Exact Persistence Exponent for the 2D-Diffusion Equation and Related Kac Polynomials

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We compute the persistence for the 2D-diffusion equation with random initial condition, i.e., the probability $p_0(t)$ that the diffusion field, at a given point $\mathbf{x}$ in the plane, has not changed sign up to time t. For large t, we show that $p_0(t) \sim t^{-\theta(2)}$ with $\theta(2) = 3/16$. Using the connection between the 2D-diffusion equation and Kac random polynomials, we show that the probability $q_0(n)$ that Kac's polynomials, of (even) degree n, have no real root decays, for large n, as $q_0(n) \sim n^{-3/4}$. We obtain this result by using yet another connection with the truncated orthogonal ensemble of random matrices. This allows us to compute various properties of the zero crossings of the diffusing field, equivalently of the real roots of Kac's polynomials. Finally, we unveil a precise connection with a fourth model: the semi-infinite Ising spin chain with Glauber dynamics at zero temperature.

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Persistence and first-passage properties have attracted a lot of interest during the last decades in physics, both theoretically [1–3] and experimentally [4–7], as well as in mathematics [8]. For a stochastic process $X(t)$, the persistence $P_0(t)$ is the probability that it has not changed sign up to time t. In nonequilibrium statistical physics, this is an interesting observable which is nonlocal in time and thus carries useful information on the full history of the system on a given time interval [9].

In many physically relevant situations, $P_0(t)$ decays algebraically at late time $t \gg 1$, $P_0(t) \sim t^{-\theta}$, where θ is called the *persistence exponent* [1–3]. For instance, for Brownian motion or Lévy flights, which are Markov processes, $\theta = 1/2$. But in many cases, in particular, for coarsening dynamics [10], and more generally for non-Markov processes, the exponent θ is nontrivial and extremely hard to compute [1–3]. Consequently, there are very few non-Markov processes, for which θ is known exactly. One notable example is the 1D Ising chain with Glauber dynamics. In this case, at temperature $T = 0$, the persistence exponent for the local magnetization can be computed exactly, yielding $\theta_{\text{Ising}} = 3/8$ [11,12].

Another example that has attracted a lot of attention [13–19] is the d-dimensional diffusion equation where the scalar field $\phi(\mathbf{x}, t)$ at point $\mathbf{x} \in \mathbb{R}^d$ and time t evolves as $\partial_t \phi(\mathbf{x}, t) = \Delta \phi(\mathbf{x}, t)$, where initially $\phi(\mathbf{x}, t = 0)$ is a Gaussian random field, with zero mean and short-range correlations $\langle \phi(\mathbf{x}, 0) \phi(\mathbf{x}', 0) \rangle = \delta^d(\mathbf{x} - \mathbf{x}')$. For a system of linear size L, the persistence $p_0(t, L)$ is the probability that $\phi(\mathbf{x}, t)$, at some fixed point $\mathbf{x}$ in space, does not change sign up to time t [13,14]. We assume that $\mathbf{x}$ is far enough from the boundary, where the system is invariant under translations, and $p_0(t, L)$ is thus independent of $\mathbf{x}$. It was shown [13,14] that $p_0(t, L)$ takes the scaling form, for large t and large L, with t/L^2 fixed

$$p_0(t, L) \sim L^{-2\theta(d)} h(L^2/t), \tag{1}$$

with $h(u) \to c_1$, a constant, when $u \to 0$ and $h(u) \propto u^{\theta(d)}$ when $u \to \infty$, where $\theta(d)$ was found, numerically, to be nontrivial, e.g., $\theta(1) = 0.120\,50(5)\ldots$, $\theta(2) = 0.187\,5(1)\ldots$ [13,14,16]. Remarkably, the persistence for $d = 1$ was observed in experiments on magnetization of spin polarized Xe gas and the exponent $\theta_{\exp}(1) \simeq 0.12$ was measured [7]. This scaling form (1) shows that $p_0(t, L) \sim t^{-\theta(d)}$ for an infinite system. Alternatively, $\theta(d)$ can also be obtained, in a finite system of size L, from $p_0(t, L) \sim L^{-2\theta(d)}$ for $t \gg L^2$. To study $p_0(t, L)$ it is useful to introduce the normalized process $X(t) = \phi(\mathbf{0}, t)/\langle \phi(\mathbf{0}, t)^2 \rangle$ [20]. Being Gaussian, $X(t)$ is completely characterized by its autocorrelation function which, for an infinite system $L \to \infty$, behaves like $C(t, t') = \langle X(t) X(t') \rangle = [2\sqrt{tt'}/(t + t')]^{d/2}$. In terms of logarithmic time $T = \ln t$, $Y(T) = X(e^T)$ is a Gaussian stationary process with covariance $c(T) = [\text{sech}(T/2)]^{d/2}$ (see Fig. 1). In particular, $c(T) \approx 1 - dT^2/16$ for $T \to 0$, indicating a smooth process with a finite density of zero crossings $\rho_0 = (2\pi)^{-1} \sqrt{d/2}$ [21]. Although several very accurate approximation schemes exist to compute $\theta(d)$ [13–15,22,23], there is not a single value of d for which this persistence exponent could be computed exactly. In this Letter, we focus on the case $d = 2$. As we show below, this case is particularly interesting because it is related to a variety of other interesting models (see Fig. 1), in particular to the celebrated Kac's random polynomials [17–19,24]. These are polynomials of degree n

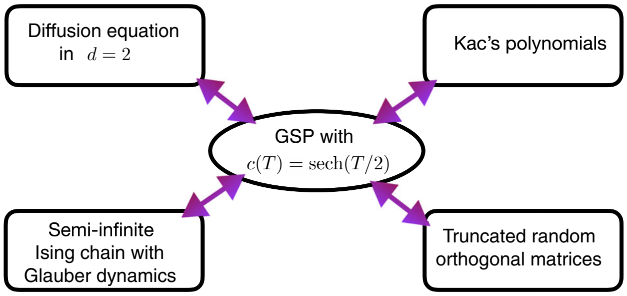

FIG. 1. Connections between the four models studied here: they are related to the same Gaussian stationary process (GSP) with correlator $c(T) = \mathrm{sech}(T/2)$. Our main result is the exact value of the persistence exponent for this GSP, $\theta(2) = b = 3/16$, together with the full statistics of its zero crossings (5)–(7).

$$K_n(x) = \sum_{i=0}^{n} a_i x^i, \tag{2}$$

where the coefficients a_i's are independent and identically distributed (i.i.d.) real Gaussian random variables of zero mean and unit variance. Of course, $K_n(x)$ has n roots in the complex plane. These roots tend to cluster, when $n \to \infty$, close to the unit circle centered at 0. But because the coefficients a_i's are real, the statistics of the number of real roots is singular, and it has thus generated a lot of interest [25,26]. In particular, the average number of real roots grows, for $n \gg 1$, like $\sim(2/\pi)\ln n$, hence much smaller than n. It is thus natural to ask: what is the probability $q_0(n)$ that $K_n(x)$ has no real roots for an even n? It was shown in Ref. [24] that $q_0(n)$ decays to zero as $q_0(n) \sim n^{-4b}$ where, remarkably, b turns out to be the persistence exponent for the diffusion equation in $d = 2$ [17,18], i.e., $b = \theta(2)$.

To establish the connection between these two problems, one first notices that almost all the real roots of $K_n(x)$ lie very close to ± 1, in a window of size $\mathcal{O}(1/n)$ [27]. In addition, it was shown in Ref. [24] (see also Ref. [28]) that, for large n, the real roots of $K_n(x)$ behave independently and identically within each of the four subintervals $(-\infty, -1]$, $[-1, 0]$, $[0, 1]$, and $[1, +\infty)$. One can thus focus on one of these intervals, say [0, 1], and consider $\tilde{q}_0(x, n)$, which is the probability that K_n has no real root in $[0, x]$, with $0 < x \le 1$. Clearly, $q_0(n) \sim [\tilde{q}_0(1, n)]^4$ for large n. For $x \to 1^-$, it was shown in Refs. [17–19,24] that the behavior of $\tilde{q}_0(x, n)$ is governed by the zero-crossings properties of the GSP $Y(T)$ with covariance $c(T) = \mathrm{sech}(T/2)$, i.e., the same GSP that governs the zero crossings of the 2*D*-diffusion equation (see Fig. 1). In particular, in the scaling limit $n \to \infty$, $x \to 1^-$ with $n(1 - x)$ fixed [recall that the scaling region around ± 1 is of order $\mathcal{O}(1/n)$], one can show that $\tilde{q}_0(x, n)$ takes the scaling form [17,18],

$$\tilde{q}_0(x, n) \sim n^{-b} \tilde{h}[n(1 - x)], \tag{3}$$

with $\tilde{h}(u) \to c_2$, a constant, for $u \to 0$ and $\tilde{h}(u) \sim u^b$ for $u \to \infty$. The large u behavior follows from the fact that $\tilde{q}_0(x, n \to \infty)$ is well defined. This form (3) is the exact analogue of the finite size scaling form in Eq. (1), with n playing the role of L^2 and $(1 - x)$ the role of inverse time $1/t$ [17,18]. This implies that $b = \theta(2)$ can be extracted either for finite n, from $\tilde{q}_0(1, n) \sim n^{-b}$, or for $n \to \infty$ (i.e., for the Gaussian power series) from $\tilde{q}_0(x, n \to \infty) \sim (1 - x)^b$, as $x \to 1^-$. The study of this exponent b has generated a lot of interest in the math literature [8,19,24,29–31] and the best existing bounds are $0.144\,38\ldots = 1/(4\sqrt{3}) \le b \le 1/4 = 0.25$ [29,31].

Main results.—Here, we exploit a connection between the Kac's polynomials and the so-called truncated real orthogonal ensemble of random matrices [32–34] (see below) to obtain the exact result

$$b = \theta(2) = 3/16 = 0.1875, \tag{4}$$

which is fully consistent with numerical simulations [13,14,16] and the above exact bounds [29,31] as well as with a recent conjecture in number theory [35]. We also compute the full probability distribution of the number of zero crossings N_t of $\phi(\mathbf{0}, t)$ up to time t. Let $p_k(t, L) = \mathrm{Prob}(N_t = k)$ and $p_k(t) = p_k(t, L \to \infty)$. We show that, for large t and k, with $k/\ln t$ fixed, $p_k(t)$ takes the large deviation form proposed in Refs. [17,18],

$$p_k(t) \sim t^{-\varphi(k/\ln t)}, \tag{5}$$

where the large deviation function $\varphi(x)$ is computed exactly. Its asymptotic behaviors are given by

$$\varphi(x) \sim \begin{cases} \frac{3}{16} + x\ln x, & x \to 0 \\ \frac{1}{2\sigma^2}(x - \frac{1}{2\pi})^2, & |x - \frac{1}{2\pi}| \ll 1 \\ \frac{\pi^2}{4}x^2 - \frac{\ln 2}{2}x, & x \to \infty, \end{cases} \tag{6}$$

with $\sigma^2 = 1/\pi - 2/\pi^2$. Close to the center, for $x = 1/(2\pi)$, the quadratic behavior in Eq. (6) shows that $p_k(t)$ has a Gaussian peak, of width $\sigma \ln(t)$, close to its maximum $\langle N_t \rangle \approx \ln(t)/(2\pi)$. However, away from this central Gaussian regime, $p_k(t)$ is flanked, on both sides of $\langle N_t \rangle$, by nontrivial tails (6)—the right one being, however, still Gaussian (at leading order), though different from the Gaussian central part. Finally, we also obtain the large t behavior of the cumulants $\langle N_t^p \rangle_c$ of arbitrary order p

$$\langle N_t^p \rangle_c \sim \kappa_p \ln t, \quad \kappa_p = \frac{2^{p-2}}{\pi^2} \sum_{m=1}^{p} (-2)^{m-1} \Gamma^2\left(\frac{m}{2}\right) \mathcal{S}_p^{(m)}, \tag{7}$$

where $\mathcal{S}_p^{(m)}$ is the Stirling number of the second kind [36]. In particular, one recovers $\kappa_1 = 1/(2\pi)$ and $\kappa_2 = \sigma^2 = 1/\pi - 2/\pi^2$ (see Ref. [22]) and obtains, for instance,

$\kappa_3 = 4/\pi - 12/\pi^2$. Our main results in Eqs. (4), (6), and (7) are not only relevant for the $2D$-diffusion equation, but also for the whole class of models discussed in this Letter that can me mapped onto the GSP, $Y(T)$, with correlator $c(T) = \mathrm{sech}(T/2)$ (see Fig. 1). In particular, the probability $\mathcal{P}_k(T)$ that it has exactly k zeros up to T is given, for large T and $k = \mathcal{O}(T)$, by $\mathcal{P}_k(T) \sim e^{-T\varphi(k/T)}$, with the same function $\varphi(x)$ (6). Similarly, the cumulants of the number of zero crossings are given by Eq. (7), with the substitution $\ln t \to T$ and the same coefficients κ_p. We further show that this GSP has a Pfaffian structure: the multitime correlation functions of $\mathrm{sgn}[Y(T)]$ can be written as Pfaffians [37]. Besides, we demonstrate that the zeros of $Y(T)$ form a Pfaffian point process [37]. Finally, we establish an exact mapping between the $2D$-diffusing field and the semi-infinite Ising spin chain with Glauber dynamics at zero temperature. As we will see, using the exact result for the persistence exponent of the full chain, $\theta_{\mathrm{Ising}} = 3/8$ [11,12], this connection provides an alternative derivation of the exact result $\theta(2) = b = \theta_{\mathrm{Ising}}/2 = 3/16$.

Truncated random orthogonal matrices.—We consider the set of real orthogonal matrices, of size $(2n+1) \times (2n+1)$ (with n a positive integer), uniformly distributed, with the Haar measure, on the orthogonal group $O(2n+1)$. Let O be such a real random orthogonal matrix, such that $OO^T = \mathbb{I}$. We define its truncation M_{2n} as the $2n \times 2n$ random matrix obtained by removing the last column and row from the matrix O

$$O = \begin{pmatrix} M_{2n} & \mathbf{u} \\ \mathbf{v}^T & a \end{pmatrix}, \tag{8}$$

where $\mathbf{u}$, $\mathbf{v}$ are column vectors and a is a scalar. Such truncated matrices, together with their unitary counterpart, were studied in the context of mesoscopic physics [34,42] and extreme statistics [43]. The orthogonality condition $OO^T = \mathbb{I}$ implies that $M_{2n}M_{2n}^T = \mathbb{I} - \mathbf{u}\mathbf{u}^T$ and, hence, all the eigenvalues of M_{2n} lie in the unit disk (since their norm is less than unity). They are the roots of the characteristic polynomial $g_M(z) = \det(z\mathbb{I} - M_{2n})$, which after some manipulations, can be written as [44] [Lemma 6.7.2] (see also [37])

$$g_M(z) = \det O \det(zM_{2n} - \mathbb{I})(a + z\mathbf{v}^T(\mathbb{I} - zM_{2n})^{-1}\mathbf{u}). \tag{9}$$

Since the eigenvalues z_i's of M_{2n} are such that $0 < |z_i| < 1$, one has necessarily that $\det(z_i M_{2n} - \mathbb{I}) = z_i^N g_M(1/z_i) \neq 0$ in the right-hand side of Eq. (9). This implies that the z_i's are the zeros of $[a + z\mathbf{v}^T(\mathbb{I} - zM_{2n})^{-1}\mathbf{u}]$ [see Eq. (9)]. Expanding in powers of z shows that the eigenvalues of M_{2n} are the zeroes of the series

$$F_{2n}(z) = a + \sum_{k=1}^{\infty} z^k \mathbf{v}^T M_{2n}^{k-1}\mathbf{u}, \tag{10}$$

with $|z| < 1$ (note that $\mathbf{v}^T M_{2n}^{k-1}\mathbf{u}$ are real numbers). Quite remarkably, one can show [44] that the scaled sequence of the real coefficients of the series in Eq. (10), i.e., $\sqrt{2n}\{a, \mathbf{v}^T\mathbf{u}, \mathbf{v}^T M_{2n}\mathbf{u}, \mathbf{v}^T M_{2n}^2\mathbf{u}, \ldots\}$, converges, as $n \to \infty$, to a sequence of i.i.d. Gaussian random variables, with zero mean and unit variance. This implies that, for $n \to \infty$, the real eigenvalues of M_{2n} in Eq. (8) and the real zeroes of $K_n(x)$ in Eq. (2) in the interval $[-1, 1]$ share the same statistics.

But what about the connection between these two models for finite n? In fact, it is known that the real eigenvalues of M_{2n} accumulate close to $x = \pm 1$, also on a window of size $\mathcal{O}(1/n)$ [34], like for the Kac's polynomials [27]. Hence, if one considers the probability $\tilde{Q}_0(x, n)$ that M_{2n} has no real eigenvalue in $[0, x]$, it is natural to expect that, as for Kac's polynomials (3), for large n and $x \to 1$ keeping $n(1-x)$ fixed, $\tilde{Q}_0(x, n)$ behaves as

$$\tilde{Q}_0(x, n) \sim n^{-\gamma}\tilde{H}(n(1-x)), \tag{11}$$

where the exponent γ is yet unknown and, *a priori*, the scaling function $\tilde{H}(u)$ is different for $\tilde{h}(u)$ in Eq. (3). However, for $n \to \infty$, we have seen that $\tilde{q}_0(x, n)$ and $\tilde{Q}_0(x, n)$ do coincide, since they both correspond to the probability that the (infinite) Gaussian power series has no real root in $[0, x]$. This implies that $\tilde{Q}_0(x, n \to \infty) = \tilde{q}_0(x, n \to \infty) \sim (1-x)^b$, which, together with the scaling form (11), shows that $\gamma = b$. Finally, since we expect that $\tilde{Q}_0(1, n)$ exists, one has $\tilde{H}(u) \to c_3$, a constant, when $u \to 0$, and therefore $\tilde{Q}_0(1, n) \sim n^{-b}$ for large n. One can also consider the probability $Q_0(x, n)$ that M_{2n} has no real eigenvalue in $[-x, x]$. Using the statistical independence of the positive and negative real eigenvalues for large n, one has $Q_0(x, n) \sim [\tilde{Q}_0(x, n)]^2$, and in particular $Q_0(1, n) \sim n^{-2b}$ for large n. Using similar arguments, one can show that the full statistics of the zero-crossings of the diffusion equation (equivalently of the real roots of $K_n(x)$) can be obtained, at leading order for large n, from the statistics of the number of real eigenvalues $\mathcal{N}_n$ of the random matrix M_{2n}, which we now study. Our analysis follows the line developed in Ref. [45] where the real eigenvalues of real Ginibre matrices were studied.

We start with the full joint distribution of the eigenvalues of M_{2n} (8). Since M_{2n} is real and of even size $2n$, it has l (with l even) real eigenvalues (and possibly $l = 0$), denoted by $\lambda_1 \leq \ldots \leq \lambda_l$, and $m = n - l/2$ pairs of complex conjugate eigenvalues $z_1 = x_1 + iy_1, z_2 = \overline{z_1}, \ldots, z_{2m-1} = x_m + iy_m, z_{2m} = \overline{z_{2m-1}}$ with $x_1 \leq \ldots \leq x_m$. Then the ordered eigenvalues of M_{2n} conditioned to have l real eigenvalues have the joint distribution [33,34]

$$p^{(l,m)}(\vec{\lambda}, \vec{z}) = C|\Delta(\vec{\lambda}, \vec{z})| \prod_{j=1}^{l} w(\lambda_j) \prod_{j=1}^{2m} w(z_j), \tag{12}$$

where $C \equiv C_{m,n}$ is a normalization constant,

$$w^2(z) = (2\pi|1-z^2|)^{-1}, \tag{13}$$

and Δ is a Vandermonde determinant. The generating function (GF) of $\mathcal{N}_n$ (the number of real roots) reads

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} = \left\langle \prod_{i=1}^{2n} 1-(1-e^s)\chi_{\mathbb{R}}(\zeta_i)\right\rangle_{M_{2n}}, \tag{14}$$

for $s<0$, where the product runs over all the eigenvalues ζ_i's (both real and complex) of M_{2n}. In Eq. (14), $\chi_{\mathbb{R}}(z)=1$ if z is real and 0 otherwise and $\langle\cdots\rangle_{M_{2n}}$ denotes an average over the joint distribution (12), further summed over all possible (l,m) [46,47]. It turns out that such averages (14) can be computed explicitly in terms of Pfaffians [46–52], as follows. Let $f(z)$ be any smooth integrable complex function, and $\{p_j(z)\}_{j=0}^{2n-1}$ be an arbitrary sequence of monic polynomials of degree j, then

$$\left\langle \prod_{i=1}^{2n} f(\zeta_i)\right\rangle_{M_{2n}} = \frac{\mathrm{Pf}(U_f)}{\mathrm{Pf}(U_1)}, \tag{15}$$

where Pf denotes a Pfaffian [53] and U_f is a skew symmetric (i.e., antisymmetric) matrix of size $2n\times 2n$ with entries $u_{j,k}=(p_{j-1}f,p_{k-1}f)_w$ and skew product

$$(h,g)_w = \int_{\mathbb{R}^2} h(x)g(y)\mathrm{sgn}(y-x)w(x)w(y)dxdy + 2i\int_{\mathbb{C}} h(z)g(\bar{z})\mathrm{sgn}[\mathrm{Im}(z)]w(z)w(\bar{z})d^2z, \tag{16}$$

where $w(z)$ is given in Eq. (13). To compute the ratio in Eq. (15), it is convenient to choose the monic polynomials $p_j(z)$ to be skew orthogonal with respect to the product (16) [with this choice, the denominator in Eq. (15) is easy to compute [37]]. Using these polynomials [33], the GF in Eq. (14) can be evaluated explicitly using Eq. (15), leading to [54] (see also [55])

$$\langle e^{s\mathcal{N}_n}\rangle_{M_N} = \det_{0\le j,k\le n-1}\left(\delta_{j,k} - \frac{1-e^{2s}}{\pi(j+k+1/2)}\right). \tag{17}$$

Let us denote by H_n the $n\times n$ matrix with entries $h_{j,k}=[\pi(j+k+1/2)]^{-1}$. We write the determinant in Eq. (17) as $\det(\mathbb{I}-\alpha H_n)=\exp\mathrm{Tr}[\ln(\mathbb{I}-\alpha H_n)]$, with $\alpha=(1-e^{2s})$ and then expand the logarithm, to get $\det(\mathbb{I}-\alpha H_n)=\exp[\sum_{m\ge 1}(\alpha^m/m)\mathrm{Tr}(H_n^m)]$. The asymptotic analysis of the traces yields [56]

$$\mathrm{Tr}H_n^m = \frac{1}{2\pi}\int_0^\infty \mathrm{sech}^m\left(\frac{\pi u}{2}\right)du\log n[1+o(1)], \quad n\to\infty.$$

By summing up these traces, we obtain

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} = n^{[1/(2\pi)]\int_0^\infty \log\{1-(1-e^{2s})\mathrm{sech}[(\pi u)/2]\}du+o(1)}. \tag{18}$$

For $s<0$, the integral can be calculated explicitly as

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} \sim n^{\psi(s)},$$
$$\psi(s) = \frac{1}{8} - \left[\frac{\sqrt{2}}{\pi}\cos^{-1}\left(\frac{e^s}{\sqrt{2}}\right)\right]^2. \tag{19}$$

By taking $s\to-\infty$ we get the probability that M_{2n} has no real eigenvalues, using $Q_0(1,n)=\mathrm{Prob}(\mathcal{N}_n=0)=\lim_{s\to-\infty}\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}}\sim n^{-2b}$. From Eq. (19), we thus obtain $b=(-1/2)\lim_{s\to-\infty}\psi(s)=3/16$, as announced in Eq. (4). From the GF in Eq. (19), we also obtain the cumulants of $\mathcal{N}_n$. To export these results to the $2D$-diffusion equation, we recall that the number of zero crossings N_t identifies with the *positive* real eigenvalues $\mathcal{N}_n^+$ of M_{2n}. Hence, for $n\gg 1$, the number of positive and negative $\mathcal{N}_n^\pm$ real eigenvalues are both independent and identically distributed [54], one obtains that $\langle e^{s\mathcal{N}_n^+}\rangle_{M_{2n}}\sim n^{\psi(s)/2}$. By further expanding $\psi(s)$ close to $s=0$ [37], one finally obtains the result announced in Eq. (7). Similarly, transposing this result $\langle e^{s\mathcal{N}_n^+}\rangle_{M_{2n}}\sim n^{\psi(s)/2}$ to the diffusion equation, one obtains the large deviation form in Eq. (5) with $\varphi(x)=\max_{s\in\mathbb{R}}[sx-\psi(s)/2]$ [57]. From this relation, together with the expression for $\psi(s)$ in Eq. (19), we obtain the asymptotic behaviours given in Eq. (6) [37].

Several results found so far point to an intriguing connection with the zero temperature Glauber dynamics of the Ising spin chain [11,12]. First, $b=3/16$ is thus half of the persistence exponent, $\theta_{\mathrm{Ising}}=3/8$, found there [11,12]. In fact, 3/16 is exactly the persistence exponent corresponding to the spin at the origin of the semi-infinite Ising chain [11,12]. Furthermore, the expression found for $\psi(s)$ in Eq. (19) is strongly reminiscent of the expression found for the persistence exponent for the q-state Potts chain, with $T=0$ Glauber dynamics [see, e.g., Eq. (2) of Ref. [11]]. So what is this connection?.

To understand it, let us come back to the $2D$-diffusion field $X(t)=\phi(\mathbf{0},t)/\langle\phi(\mathbf{0},t)^2\rangle$ and consider the "clipped" process $\mathrm{sgn}[X(t)]$ [58]. As recalled above $X(t)$ has the same statistical properties as the Kac's polynomials $K_n(x)$ in the limit $n\to\infty$ and $x\to 1$. Transposing recent results obtained for Kac's polynomials in the limit $n\to\infty$ [59], we can compute the multitime correlation functions of $\mathrm{sgn}[X(t)]$, which are given by Pfaffians [53]

$$\langle \mathrm{sgn}[X(t_1)]\cdots\mathrm{sgn}[X(t_{2m})]\rangle \sim \mathrm{Pf}(A) \tag{20}$$

for $1 \ll t_1 \ll t_2 \ll \cdots \ll t_{2m}$, and where $A = (a_{i,j})_{1\le i,j\le 2m}$ is a $2m \times 2m$ antisymmetric matrix with $a_{i,i} = 0$ and for $i < j$, $a_{i,j} = -a_{j,i}$ where

$$a_{i,j} = \frac{2}{\pi}\sin^{-1}[\langle X(t_i)X(t_j)\rangle] = \frac{2}{\pi}\sin^{-1}\left(\frac{2\sqrt{t_i t_j}}{t_i + t_j}\right). \quad (21)$$

By symmetry, the even correlation functions vanish. For $m = 1$, Eqs. (20) and (21) hold for any normalized Gaussian process. However, for $m > 1$, this Pfaffian structure, which holds for the GSP $Y(T) = X(e^T)$, is nontrivial.

Let us now consider the semi-infinite Ising spin chain, whose configuration at time t is given by $\{\sigma_i(t)\}_{i\ge 0}$, with $\sigma_i(t) = \pm 1$. Initially, $\sigma_i(0) = \pm 1$ with equal probability $1/2$ and, at subsequent time, the system evolves according to the Glauber dynamics at $T = 0$ (see Refs. [11,12] for details). Using the formulation of the dynamics in terms of coalescing random walks [11,12], we show that the multitime correlation functions of σ_0 are also given by the same Pfaffian formula (20), namely, for $1 \ll t_1 \ll t_2 \ll \ldots \ll t_{2m}$ [37]

$$\langle \sigma_0(t_1)\ldots\sigma_0(t_{2m})\rangle \sim \mathrm{Pf}(A), \quad (22)$$

with precisely the same antisymmetric matrix A (21). Therefore, we conclude that $\mathrm{sgn}[X(t)]$ for the $2D$-diffusion equation and $\sigma_0(t)$ in the semi-infinite Ising chain with Glauber dynamics are actually the same process in the large time limit [60]. One can then use the known result for the persistence of $\sigma_0(t)$ [11,12] to conclude that $b = 3/16$, as found above by a completely different method. Note that the exact relation found here $\sigma_0(t) \propto \mathrm{sgn}[\phi(\mathbf{0}, t)]$ (for $t \gg 1$), where $\phi(\mathbf{x}, t)$ is the $2D$-diffusing field, is reminiscent, albeit different from, the so-called OJK approximate theory [61] in phase ordering kinetics [62], which instead approximates the $1D$ spin field by the sign of the $1D$ diffusing field.

To conclude, we have computed exactly the persistence exponent of $2D$-diffusion equation, or equivalently the one of Kac's polynomials, $\theta(2) = b = 3/16$. This was done in two different ways: (i) by using the connection to truncated random orthogonal matrices, and for which our results are actually mathematically rigorous [54], (ii) by establishing an exact mapping to the semi-infinite Ising chain with Glauber dynamics at $T = 0$ (see Fig. 1). Thanks to (i), we computed the full statistics of the number of the zero crossings (5)–(7). These RMT tools will certainly be useful to compute other properties of the GSP with correlator $c(T) = \mathrm{sech}(T/2)$ and of the different physical models associated with it (see Fig. 1).

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