Concentration-of-Measure Theory for Structures and Fluctuations of Waves

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The emergence of nonequilibrium phenomena in individual complex wave systems has long been of fundamental interest. Its analytic studies remain notoriously difficult. Using the mathematical tool of the *concentration of measure*, we develop a theory for structures and fluctuations of waves in individual disordered media. We find that, for both diffusive and localized waves, fluctuations associated with the change in incoming waves ("wave-to-wave" fluctuations) exhibit a new kind of universality, which does not exist in conventional mesoscopic fluctuations associated with the change in disorder realizations ("sample-to-sample" fluctuations), and originates from the coherence between the natural channels of waves—the *transmission eigenchannels*. Using the results obtained for wave-to-wave fluctuations, we find the criterion for almost all stationary scattering states to exhibit the same spatial structure such as the diffusive steady state. We further show that the expectations of observables at stationary scattering states are independent of incoming waves and are given by their averages with respect to eigenchannels. This suggests the possibility of extending the studies of thermalization of closed systems to open systems, which provides new perspectives for the emergence of nonequilibrium statistical phenomena.

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Recent studies on the foundations of equilibrium statistical mechanics [1–7] have shed new light on the long-standing problem of nonequilibrium phenomena in individual (quantum) wave systems, where neither fictitious ensembles nor reservoirs exist [8]. Many of them [3,4] rely on a conjecture of Berry, i.e., that in closed systems random scattering can render waves structureless on large spatial scales [9]. This wave property gives rise to a basic feature of thermal equilibrium phenomena in individual closed systems, i.e., spatial homogeneity. Yet, a major topic of non-equilibrium statistical mechanics is concerned with various spatial structures in open systems [10–14]. This contrast motivates exploring in depth spatial structures of waves, i.e., scattering states, in individual open systems, which may also pave a way for extending the studies of the relations between spatial and entanglement structures in an ensemble of open disordered systems [15] to an individual member.

In fact, spatial structures and fluctuations of waves in open disordered media are central topics of mesoscopic physics [16–20]. However, most theoretical efforts have been focused on disorder ensembles; the significance of waves in individual disordered media has been emphasized only recently [21,22]. The common wisdom of using self-averaging or the ergodic hypothesis to connect certain properties of individual disordered media to their disorder averages [23,24] essentially requires the thermodynamic limit, and cannot be applied to study fluctuations in mesoscopic scales. It remains a challenge to construct a theory for wave statistics in individual mesoscopic systems, where rich fluctuation phenomena of the wave origin can be driven, e.g., by changing the incoming wave. Such wave-to-wave fluctuations differ from well understood sample-to-sample fluctuations [16–20,23–29]. Their in-depth studies are of both fundamental and practical importance. Indeed, in individual mesoscopic systems, fluctuations and irreversibility have been known to be closely related [30]. On the other hand, wave statistics in individual open disordered media have found many optical applications [31–34].

Recently, the concentration of measure (CM) [35–38] has been adopted to study statistical phenomena in individual closed systems [1,39–41]. The CM is rooted in high-dimensional geometry. The idea can be illustrated by the unit sphere, for which the area of the sphere becomes more and more concentrated around the equator as the dimension increases. Eventually in high dimensions the entire area almost concentrates around the equator. This property can then be visualized by real-valued functions over the sphere with nice continuity properties through their concentration around some constant value. When the sphere is replaced by a general high-dimensional geometric body (e.g., the Euclidean space) and the area measure by others (e.g., the Gaussian measure), similar results follow. This idea opens new perspectives of probability theory [36–38]. It allows us not only to study variables with complicated dependence on random variables instead of being their sum, but also to obtain nonasymptotic results. A detailed introduction of CM is given in section S0 of the Supplemental Material [42].

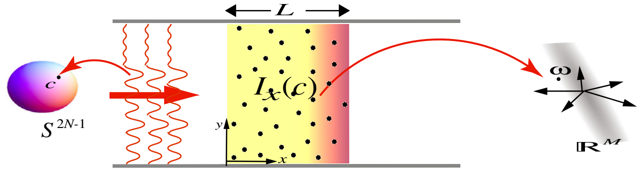

FIG. 1. In the present setting two high-dimensional geometric bodies appear: S^{2N-1} constituted by distinct incoming current amplitudes c and $\mathbb{R}^M$ by distinct disorder realizations ω.

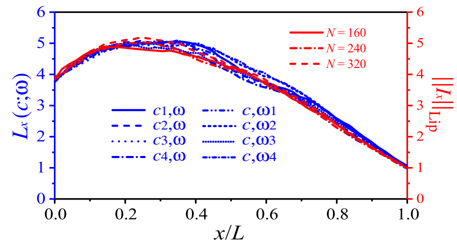

FIG. 2. Using Eqs. (5) and (12), we calculate $L_x(c;\omega)$ at $N = 800$ for 4 randomly chosen c at fixed ω and for 4 randomly chosen ω at fixed c, respectively, and calculate $\|I_x\|_{\mathrm{Lip}}$ for 3 large N. All profiles collapse into the a single curve. $L = 50$.

In this Letter, we employ CM to explore universal statistical phenomena of waves in individual open disordered media. We launch a classical wave of circular frequency Ω and carrying unit energy flux into a finite medium with N $(=(\Omega/\pi)\times$the width) channels and length L [47,48] (Fig. 1). Keeping the disorder realization fixed, but allowing the incoming wave to vary gives rise to various wave-to-wave fluctuations. Below, most attentions are paid to the fluctuations of the spatial structure of scattering state corresponding to the incoming current amplitude c, i.e., the depth (x) profile $I_x(c)$ of energy density integrated over the cross section. We develop a CM theory of wave-to-wave fluctuations. Physically, it provides information on a single stationary scattering state in a single disordered medium and the differences of behaviors between a disorder ensemble and an individual member; technically, instead of traditional impurity diagrams [16,17,20] and field theories [18,19,28], its key components are various *concentration inequalities* [37] of observables [e.g., $I_x(c)$].

By the developed theory we achieve the following results: (1) we find that, compared to conventional sample-to-sample fluctuations, wave-to-wave fluctuations exhibit a number of "anomalies." In particular, irrespective of regimes of wave propagation (diffusive, localized, etc.), the distribution of $I_x(c)$ is always *sub-Gaussian*, i.e., has an (upper) tail decaying at least as fast as a Gaussian tail [Eq. (6)]. Contrary to this, for sample-to-sample fluctuations of observables such as total transmission, as waves are more and more localized the distribution tail decays slower and slower, and the shape of the tail changes dramatically [18,49]. (2) Furthermore, we find that the wave-to-wave fluctuations of $I_x(c)$ are governed by an x-dependent curve $\|I_x\|_{\mathrm{Lip}}$, which arises from the phase coherence between distinct eigenchannels—the natural channels for wave propagation in disordered media [50,51]. In contrast, the sample-to-sample fluctuations of $I_x(c)$ (c fixed) are governed by the conductance [52] known to equal the number of open eigenchannels [53]. For diffusive waves, we find that the curve is universal with respect to disorder realizations ω at large N (cf. Fig. 2). (3) We find the criterion [Eq. (8)] for almost all stationary scattering states to exhibit the same spatial structure, i.e., a nonequilibrium steady state, and show that it can be readily satisfied for diffusive waves. (4) We show that the expectations of generic observables at stationary scattering states are independent of incoming waves and given by their averages with respect to eigenchannels [Eq. (15)], and find the corresponding criterion. These results apply to both classical and quantum waves, like Anderson localization [17,18]. In addition, owing to the nonasymptotic nature they provide new perspectives for statistics of waves launched into a disordered medium via few channels [54].

We begin with a general discussion on how the high-dimensional geometry emerges from the present setting (Fig. 1), and further provides a basis for applying CM. For simplicity we consider a two-dimensional (2D) medium. A disordered dielectric configuration $\delta\epsilon(x,y)$ is embedded into the air background. So the wave field $E(x,y)$ satisfies the Helmholtz equation [48],

$$\{\partial_x^2 + \partial_y^2 + \Omega^2[1 + \delta\epsilon(x,y)]\}E(x,y) = 0. \tag{1}$$

Given N channel bases, an incoming current amplitude is a projection represented by N complex coefficients: $(c_1, c_2, \ldots, c_N) \equiv c$. Since the incoming wave carries a unit energy flux, $\sum_{n=1}^N |c_n|^2 = 1$, and all c constitute the unit sphere S^{2N-1}. Next, we discretize the medium into a lattice of M points. The M values $\{\omega_{(x,y)} \equiv -\Omega^2\delta\epsilon(x,y)\}$ constitute the coordinate of the Euclidean space $\mathbb{R}^M$: a point in $\mathbb{R}^M$ corresponds to a disorder realization $\omega \equiv \{\omega_{(x,y)}\}$. As observables depending on c (respectively ω) define real-valued functions over S^{2N-1} (respectively $\mathbb{R}^M$), we can apply CM to them and study wave-to-wave (respectively sample-to-sample) fluctuations.

Construction of the theory.—We divide the construction into five steps. In each step we outline the derivations and present the key results; motivations and (or) physical implications are also discussed. Technical details and expanded discussions are relegated to Ref. [42].

Step 1—formulation of the problem.—Below we choose the eigenchannel [51] introduced in the following as the basis. The transmission matrix t has matrix elements $\{t_{ab}\} = -i\sqrt{\tilde{v}_a\tilde{v}_b}\langle x = \infty a|G|x' = -\infty b\rangle$ [55], where G is Green's function and $\tilde{v}_a$ is the group velocity of ideal waveguide mode $\varphi_a(y)$ (a the mode index). By the singular value decomposition, $t = \sum_{n=1}^N u_n\sqrt{\tau_n}v_n^\dagger$, we obtain a transmission eigenvalue spectrum $\{\tau_n\}$ and mutual orthogonal unit vectors $\{u_n\}$ and $\{v_n\}$. Replacing $x = \infty$ in t_{ab} by arbitrary $x \in [0, L]$, we make the extension $t \to t(x)$ and obtain a vector field $E_n(x) \equiv \{E_{an}(x)\} = t(x)v_n$. Then $(\tau_n, v_n, E_n(x))$ defines the nth eigenchannel, determined completely by ω, Ω [56]. Each channel has a specific 2D spatial structure: the energy density profile $|E_n(x,y)|^2$ with $E_n(x,y) = \sum_{a=1}^{N} E_{na}(x)\varphi_a^*(y)$. Integrating y results in a one-dimensional (1D) structure,

$$W_{\tau_n}(x) \equiv \int dy |E_n(x,y)|^2 = E_n^\dagger(x) \cdot E_n(x), \qquad (2)$$

with $\cdot$ being a scalar product.

Treating $\omega_{(x,y)}$ as a potential, we apply the scattering theory of waves [57] to Eq. (1) and find $\hat{v}_x^{\frac{1}{2}} E(x,y) = \sum_{a=1}^{N} [t(x)c]_a \varphi_a^*(y)$, where $c = \sum_{n=1}^{N} c_n v_n$ and $\hat{v}_x \equiv (1 - \partial_{\Omega y}^2)^{\frac{1}{2}}$ is a *scalar* operator accounting for the absolute value of the group velocity in waveguide modes. Thus we reduce $I_x(c) \equiv \int dy E^*(x,y)\hat{v}_x E(x,y)$ [58] into [59]

$$I_x(c) = \sum_{n,n'=1}^{N} c_n^* c_{n'} E_n^\dagger(x) \cdot E_{n'}(x). \qquad (3)$$

Then the problem is the following: For a fixed ω, does $I_x(c)$ exhibit universal behaviors when c varies? A natural idea is to calculate all the cumulants of $I_x(c)$ and find the distribution. But, one then needs to calculate an infinite number of products of $E_n^\dagger \cdot E_{n'}$, and sum up their contributions, which is a formidable task especially for a small N. The CM allows a different route, which we follow below.

Step 2—Lipschitz continuity: a building block of CM.—This is the concept that formalizes the "nice continuity properties" of real-valued functions mentioned in the introductory part. Let a generic space $\mathcal{C}$ be equipped with the Euclidean metric $\|\cdot\|$. For $f : \mathcal{C} \to \mathbb{R}$, if

$$\|f\|_{\mathrm{Lip}} \equiv \sup_{z,z'} \frac{|f(z) - f(z')|}{\|z - z'\|} < \infty$$
$$\Leftrightarrow |f(z) - f(z')| \leq \|f\|_{\mathrm{Lip}} \|z - z'\|, \qquad (4)$$

where "sup" stands for the least upper bound, then $f(z)$ is said to have the Lipschitz continuity or be Lipschitz, and $\|f\|_{\mathrm{Lip}}$ is called the Lipschitz constant. As we will see below, even though f has a very complicated dependence on c, its wave-to-wave fluctuations are controlled by a single parameter, i.e., $\|f\|_{\mathrm{Lip}}$.

Step 3—Concentration inequality for $I_x(c)$ and results for general waves.—Now we introduce a result of CM:

Lévy's lemma [35,40]. Let μ be the uniform probability measure over S^{2N-1}, and $f : S^{2N-1} \to \mathbb{R}$ be Lipschitz. Then the probability for the deviation between f and its mean $\int f d\mu$ to exceed ε is $\leq 2e^{-\delta\varepsilon^2 N/\|f\|_{\mathrm{Lip}}^2}$, where δ is some positive absolute constant.

This means that f concentrates around $\int f d\mu$ with a rate increasing rapidly with $N/\|f\|_{\mathrm{Lip}}^2$. We stress that $\|f\|_{\mathrm{Lip}}$ depends on N generally. By the lemma the distribution of f is sub-Gaussian. But, unlike the central limit theorem, the lemma does not require the large N limit, i.e., is nonasymptotic; instead, N can be very small (cf. Fig. 4).

In S1 of Ref. [42] we use Eq. (3) to derive an analytic expression of the Lipschitz constant $\|I_x\|_{\mathrm{Lip}}$ of $I_x(c)$. The result reads

$$\|I_x\|_{\mathrm{Lip}} = \sup_c L_x(c;\omega),$$
$$L_x = \pi \left(\sum_{n=1}^{N} \left| \sum_{n'=1}^{N} c_{n'}^* E_{n'}^\dagger(x) \cdot E_n(x) \right|^2 - I_x^2 \right)^{1/2}, \qquad (5)$$

where L_x depends on c, ω in general. According to Eq. (5), $\|I_x\|_{\mathrm{Lip}} < \infty$. Combined with Lévy's lemma this gives the following concentration inequality,

$$\Pr(|I_x(c) - W(x;\omega)| > \varepsilon) \leq 2e^{-\frac{\delta\varepsilon^2 N}{\|I_x\|_{\mathrm{Lip}}^2}}. \qquad (6)$$

Here $W(x;\omega) \equiv \int I_x(c) d\mu$ and "Pr" stands for probability. After simple algebra we reduce $W(x;\omega)$ to

$$W(x;\omega) = \frac{1}{N} \sum_{n=1}^{N} W_{\tau_n}(x). \qquad (7)$$

The results of Eqs. (5)–(7) hold for general N, ω, regardless of regimes of wave propagation (diffusive, localized, etc.).

From the inequality Eq. (6) we see that provided

$$W(x;\omega) \gg \|I_x\|_{\mathrm{Lip}}/\sqrt{N}, \qquad (8)$$

the wave-to-wave fluctuations of $I_x(c)$ are negligible and almost all incoming waves behave in essentially the same way: their energies are stored in distinct channels with an equal weight of $1/N$ and a nonequilibrium steady state universal with respect to c namely $W(x;\omega)$ results, i.e., $I_x(c) \approx W(x;\omega)$. The state does not carry information on the phase coherence between eigenchannels. The phase coherence, as shown by the sub-Gaussian tail in Eqs. (6) and (5), enters into $\|I_x\|_{\mathrm{Lip}}$ and influences strongly wave-to-wave fluctuations (see *Step 5* for further discussions).

From the inequality [Eq. (6)] we also see that the distribution tail of $I_x(c)$ decays at least as fast as a Gaussian tail. In contrast, in sample-to-sample fluctuations (c fixed) the distribution tail decays much slower, known for $x = L$ to be exponential for diffusive waves [52] and log-normal for deeply localized waves [18,29].

Step 4—Diffusive steady state and its fluctuations.—Below we use the general results Eqs. (5)–(7) to explore in depth diffusive waves. Previous numerical studies have shown that in quasi 1D media [60], the disorder average of

Eq. (7) gives a diffusive steady state [51], but it is difficult to (dis)prove analytically that without the averaging this remains true for general geometry. Indeed, for large N (L and Ω fixed) the medium is a slab [60] and thus high dimensional, but in high dimension the explicit form of $W_{\tau_n}(x)$, even its disorder average, is unknown; for small N, i.e., a short quasi 1D medium, the impacts of the sample-to-sample fluctuations on $W_{\tau_n}(x)$ have not yet been studied.

We first establish a concentration inequality of $W(x;\omega)$. To this end we show in S2 of Ref. [42] that, even for a single disordered slab, distinct structures $W_{\tau_n}(x)$ are described by a single formula [51], which depends smoothly on τ_n and was derived originally for an ensemble of quasi 1D disordered media. Using this fact, we show in S3 of Ref. [42] that the real-valued function $W(x;\omega)$ over $\mathbb{R}^M$ is Lipschitz, i.e.,

$$|W(x;\omega) - W(x;\omega')| \leq \tilde{c}(x) N^{-\frac{1}{2}} \|\omega - \omega'\|. \tag{9}$$

Here $\tilde{c}(x) = \mathcal{O}(1)$; its explicit form is unimportant and given in Ref. [42]. The property Eq. (9) allows us to use Pisier's theorem in CM [35] to show in S4 of Ref. [42] the following:

Theorem. If $\omega = \{\omega_{(x,y)}\}$ is drawn randomly from an ensemble of disorder realizations with $\omega_{(x,y)}$ being independent Gaussian variables of zero mean and variance σ^2, then the probability for the deviation between $W(x;\omega)$ and its disorder mean $\mathrm{E}[W(x;\omega)]$ to exceed ε satisfies

$$\Pr\left(|W(x;\omega) - \mathrm{E}[W(x;\omega)]| > \varepsilon\right) \leq 2e^{-\frac{2N\varepsilon^2}{(\tilde{c}(x)\pi\sigma)^2}}. \tag{10}$$

Thus the concentration is strong for large N, i.e.,

$$W(x;\omega) \approx \mathrm{E}[W(x;\omega)] \tag{11}$$

for almost all ω. Importantly, the factor N in the sub-Gaussian bound of the concentration inequality [Eq. (10)] comes from the Lipschitz constant of $W(x;\omega)$, i.e., the coefficient on the right-hand side of the inequality [Eq. (9)].

Noticing that the detailed structures of $\{W_{\tau_n}(x)\}$ enter into the inequality [Eq. (10)] only through the unimportant factor of $\tilde{c}(x)$, we conjecture that the inequality applies for small N also. While to prove this conjecture rigorously is beyond the present work, in S4 of Ref. [42] we confirmed the conjecture numerically for N being as small as 20.

So Eq. (11) holds for both large and small N. Due to $I_x(c) \approx W(x;\omega)$ the disorder average of $I_x(c)$ (c fixed) $\mathrm{E}[I_x(c)] \approx \mathrm{E}[W(x;\omega)]$. Together with Eq. (11) this gives $W(x;\omega) \approx \mathrm{E}[I_x(c)]$. As $\mathrm{E}[I_x(c)]$ is known to be the solution to the diffusion equation [16,17,20], $W(x;\omega)$ is a diffusive steady state (for almost all ω), which decreases linearly in x. This result renders the transport mean free path ℓ well defined for single ω—because for a diffusive steady state the total transmission $W(L;\omega) = \ell/L$ [16]—and identical to that defined for a disorder ensemble.

Step 5—Wave-to-wave fluctuations in diffusive regime.—To study these fluctuations we need to better understand $\|I_x\|_{\mathrm{Lip}}$. For large N we calculate Eq. (5) in S1 of Ref. [42] and obtain

$$\|I_x\|_{\mathrm{Lip}} = \int L_x(c;\omega) d\mu = \mathcal{O}(1). \tag{12}$$

Surprisingly, as shown in Fig. 2, the profiles of $\|I_x\|_{\mathrm{Lip}}$ at distinct N collapse into a single curve; moreover, the profile of L_x is universal with respect to c and ω, and the universal curve is identical to that of $\|I_x\|_{\mathrm{Lip}}$.

This universality of L_x, together with Eq. (12), implies the universality of $\|I_x\|_{\mathrm{Lip}}$ with respect to ω. As shown in S1 of Ref. [42], it even leads to an explicit expression of $\|I_x\|^2_{\mathrm{Lip}}$:

$$\|I_x\|^2_{\mathrm{Lip}} = \frac{\pi^2}{N+1}\left(\sum_{n=1}^{N}[W_{\tau_n}(x) - W(x;\omega)]^2 + \sum_{n\neq n'}^{N}\left|\sum_{a=1}^{N} E^*_{na}(x)E_{n'a}(x)\right|^2\right). \tag{13}$$

By using Eq. (13) we find in S5 of Ref. [42] that

$$\mathrm{var}(I_x) = \|I_x\|^2_{\mathrm{Lip}}/(\pi^2 N), \tag{14}$$

and thus includes both incoherent and coherent contributions of eigenchannels, corresponding respectively to the first and second term in Eq. (13).

For small N the universality above is violated. But numerical calculations show that $\|I_x\|_{\mathrm{Lip}} = \mathcal{O}(1)$ still holds (see the symbols corresponding to $N = 20$ in the left panel of Fig. 3). Due to this and $W(x;\omega) = \mathcal{O}(1)$, for diffusive waves, the criterion Eq. (8) can be readily satisfied.

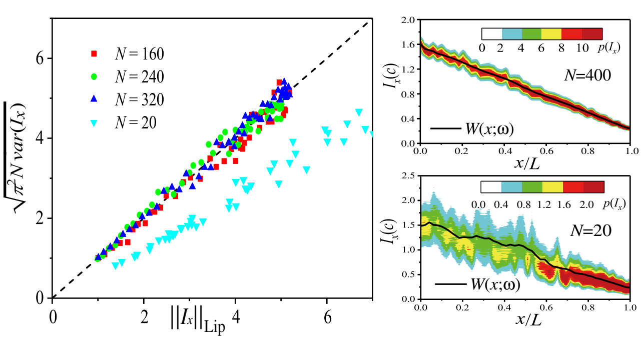

FIG. 3. Simulations show that in a single slab ω the profiles $I_x(c)$ for distinct c concentrate around $W(x;\omega)$ (right panels), and the data: $[\|I_x\|_{\mathrm{Lip}}, \sqrt{\pi^2 N \mathrm{var}(I_x)}]$ (symbols) collapse into a straight line of unit slope for distinct large N while deviate from this line for small N (left panel). $L = 50$.

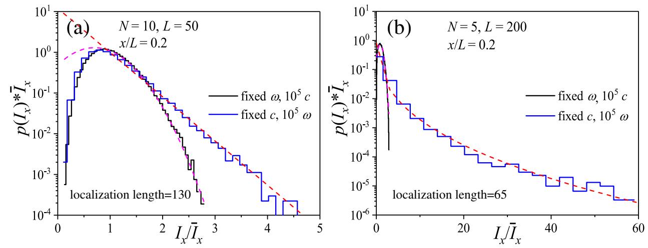

FIG. 4. Quasi 1D simulations show that, for both diffusive (a) and localized (b) waves, the distribution of wave-to-wave fluctuations of $I_x(c)$ (black histograms) displays a tail well fit by a Gaussian distribution (pink dashed lines), while that of sample-to-sample fluctuations (blue histograms) is exponential (a) and stretched exponential with a stretching exponent of 0.4 (b), respectively (red dashed lines) [62]. $\bar{I}_x = \int I_x p(I_x) dI_x$.

Numerical confirmations.—We put the theory into numerical tests. The methods of numerical experiments are described in S7 of Ref. [42]. First, we simulate wave propagation in a single slab for 10^4 randomly chosen c. Simulations confirm that the profiles $I_x(c)$ concentrate around a linear decrease for both large and small N (Fig. 3, right panels). This also gives $\ell = 13$ for single ω [47]. Moreover, from the distribution $p(I_x)$ of wave-to-wave fluctuations, we compute $var(I_x)$, and find that the relation Eq. (14) holds for large N but is violated for small N (Fig. 3, left panel), as expected by our theory. Second, we simulate propagation of diffusive and localized waves in quasi 1D media. We perform the statistics of wave-to-wave and sample-to-sample fluctuations of I_x. As shown in Fig. 4, for both diffusive and localized waves the distribution of wave-to-wave fluctuations displays a Gaussian tail, in agreement with the inequality Eq. (6). In contrast, the distribution of sample-to-sample fluctuations is much broader, which is exponential for diffusive waves and stretched-exponential for localized waves [61].

Our theory provides new perspectives for the long-standing problem of the emergence of irreversibility in individual systems. In particular, in S6 of Ref. [42] we study the expectation: $O(c) \equiv \langle E|\hat{O}|E\rangle = \sum_{n,n'} c_n^* c_{n'} \langle E_n|\hat{O}|E_{n'}\rangle$ of a generic Hermitian operator $\hat{O}$ at the stationary scattering state $E(x,y)$ (determined by c), where $E_n = E_n(x,y)$. Repeating the analysis above we find

$$O(c) \approx \frac{1}{N} \sum_{n=1}^{N} \langle E_n|\hat{O}|E_n\rangle \equiv \bar{O}, \qquad (15)$$

for almost all c, if $\bar{O} \gg \|O\|_{\mathrm{Lip}}/\sqrt{N}$. This independence of observables on the incoming wave resembles thermalization in closed systems [1–8]. But, as the systems here are open, conceptual differences exist. Notably, bound states and equilibrium thermal ensembles in closed systems are replaced respectively by stationary scattering states and $N^{-1}\sum_n |E_n\rangle\langle E_n|$, which may be called the eigenchannel ensemble.

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[47] Throughout this work we require the sample size be much larger than the transport mean free path ℓ. The latter is usually defined for a disorder ensemble. As we show in this Letter, ℓ is well defined for a single disordered medium also, and is identical to that defined for an ensemble. In S2 of Ref. [42] we further show that, when N is large, for a single disordered medium ℓ can also be defined by the transparent eigenchannel structure $W_{\tau=1}(x)$.

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[55] Here we write G in the representation which is a mixture of the real coordinate x and the waveguide mode. In other words, we perform the Fourier transform of $\langle xy|G|x'y'\rangle$ with respect to y, y'.

[56] To make formulae compact we suppress ω and Ω in $I_x(c)$ and E_n throughout.

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[58] Strictly speaking, the definition of energy density excludes the scalar operator $\hat{v}_x$. However, we have found that omitting $\hat{v}_x$ in the definition only leads to the appearance of an irrelevant overall factor.

[59] Note that at $x < L$ the vectors $E_n(x)$ are not orthogonal in general.

[60] For a slab the width $\gtrsim L$ while for a quasi 1D medium the width $\ll L$.

[61] For the sample-to-sample fluctuations of $I_x(c)$ (with the incoming current amplitude c fixed), we are not aware of any reports of either the exponential distribution of diffusive waves (except for the special case of $x = L$ and c chosen to correpond to a plane wave or a Gaussian beam [52]) or the stretched-exponential distribution of localized waves.

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