthogonal projectors

$$\Pi_s = \prod_{j=1}^{n-k}\frac{1}{2}[I + (-1)^{s_j}g_j], \tag{11}$$

where s_j is the jth entry of the syndrome s, and the code space is the support of Π_0. An error is detectable if it maps the support of Π_0 outside of Π_0 and has no effect if it acts trivially on Π_0, that is, if it is in $\mathbb{S}$. The distance of the code is the minimal Pauli weight of an undetectable error that acts nontrivially on Π_0. For each error syndrome $s\in\mathbb{Z}_2^{n-k}$ we can find a Pauli operator R_s satisfying $R_s\Pi_s R_s = \Pi_0$ which corrects the error.

We can find a set of operators $\{\bar{X}_j,\bar{Z}_j : j = 1,\ldots,k\}$ such that for all $S\in\mathbb{S}$ and $j\neq k$,

$$\begin{aligned}&[\bar{X}_j, S] = [\bar{Z}_j, S] = 0,\\ &[\bar{X}_j,\bar{X}_k] = [\bar{X}_j,\bar{Z}_k] = [\bar{Z}_j,\bar{Z}_k] = 0,\\ &\bar{X}_j\bar{Z}_j = -\bar{Z}_j\bar{X}_j.\end{aligned} \tag{12}$$

Let $\mathbb{L}$ be the projective group generated by $\{\bar{X}_j,\bar{Z}_j : j = 1,\ldots,k\}$. Then $2^{-k/2}\mathbb{L}\Pi_0$ is a trace-orthonormal set of operators that span the code space. Therefore, any operator $\bar{\rho}$ in the code space can be written as

$$\bar{\rho} = 2^{-k}\sum_{L\in\mathbb{L}}\mathrm{Tr}(L\Pi_0\bar{\rho})L\Pi_0. \tag{13}$$

Effective noise under error correction.—We now prove that, even with bad decoders (or no correction), encoding in an error-correcting code decoheres local errors.

For ideal encoding and correction operations, preparing an initial state in the code space, applying a general local n-qubit noise process $\mathcal{N} = \mathcal{N}^{(1)}\otimes\mathcal{N}^{(2)}\otimes\cdots\otimes\mathcal{N}^{(n)}$, and performing a syndrome measurement with the outcome s maps the system from the support of Π_0 to that of Π_s. Let $p(s)$ be the probability of observing the syndrome s, which will generally depend upon the input state. Then by Eq. (6) the effective noise map from Π_0 to Π_s is

$$\bar{\mathcal{N}}(s)_{L,L'} = \frac{\langle\langle L\Pi_s|\mathcal{N}|L'\Pi_0\rangle\rangle}{p(s)2^k}, \tag{14}$$

where the factor of 2^{-k} comes from the normalization of $\mathbb{L}\Pi_s$ [6]. Note that it is conventional to apply a "pure error" [27] to map back to the code space. We omit this step to highlight the fact that syndrome measurements alone decohere the noise.

Theorem 2.—For any $[\![n,k,d]\!]$ stabilizer code, the average off-diagonal elements of the logical noise under a local noise process $\mathcal{N} = \bigotimes_{j=1}^{n}\mathcal{N}^{(j)}$ scales as

$$\sum_s p(s)\bar{\mathcal{N}}(s)_{L,L'} \in \mathcal{O}(r^{d/2}) \quad \text{as } r\to 0, \tag{15}$$

where $r = \max_j r(\mathcal{N}^{(j)})$.

Proof.—By Eq. (11), Eq. (14) can be rewritten as

$$\bar{\mathcal{N}}(s)_{L,L'} = \sum_{S,S'\in\mathbb{S}}\frac{\phi(S|s)\langle\langle LS|\mathcal{N}|L'S'\rangle\rangle}{p(s)2^{2n-k}}, \tag{16}$$

where $\phi(S|s)$ is the sign of S in the expansion of Eq. (11). As $\mathcal{N}$ and the stabilizers are all tensor products, terms of the form $\langle\langle LS|\mathcal{N}|L'S'\rangle\rangle$ can be factorized. However, this introduces a subtlety as LS may be a phase multiple of an element of $\{I,X,Y,Z\}^{\otimes n}$, which needs to be accounted for when factoring the tensor product. Let $\chi(A)\in\{\pm,\pm i\}$ be the phase multiple of A relative to its representative element A' in the projective Pauli group $\{I,X,Y,Z\}^{\otimes n}$ so that $A = \chi(A)A'$. Note that we can ignore the $\pm i$ case as all operators under consideration are Hermitian. Then, using $\mathcal{N}_{P,Q} = \langle\langle P|\mathcal{N}^{(j)}|Q\rangle\rangle/2$ for $P,Q\in\{I,X,Y,Z\}$,

$$\bar{\mathcal{N}}(s)_{L,L'} = \sum_{S,S'\in\mathbb{S}}\frac{\phi(S|s)\chi(LS)\chi(L'S')}{p(s)2^{n-k}}\prod_{j=1}^{n}\mathcal{N}^{(j)}_{L_jS_j,L'_jS'_j}. \tag{17}$$

By the definition of the code distance, SL and $S'L'$ differ on at least d qubits for $S \in \mathbb{S}L$, $S' \in \mathbb{S}L'$, and $L \neq L'$. Therefore, for any $L \neq L'$, each term on the right-hand side of Eq. (17) is in $\mathcal{O}(r^{d/2})$ by Lemma 1 after syndrome measurements. Averaging over the syndromes cancels the $p(s)$ in the denominator.

Intuitively, syndrome measurements decohere errors because the act of measuring projects out any Pauli in the expansion of the output state that is not of the form LS, thus removing the components of the output state corresponding to the additional Pauli operators introduced by coherent noise.

In Theorem 1, we proved that any errors are suppressed exponentially with the code distance. To conclude that the noise is decohered, we need to show that the off diagonals of the logical error matrix E do not scale as the square root of the diagonals, so that the ratio of the off diagonals to diagonals decreases with code distance (i.e., the ratio of the off-diagonal elements to the diagonal elements of the logical noise is less than the corresponding ratio for the physical noise). To see that this holds, at least for typical noise in nondegenerate stabilizer codes, note that Eq. (16) is linear in $\mathcal{N}$. Writing $\mathcal{N} = \sum_{x\subset\mathbb{Z}_n} E(x)$ where $E(x)$ is an error that only acts nontrivially on qubits in x and $E(\varnothing) = I$,

$$\bar{\mathcal{N}}(s)_{L,L'} = \sum_{S,S'\in\mathbb{S},x\subset\mathbb{Z}_n} \frac{\phi(S|s)\chi(SL)\chi(S'L')}{p(s)2^{n-k}} \prod_{j\in x} E(x)^{(j)}_{L_jS_j,L'_jS'_j}. \tag{18}$$

For a nondegenerate distance d stabilizer code, there exists some set x of at most $\lceil d/2 \rceil$ qubits such that $E(x)$ cannot be corrected, that is, canceled out when averaged over syndromes. This set contributes a term $\sum_{S\in\mathbb{S}} \prod_{j\in x} E(x)^{(j)}_{L_jS_j,L_jS_j}$. By reducing the generators so that at most one generator acts nontrivially as σ on each $j \in x$ for each $\sigma \in \vec{\sigma}$, we can find some stabilizer such that $L_jS_j \neq \sigma_0$ for all $j \in x$. Let

$$r' = \min_{j,\sigma\in\vec{\sigma}} E(x)^{(j)}_{\sigma,\sigma}, \tag{19}$$

which will be $\mathcal{O}(r)$ for typical noise. Then x contributes a term that scales as at least $r'^{|x|}$ to the effective logical error and so the logical infidelity scales as $r'^{\lceil d/2 \rceil}$ or worse, so that the off diagonals are, at worst, proportional to the diagonals of the logical error matrix.

As d increases, the scaling described above causes the effective logical noise to become progressively less coherent so that the Pauli twirl approximation captures the logical noise more effectively. However, due to contributions from the coherent part of the physical noise to the Pauli part of the logical noise, approximating the physical noise as Pauli in order to calculate the logical noise produces inaccurate results as observed previously [3,21]. Reference [21] demonstrated that the coherent contribution dominates the Pauli part of the logical noise after many rounds of error correction. We now apply our bounds on the scaling of errors to a more general analysis of error accumulation in a scheme with rounds of error correction. The effective logical noise after h rounds of error correction is

$$(I - \bar{E})^h \approx I - h\bar{E} + \binom{h}{2}\bar{E}^2, \tag{20}$$

where we have taken a binomial expansion to second order in $\bar{E}$. Assuming typical noise, the off diagonals of $\bar{E}$ scale at worst as $\mathcal{O}(r^{(d+1)/2})$, and the diagonals as $\mathcal{O}(r^{d/2})$. When the noise is Pauli, the effective logical noise on the diagonal after h rounds of error correction will be at worst

$$(I - \bar{E})^h_{\sigma,\sigma} \approx 1 - \mathcal{O}(hr^{(d+1)/2}) + \mathcal{O}(h^2r^{d+1}). \tag{21}$$

If coherent noise is present,

$$(I - \bar{E})^h_{\sigma,\sigma} \approx 1 - \mathcal{O}(hr^{(d+1)/2}) + \mathcal{O}(h^2r^{d}). \tag{22}$$

Taking the ratio of the first and second order terms, quadratic errors start to accumulate from Pauli noise at $h_P \approx 1/r^{(d+1)/2}$ and from coherent noise at $h_c \approx 1/r^{(d-1)/2}$. The coherent noise begins to dominate the Pauli part of the effective logical noise at $h_{\text{crit}} \approx 1/r$, independent of the code distance. This critical value is consistent with the value observed in Ref. [21] of $1/\epsilon^2$, where, ϵ is the angle of rotation about the x axis, and we note that all of our observations hold in their specific case when we replace r in our results with $\sqrt{\epsilon}$, as that is how the specified noise scales relative to our Lemma 1. Because the off-diagonal terms and diagonal terms produce the same scaling in a worst-case analysis with coherent noise, the ratio of off-diagonal to diagonal errors is independent of the number of rounds of error correction in the worst-case scaling of typical noise.

Conclusion.—In this Letter, we have shown that for generic local noise, coherent errors are decohered by syndrome measurements in error-correcting stabilizer codes. Consequently, error rates in logical circuits are well quantified by the logical infidelity. Therefore, it is appropriate to choose recovery operators to optimize the logical fidelity, instead of other measures such as the diamond norm. This dramatically simplifies the process of selecting recovery operators for general noise because the fidelity is a linear function of quantum channels and so we can optimize the fidelity of the logical noise for each syndrome independently, as noted in [8]. By contrast, if we tried to optimize the diamond norm of the average logical noise, we would have to simultaneously optimize all recovery operators.

While we have only explicitly considered independent errors, note that our arguments apply directly to correlated errors of the form

$$\mathcal{N} = \sum_{\alpha} p_{\alpha} \bigotimes_{j=1}^{n} \mathcal{N}^{(\alpha,j)} \tag{23}$$

by linearity. The only nontrivial issue is identifying a scaling parameter akin to the single-qubit infidelity.

Previous results have demonstrated significant logical coherent errors [3,7], namely, off diagonals that scale as $r^{3/2}$ compared to diagonals that scale as r^2. However, these results were all for distance 3 codes and are consistent with our results as for such codes, $\lceil d/2 \rceil = 2$ giving diagonals that scale as $r^{\prime 2}$ and off diagonals that scale as $r^{3/2}$ by Theorem 1. Numerically, significant discrepancies between the logical diamond norm error with and without Pauli twirling (which removes the coherent part of the noise) at the physical level have been observed for high distance surface codes [4] (up to distance 10). These discrepancies have been interpreted as suggesting significant logical coherent errors [21]. Our results show that these discrepancies are almost entirely due to contributions to the logical infidelity (and thereby diagonals) from the coherent part (i.e., off diagonals) of the physical noise, though for a specific syndrome and noise model, the effective logical noise may appear coherent. That is, the effective logical noise is generically very close to a Pauli channel on average; however, it may not be the Pauli channel one would predict from the Pauli twirl of the physical noise.

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[1] S. Aaronson and D. Gottesman, Improved simulation of stabilizer circuits, Phys. Rev. A **70**, 052328 (2004).
[2] M. Gutiérrez and K. R. Brown, Comparison of a quantum error-correction threshold for exact and approximate errors, Phys. Rev. A **91**, 022335 (2015).
[3] M. Gutiérrez, C. Smith, L. Lulushi, S. Janardan, and K. R. Brown, Errors and pseudothresholds for incoherent and coherent noise, Phys. Rev. A **94**, 042338 (2016).
[4] A. S. Darmawan and D. Poulin, Tensor-Network Simulations of the Surface Code under Realistic Noise, Phys. Rev. Lett. **119**, 040502 (2017).
[5] S. Bravyi, M. Englbrecht, R. Koenig, and N. Peard, Correcting coherent errors with surface codes, arXiv:1710.02270.
[6] B. Rahn, A. C. Doherty, and H. Mabuchi, Exact performance of concatenated quantum codes, Phys. Rev. A **66**, 032304 (2002).
[7] J. Fern, J. Kempe, S. Simic, and S. Sastry, Generalized performance of concatenated quantum codes—A dynamical systems approach, IEEE Trans. Autom. Control **51**, 448 (2006).
[8] C. Chamberland, J. J. Wallman, S. Beale, and R. Laflamme, Hard decoding algorithm for optimizing thresholds under general Markovian noise, Phys. Rev. A **95**, 042332 (2017).
[9] J. Emerson, R. Alicki, and K. Życzkowski, Scalable noise estimation with random unitary operators, J. Opt. B **7**, S347 (2005).
[10] J. Emerson, M. Silva, O. Moussa, C. A. Ryan, M. Laforest, J. Baugh, D. G. Cory, and R. Laflamme, Symmetrized characterization of noisy quantum processes, Science **317**, 1893 (2007).
[11] C. Dankert, R. Cleve, J. Emerson, and E. Livine, Exact and approximate unitary 2-designs and their application to fidelity estimation, Phys. Rev. A **80**, 012304 (2009).
[12] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. B. Blakestad, J. D. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland, Randomized benchmarking of quantum gates, Phys. Rev. A **77**, 012307 (2008).
[13] E. Magesan, J. M. Gambetta, and J. Emerson, Scalable and Robust Randomized Benchmarking of Quantum Processes, Phys. Rev. Lett. **106**, 180504 (2011).
[14] A. Kitaev, Quantum computations: Algorithms and error correction, Russ. Math. Surv. **52**, 1191 (1997).
[15] S. Beigi and R. König, Simplified instantaneous non-local quantum computation with applications to position-based cryptography, New J. Phys. **13**, 093036 (2011).
[16] J. J. Wallman and S. T. Flammia, Randomized benchmarking with confidence, New J. Phys. **16**, 103032 (2014).
[17] Y. R. Sanders, J. J. Wallman, and B. C. Sanders, Bounding quantum gate error rate based on reported average fidelity, New J. Phys. **18**, 012002 (2015).
[18] R. Kueng, D. M. Long, A. C. Doherty, and S. T. Flammia, Comparing Experiments to the Fault-Tolerance Threshold, Phys. Rev. Lett. **117**, 170502 (2016).
[19] J. J. Wallman and J. Emerson, Noise tailoring for scalable quantum computation via randomized compiling, Phys. Rev. A **94**, 052325 (2016).
[20] P. S. Iyer and D. Poulin, A small computer is needed to optimize fault-tolerant protocols, arXiv:1711.04736.
[21] D. Greenbaum and Z. Dutton, Modeling coherent errors in quantum error correction, Quantum Sci. Technol. **3**, 015007 (2018).
[22] E. Huang, A. C. Doherty, and S. T. Flammia, Performance of quantum error correction with coherent errors, arXiv:1805.08227.
[23] S. Kimmel, M. P. da Silva, C. A. Ryan, B. R. Johnson, and T. A. Ohki, Robust Extraction of Tomographic Information via Randomized Benchmarking, Phys. Rev. X **4**, 011050 (2014).

[24] M. B. Ruskai, S. J Szarek, and E. Werner, An analysis of completely-positive trace-preserving maps on M_2, Linear Algebra Appl. **347**, 159 (2002).

[25] J. J. Wallman, C. Granade, R. Harper, and S. T. Flammia, Estimating the coherence of noise, New J. Phys. **17**, 113020 (2015).

[26] D. Gottesman, An introduction to quantum error correction and fault-tolerant quantum computation, arXiv: 0904.2557.

[27] D. Poulin, Optimal and efficient decoding of concatenated quantum block codes, Phys. Rev. A **74**, 052333 (2006).