Quantization of the Thermal Hall Conductivity at Small Hall Angles

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We consider the effect of coupling between phonons and a chiral Majorana edge in a gapped chiral spin liquid with Ising anyons (e.g., Kitaev's non-Abelian spin liquid on the honeycomb lattice). This is especially important in the regime in which the longitudinal bulk heat conductivity κ_{xx} due to phonons is much larger than the expected quantized thermal Hall conductance $\kappa_{xy}^{q} = (\pi T/12)(k_B^2/\hbar)$ of the ideal isolated edge mode, so that the thermal Hall angle, i.e., the angle between the thermal current and the temperature gradient, is small. By modeling the interaction between a Majorana edge and bulk phonons, we show that the exchange of energy between the two subsystems leads to a transverse component of the bulk current and thereby an *effective* Hall conductivity. Remarkably, the latter is equal to the quantized value when the edge and bulk can thermalize, which occurs for a Hall bar of length $L \gg \ell$, where ℓ is a thermalization length. We obtain $\ell \sim T^{-5}$ for a model of the Majorana-phonon coupling. We also find that the quality of the quantization depends on the means of measuring the temperature and, surprisingly, a more robust quantization is obtained when the lattice, not the spin, temperature is measured. We present general hydrodynamic equations for the system, detailed results for the temperature and current profiles, and an estimate for the coupling strength and its temperature dependence based on a microscopic model Hamiltonian. Our results may explain recent experiments observing a quantized thermal Hall conductivity in the regime of small Hall angle, $\kappa_{xy}/\kappa_{xx} \sim 10^{-3}$, in α-$RuCl_3$.

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Non-Abelian statistics is a deep generalization of quantum statistics in two dimensions, in which the final state depends upon the order in which exchanges of particles—non-Abelian anyons—are performed [1–3]. In addition to its fundamental interest, this provides a powerful paradigm for quantum computing, allowing for fault-tolerant processes [4,5]. The main platforms in which non-Abelian topological phases have been sought are the $\nu = 5/2$ fractional quantum Hall effect [1,2], where non-Abelian anyons are suspected but have not been established, and hybrid semiconductor-superconductor structures, to which quantum computing groups are devoting massive efforts [6], but where confirmation is still awaited.

A third possible route to non-Abelian anyons is via a quantum spin liquid [7]. In his seminal work [8], Kitaev presented a spin-1/2 model on the honeycomb lattice with bond-dependent anisotropy which, in a magnetic field, realizes a non-Abelian topological phase. This phase hosts *Ising anyons*, topologically the same anyon type which is targeted by the hybrid efforts. A key and general characteristic of a topological phase is the *chiral central charge c*, which characterizes its gapless edge modes. It is directly measurable as a quantized thermal Hall conductivity, $\kappa_{xy}^{q} = \pi c T/6$ ($\hbar = k_B = 1$). A noninteger value is an unambiguous indicator of a non-Abelian phase, and $c = 1/2$ for Ising anyons.

Stimulated by the recognition that Kitaev's anisotropic interactions arise naturally in certain strongly spin-orbit coupled Mott insulators [9,10], mounting efforts have targeted such systems in the laboratory. There is now strong evidence that Kitaev interactions are substantial in several 2D honeycomb lattice materials [11]: α-Na_2IrO_3 [12], α-Li_2IrO_3 [13], and α-$RuCl_3$ [14]. While it is clear that none of these materials are exactly described by Kitaev's model, the beauty of a topological phase is its robustness: once obtained, it is stable to an arbitrary weak perturbation and its essential properties are completely independent of the details of the Hamiltonian. A very recent experiment [15] presents observations of an apparent plateau with a quantized thermal Hall conductivity with $c = 1/2$ in α-$RuCl_3$ in an applied field of 9–10 T, at temperatures of 3–5 K. If confirmed, it could be a revolutionary discovery not only in the non-Abelian context, but also as the first truly unambiguous signature of a quantum spin liquid phase in experiment. These results appear to complement recent experiments on quantum Hall systems which have observed half-integer thermal conductance, but through rather different means [16].

The α-$RuCl_3$ experiments do, however, present at least one major puzzle. The thermal *Hall angle* $\theta_H = \tan^{-1}(\kappa_{xy}/\kappa_{xx}) = 10^{-3}$ is small, i.e., $\kappa_{xx} \gg \kappa_{xy}$. This is incompatible with conduction solely through a Majorana edge mode. Indeed, in two-dimensional electron gases, a quantized Hall effect is only observed when the Hall angle is large. This raises the fundamental question of whether the thermal Hall effect is different: is quantization even expected and possible at small Hall angles? We consider here a universal effective model for an Ising anyon phase, in which the chiral Majorana edge mode is augmented by acoustic bulk phonons, which can provide a diagonal bulk thermal conductivity. Remarkably, we find that not only does the quantized thermal Hall effect persist in the presence of the phonons, but it *relies* upon them. The ultimate view of the quantized transport is distinctly different from the usual isolated edge mode picture, and we predict notable experimental consequences of the mixing of edge and bulk heat propagation. Our considerations are quite general and we expect that similar physics applies to thermal transport in other systems with edge modes, such as topological superconductors and quantum Hall systems.

We formulate the problem in terms of hydrodynamic equations describing the energy transport. We consider the following two subsystems: the phonons, or lattice, located in the bulk, and denoted with the index "ph", and the Majorana fermions, or spins, confined to the edge and indexed by "f", as well as a coupling between them. For simplicity, we assume an isotropic bulk, with the relation

$$\mathbf{j}_{\mathrm{ph}} = -\kappa \boldsymbol{\nabla} T_{\mathrm{ph}}; \tag{1}$$

i.e., the energy current density in the bulk is parallel to the thermal gradient, with κ a characteristic of the lattice. The clockwise edge current is that of a chiral fermion with central charge $c = 1/2$, i.e.,

$$I_f = \frac{\pi c T_f^2}{12}. \tag{2}$$

The heat exchange between phonons and Majorana fermions can be modeled phenomenologically through an energy current j_{ex} between the two subsystems (see the arrows in Fig. 1). Microscopically, it is due to the scattering events between edge Majorana fermions and bulk phonons, and is the rate of energy transfer at the edge per unit length, i.e., $j_{\mathrm{ex}} \equiv (1/L)(\partial\mathcal{E}/\partial t)_{\mathrm{ph}\to f} = -(1/L)(\partial\mathcal{E}/\partial t)_{f\to\mathrm{ph}}$, where L is the length of the edge in evaluating $(\partial\mathcal{E}/\partial t)_{\mathrm{ph}\to f}$ [17]. This, in turn, implies that the phonons and Majorana fermions have not fully thermalized with one another. Assuming, however, that thermalization is almost complete, i.e., $T_f \approx T_{\mathrm{ph}}$, and that the fermions are strictly confined to the edge, j_{ex} can be linearized in the temperature difference $T_{\mathrm{ph}} - T_f$ at the edge,

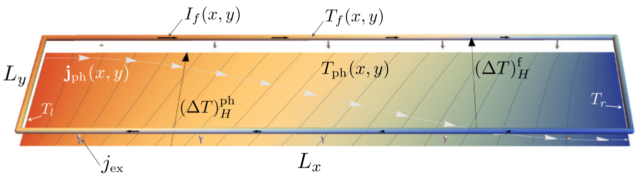

FIG. 1. Temperature maps of our rectangular system with dimensions L_x and L_y consisting of a phonon bulk (lower box) and a Majorana fermion edge (upper edge). The phonon temperatures at the left and right edges are assumed to be fixed as $T_{l,r}$, respectively, due to the coupling of the lattice with the heater and thermal bath. The black arrows for I_f along the edge denote the direction and magnitude of the "clockwise" energy current associated with the chiral Majorana mode. The white arrows in the bulk show a stream line of $\mathbf{j}_{\mathrm{ph}}$. The 3D white arrows for j_{ex} indicate the energy current between the Majorana edge and bulk phonons. $(\Delta T)^{\mathrm{ph}}_H$ and $(\Delta T)^{f}_H$ are the measured "Hall" temperature differences when the contacts are coupled to the lattice or spins, respectively.

$$j_{\mathrm{ex}} = \lambda(T)(T_{\mathrm{ph}} - T_f), \tag{3}$$

where, crucially, $\lambda > 0$ is a function of the overall constant temperature $T \approx T_{\mathrm{ph},f}$, and can be parametrized as $\lambda(T) \sim T^{\alpha}$. We will determine α from a phase space analysis of the scattering events.

Hydrodynamic equations.—We assume our (two-dimensional) system to be a rectangular slab of width L_y and length $L_x \gtrsim L_y$ (see Fig. 1), and choose coordinates with $|x| < x_0 = L_x/2$ and $|y| < y_0 = L_y/2$.

The continuity equation in the bulk in a steady state is $\boldsymbol{\nabla} \cdot \mathbf{j}_{\mathrm{ph}}(x,y) = 0$ which implies the Laplace equation

$$\nabla^2 T_{\mathrm{ph}}(x,y) = 0. \tag{4}$$

Energy conservation at the edges gives rise to appropriate boundary conditions. At the left and right edges, we assume that only the *lattice* is coupled to thermal leads and the phonons have fixed constant temperatures, $T_{l,r}$, respectively. At the top and bottom edges, the current out of the phonon subsystem must equal the exchange current, hence $\pm j^y_{\mathrm{ph}}(x, \pm y_0) = j_{\mathrm{ex}}(x, \pm y_0)$. Moreover, the continuity equations for the edges imply $\pm\partial_x I_f(x, \pm y_0) = j_{\mathrm{ex}}(x, \pm y_0)$. Together these yield, given Eqs. (1) and (2),

$$\kappa \partial_y T_{\mathrm{ph}}(x, \pm y_0) = -\kappa^q_{xy} \partial_x T_f(x, \pm y_0). \tag{5}$$

Note the appearance of the ideal quantized Hall conductivity $\kappa^q_{xy} = \pi c T/6 = \pi T/12$ here, using $T_f \approx T$ (valid within our linearized treatment).

Quantization in the infinitely long limit.—For simplicity, we first solve our hydrodynamic equations in the limit of an infinitely long system ($L_x \to \infty$). Note that, even for finite systems with $L_x \gg L_y$, this infinitely long limit is expected to be relevant far away from the left and right edges.

Since there is translation symmetry in the x direction, the boundary conditions $T_{\mathrm{ph}}(\pm x_0, y) = T_{r,l}$ lead to a uniform temperature gradient $(\overline{dT/dx}) = \lim_{L_x\to\infty}(T_r - T_l)/L_x$, and the phonon and Majorana temperatures must take the forms

$$T_{\mathrm{ph}}(x,y) = \frac{\overline{dT}}{dx}x + \hat{T}(y) + \mathrm{const},$$
$$T_f(x, \pm y_0) = \frac{\overline{dT}}{dx}x + \mathrm{const}. \tag{6}$$

Laplace's equation, Eq. (4), immediately implies that $\hat{T}(y)$ must be a linear function of y which we write $\hat{T}(y) = [(\Delta T)_H^{\mathrm{ph}}/L_y]y$. Therefore, from Eq. (5), we get

$$\partial_y T_{\mathrm{ph}}(x,y) = -\frac{\kappa_{xy}^q}{\kappa}\frac{\overline{dT}}{dx}, \tag{7}$$

since $\partial_y T_{\mathrm{ph}}(x,y) = \partial_y T_{\mathrm{ph}}(x, \pm y_0) = \mathrm{const}$. From a phenomenological perspective, the *total* current in the Hall bar geometry must flow only along x, but Eq. (7) implies that the phonon thermal gradient is tilted from the current axis by a small Hall angle of $|\tan\theta_H| = \kappa_{xy}^q/\kappa \ll 1$.

Next, consider the view of Alice the experimentalist. She measures the temperature gradients via three contacts, and assumes for the moment that these measurements give the phonon temperature (the most reasonable assumption). To deduce the Hall conductivity, she posits a bulk heat current satisfying $\boldsymbol{j} = -\boldsymbol{\kappa}^{\mathrm{ph,expt}}\cdot\boldsymbol{\nabla}T$, and tries to deduce the tensor $\boldsymbol{\kappa}^{\mathrm{ph,expt}}$ [the ph (f) superscript means this quantity is obtained from a measurement of the phonon (Majorana fermion) temperature]. By measuring the longitudinal temperature gradient, she obtains $\kappa_{xx}^{\mathrm{ph,expt}} = \kappa$ as expected, and then, imposing $j^y = 0$, she equates the experimental Hall angle $\tan\theta_H = \{[(\Delta T)_H^{\mathrm{ph}}]/L_y\}/(\overline{dT/dx})$ to $\kappa_{xy}^{\mathrm{ph,expt}}/\kappa_{xx}^{\mathrm{ph,expt}}$. By comparing this equation to the theoretical result in Eq. (7), we immediately recognize that the magnitude of the effective Hall conductivity (denoted simply as $\kappa_{xy}^{\mathrm{expt}}$ in the rest of the text) is $|\kappa_{xy}^{\mathrm{ph,expt}}| = \kappa_{xy}^q$; i.e., the experimentally measured thermal Hall conductivity takes the quantized value.

A few remarks are in order. First, a transverse temperature difference $(\Delta T)_H^{\mathrm{ph}}$ leading to a "Hall thermal gradient" $(\Delta T)_H^{\mathrm{ph}}/L_y = -(\kappa_{xy}^q/\kappa)(\overline{dT/dx})$ develops, which allows us to compensate the transverse energy current j_{ex} at the edges and leads to a zero net transverse current. Second, the effective thermal Hall conductivity is only found to be quantized if the transverse temperature gradient is obtained from the *phonon* temperatures at the top and bottom edges. In contrast, if Bob somehow measures the Majorana temperatures, the transverse temperature gradient is identified as $(\Delta T)_H^f/L_y$ and thus, from Eqs. (3) and (5), he finds a different effective thermal Hall conductivity [see also Fig. 2(a)]:

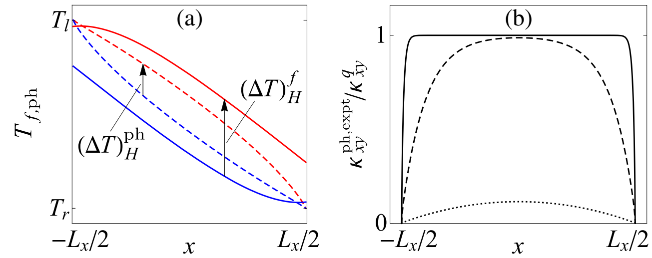

FIG. 2. (a) Temperature profiles of the Majorana fermions (solid lines) and phonons (dashed lines) at the top (red lines) and bottom (blue lines) edges, $T_{f,\mathrm{ph}}(x,\pm y_0)$. The measured "Hall" temperature differences $(\Delta T)_H^{\mathrm{ph},f}(x) \equiv T_{\mathrm{ph},f}(x,y_0) - T_{\mathrm{ph},f}(x,-y_0)$ are shown with the black arrows. (b) Measured thermal Hall conductivity $\kappa_{xy}^{\mathrm{ph,expt}}$ [Eq. (13)] as a function of the longitudinal position x at which $(\Delta T)_H^{\mathrm{ph}}$ is measured for dimensionless thermal couplings $\lambda L_x/\kappa_{xy}^q = 100$ (solid line), 10 (dashed line), and 1 (dotted line) at fixed $L_x/L_y = 100$.

$$\kappa_{xy}^{f,\mathrm{expt}} = -\frac{\kappa(\Delta T)_H^f}{L_y\frac{\overline{dT}}{dx}} = \kappa_{xy}^q\left(1 + \frac{2\kappa}{\lambda(T)L_y}\right). \tag{8}$$

Note that $\kappa_{xy}^{f,\mathrm{expt}} \approx \kappa_{xy}^{\mathrm{ph,expt}}$ only for a large enough phonon-Majorana coupling $\lambda(T) \gg \kappa/L_y$.

General conditions for quantization.—To understand how the quantization of the effective thermal Hall conductivity can break down and determine the range of its applicability, we now extend the solution of our hydrodynamic equations to a finite system with $L_x \gtrsim L_y$, where we must take into account all boundary conditions, i.e., include the right and left boundary conditions on top of those in Eq. (5). Again assuming that the leads are coupled to the phonons only, those are

$$T_{\mathrm{ph}}(\pm x_0, y) = T_{r,l},$$
$$j_{\mathrm{ex}}(\pm x_0, y) = \lambda(T)(T_{\mathrm{ph}} - T_f) = \mp\,\kappa_{xy}^{\mathrm{q}}\partial_y T_f. \tag{9}$$

Considering a small enough phonon-Majorana coupling λ, we aim to obtain a perturbative solution of the hydrodynamic equations. To this end, we write

$$T_{\mathrm{ph},f}(x,y) = T + \tilde{T}_{\mathrm{ph},f}(x,y), \tag{10}$$

with $\tilde{T}_{\mathrm{ph},f}(x,y) \ll T$. We express the temperature variations in series expansions as $\tilde{T}_{\mathrm{ph},f} = \sum_{n=0}^{\infty}\tilde{T}_{\mathrm{ph},f}^{(n)}$ and assume that terms of increasing order n are progressively less important. Note also that $\tilde{T}_{\mathrm{ph},f}(x,y) = -\tilde{T}_{\mathrm{ph},f}(-x,-y)$ generally follows from the symmetries of the hydrodynamic equations. Starting from the $\lambda = 0$ solution, $\tilde{T}_{\mathrm{ph}}^{(0)}(x,y) = (\overline{dT/dx})x$ and $\tilde{T}_f^{(0)}(x,y) = 0$, the temperature variations can then be found by an iterative procedure. At each iteration step $n > 0$, we first solve the ordinary differential equations [see Eqs. (5) and (9)]

$$\kappa_{xy}^{q}\partial_x\tilde{T}_f^{(n)} = \pm\lambda[\tilde{T}_{\rm ph}^{(n-1)} - \tilde{T}_f^{(n)}], \quad \text{for } y = \pm y_0,$$
$$\kappa_{xy}^{q}\partial_y\tilde{T}_f^{(n)} = \mp\lambda[\tilde{T}_{\rm ph}^{(n-1)} - \tilde{T}_f^{(n)}], \quad \text{for } x = \pm x_0, \qquad (11)$$

for the Majorana temperature $\tilde{T}_f^{(n)}$ along the edge. Then, using this solution, we obtain an appropriate Laplace equation $\nabla^2\tilde{T}_{\rm ph}^{(n)} = 0$ for the phonon temperature $\tilde{T}_{\rm ph}^{(n)}$ in the bulk, along with Dirichlet boundary conditions $\tilde{T}_{\rm ph}^{(n)}(\pm x_0, y) = 0$ at the left and right edges, and Neumann boundary conditions

$$\partial_y\tilde{T}_{\rm ph}^{(n)} = \pm\frac{\lambda}{\kappa}[\tilde{T}_f^{(n)} - \tilde{T}_{\rm ph}^{(n-1)}] \quad \text{for } y = \pm y_0, \qquad (12)$$

at the top and bottom edges. It is well known that such a Laplace equation with mixed Dirichlet and Neumann boundary conditions has a unique solution that can be obtained by standard methods. Our perturbative solution is convergent whenever $\lambda \ll \kappa/L_y$ (see [18] for the error analysis).

Assuming this condition, we perform the first iteration step (see [18]) to calculate the phonon temperature $\tilde{T}_{\rm ph}^{(1)}$ and obtain the effective thermal Hall conductivity in terms of the transverse temperature difference $(\Delta T)_H^{\rm ph}(x)$ [see Fig. 2(a)]:

$$\kappa_{xy}^{\rm ph,expt}(x) = -\frac{\kappa}{\frac{dT}{dx}L_y}[\tilde{T}_{\rm ph}^{(1)}(x, y_0) - \tilde{T}_{\rm ph}^{(1)}(x, -y_0)]. \qquad (13)$$

Note that $\kappa_{xy}^{\rm ph,expt}(x)$ generally depends on the position x at which the temperatures are measured [see Fig. 2(b)]. Indeed, we find that $\kappa_{xy}^{\rm ph,expt}(x)$ only takes a quantized (or even constant) value if $L_x \gg L_y$ and $L_x \gg \ell \equiv \kappa_{xy}^q/\lambda$. First, an accurate measurement of the thermal Hall conductivity generally requires an elongated system with $L_x \gg L_y$. Second, the system size L_x must be larger than the characteristic length ℓ associated with the thermalization of the Majorana edge mode (see Table I for a summary). Indeed, even for $L_x \gg L_y$, there are two regimes for the effective thermal Hall conductivity (see [18]):

$$\kappa_{xy}^{\rm ph,expt}(x) \approx \begin{cases} \frac{\pi T}{12} & (L_x \gg \ell), \\ \frac{\pi T(L_x^2 - 4x^2)}{96\ell^2} & (L_x \ll \ell). \end{cases} \qquad (14)$$

In the second regime we find that $\kappa_{xy}^{\rm ph,expt}(x)$ has a strong dependence on x and is smaller than $\kappa_{xy}^q = (\pi/12)T$ by a factor $\sim(L_x/\ell)^2 \ll 1$.

Estimation of the spin-lattice thermal coupling λ.—The phenomenological spin-lattice coupling $\lambda(T)$ defined in Eq. (3) can be obtained microscopically from, e.g., the Boltzmann equation. We calculate the rate of energy exchange per unit length $j_{\rm ex} = (1/L)(\partial\mathcal{E}/\partial t)_{{\rm ph}\to f}$ due to the scattering at the edge. Comparing to the form in Eq. (3), we extract $\lambda(T) = \lambda_0 T^\alpha$, i.e., the exponent α and the coefficient λ_0.

TABLE I. Values of the effective thermal Hall conductivities extracted by measuring the temperatures of the phonon ($\kappa_{xy}^{\rm ph,expt}$) or Majorana ($\kappa_{xy}^{\rm f,expt}$) subsystems in three coupling regimes, defined by the value of λ relative to $\lambda_f = \kappa_{xy}^q/L_x$ and $\lambda_{\rm ph} = \kappa/L_y$. The three coupling regimes can also be identified by comparing the system dimensions L_x, L_y to the characteristic lengths $\ell = \kappa_{xy}^q/\lambda$ and κ/λ.

Coupling regime	Weak	Intermediate	Strong
$\lambda \sim T^\alpha$	$\lambda \lesssim \lambda_f$	$\lambda_f \ll \lambda \ll \lambda_{\rm ph}$	$\lambda_{\rm ph} \ll \lambda$
L_x	$L_x \lesssim \ell$	$L_x \gg \ell$	
L_y	$L_y \ll \kappa/\lambda$		$L_y \gg \kappa/\lambda$
$\kappa_{xy}^{\rm ph,expt}$	$\kappa_{xy}^{\rm ph,expt} \ll \kappa_{xy}^q$	κ_{xy}^q	κ_{xy}^q
$\kappa_{xy}^{\rm f,expt}$	Ref. [19]	$\kappa_{xy}^{\rm f,expt} \gg \kappa_{xy}^q$	κ_{xy}^q

We consider a coupling at the top edge $y = y_0 = L_y/2$ of the form

$$H_{\rm int} = \frac{-igv_f}{4}\int dx\zeta(x)K_{ij}\partial_i u_j(x, y_0)\eta(x)\partial_x\eta(x), \qquad (15)$$

where $\eta(x)$, $\vec{u}(x, y)$, $\zeta(x)$ are the Majorana edge mode, the lattice displacement field, and disorder potential, respectively, g parametrizes the spin-lattice coupling, and v_f is the fermion velocity. $K_{ij}\partial_i u_j$ with i, $j = x$, y is some linear combination of the elastic tensor for u. Physically, Eq. (15) may be understood from the observation that the lattice displacement modifies the velocity of the Majorana edge mode by affecting the strength of the Kitaev coupling.

Using Eq. (15) and calculating the energy transfer rate using a Boltzmann equation, we obtain a large power $\alpha = 6$. The reason for the large exponent is twofold. First, the dispersions of both bulk phonons and edge Majoranas are linear which reduces the low energy phase space. Second, the vertex necessarily involves two gradients: one because $\eta(x)\eta(x) = \delta(0)$ is a c-number for Majorana fermions, and another because the strain tensor includes a gradient. We note that, without disorder, two-phonon processes are necessary to satisfy kinematic constraints in the physical regime, where the velocity of the acoustic phonon $v_{\rm ph}$ is larger than v_f. In that case one obtains an even larger $\alpha = 8$.

To estimate the coefficient λ_0, we further assume that the averaged disorder potential satisfies $\langle\zeta(x)\zeta(x')\rangle_{\rm dis} = \zeta^2\delta(x - x')$, and consider an isotropic acoustic phonon mode only. From the Boltzmann equation solution (see [18]), we obtain

$$\lambda = \frac{g^2\zeta^2}{32(2\pi)^3 v_{\rm ph}^4 v_f^2\rho_0} fT^6, \qquad (16)$$

where ρ_0 is the mass density of the lattice. In the model we consider $f = 4.2 \times 10^4$. Unfortunately, at this time an accurate quantitative estimate of λ for α-RuCl$_3$ is not possible due to the lack of knowledge of microscopic details of g, v_f, and ζ. However, crudely applying Eq. (16), we estimate the characteristic length $\ell = \kappa^q_{xy}/\lambda$ to be several orders of magnitude larger than the lattice spacing at temperatures of a few kelvins. Importantly, due to the large exponent α, we expect that upon lowering the temperature of the sample ℓ grows rapidly and that the system enters the regime where $L_x \ll \ell$ in Eq. (14) and thus the quantization of the thermal Hall conductivity breaks down.

Summary and discussion.—By carefully analyzing the interplay between the chiral Majorana edge mode of an Ising anyon phase and the energy currents carried by bulk phonons, we have demonstrated that the thermal Hall conductivity of such a non-Abelian topological phase can be effectively quantized in the presence of a much larger longitudinal thermal conductivity. This is in accordance with recent experiments on α-$RuCl_3$ [15]. However, this quantization only survives under certain conditions. The main results are summarized in Table I.

In words, those results are as follows. The quantization survives for a sufficiently strong spin-lattice coupling $\lambda \gg \lambda_f \equiv \kappa^q_{xy}/L_x$, while it immediately disappears in the weak-coupling regime defined by $\lambda \lesssim \lambda_f$ [see Fig. 2(b)]. Importantly, since $\lambda \propto T^\alpha$ is strongly dependent on the temperature, with $\alpha \geq 6$ for the mechanisms considered in this work, we predict that the observed quantization of the thermal Hall conductivity should eventually break down as the temperature is lowered.

Even within the range of quantization ($\lambda \gg \lambda_f$), we can identify two separate regimes, depending on how λ compares to $\lambda_{\mathrm{ph}} \equiv \kappa/L_y \gg \lambda_f$. In the strong-coupling regime, defined by $\lambda \gg \lambda_{\mathrm{ph}}$, the spins and the lattice share the same temperature, and the quantization of the thermal Hall conductivity follows from effectively having a system with a diagonal conductivity $\kappa^{\mathrm{expt}}_{xx} = \kappa^{\mathrm{expt}}_{yy} = \kappa$ of the phonons and an off-diagonal $\kappa^{\mathrm{expt}}_{xy} = \kappa^q_{xy}$ of the Majoranas. Surprisingly, however, in the intermediate regime defined by $\lambda_f \ll \lambda \ll \lambda_{\mathrm{ph}}$, the thermal Hall conductivity appears to be quantized *despite* a large temperature mismatch between the spins and the lattice. This is only true, however, if it is obtained by measuring the *lattice* temperatures along the edge. If one could directly measure the local temperature of the Majorana edge mode, it would appear to give a much larger thermal Hall conductivity.

Finally, we emphasize that our hydrodynamic equations are applicable far beyond the scope of the present work. Here, by solving them, we obtained a wide range of experimentally measurable quantities, such as detailed temperature profiles of various degrees of freedom (e.g., spins and lattice) across the system. However, due to their phenomenological nature, the hydrodynamic equations we derived should readily extend to a rich variety of chiral topological phases and thus may find applications far away from the field of quantum spin liquids.

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[17] Note that $L = L_x$, respectively $L = L_y$, for j_{ex} (top and bottom), respectively j_{ex} (left and right).
[18] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.147201 for calculation details.
[19] "–" in the last line means that the quantization $\kappa^{f,\mathrm{expt}}_{xy}$ relative to κ^q_{xy} is not generic ($\kappa^{\mathrm{f,expt}}_{xy} \gtrless \kappa^q_{xy}$) in the weak coupling regime $\lambda \lesssim \lambda_f$, i.e., it depends on the strength of λ and the position x where the temperature is measured.