Sensitivity Bounds for Multiparameter Quantum Metrology

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We identify precision limits for the simultaneous estimation of multiple parameters in multimode interferometers. Quantum strategies to enhance the multiparameter sensitivity are based on entanglement among particles, modes, or combining both. The maximum attainable sensitivity of particle-separable states defines the multiparameter shot-noise limit, which can be surpassed without mode entanglement. Further enhancements up to the multiparameter Heisenberg limit are possible by adding mode entanglement. Optimal strategies that saturate the precision bounds are provided.

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A central problem of quantum metrology is to identify fundamental sensitivity limits and to develop strategies to enhance the precision of parameter estimation [1–5]. Quantum noise poses an unavoidable limitation even under ideal conditions, in the absence of environmental coupling. Nevertheless, quantum noise can be reduced by adjusting the properties of the probe state and the output measurement. Knowing the sensitivity limits of different classes of probe states is thus crucial to identify quantum resources that lead to an enhancement of sensitivity over classical strategies. The shot noise, i.e., the maximum sensitivity achievable with particle-separable states, and the Heisenberg limit, i.e., the maximum sensitivity achievable with any probe quantum state, have been clearly identified for the estimation of a single parameter [6–9]. Sub-shot-noise sensitivities have been reported in several optical [3,10–12] and atomic [5] experiments, opening up strategies to achieve quantum enhancements in matter-wave interferometers [13], atomic clocks [14], quantum sensors [15], gravitational wave detectors [16,17], and biological measurements [18]. However, much less is known about the sensitivity bounds for the simultaneous estimation of multiple parameters. What is the shot-noise and Heisenberg limit in this case? What is the role played by entanglement among the modes where the parameters are encoded? Can multiparticle and multimode entanglement enhance sensitivity?

Multiparameter estimation finds many important applications in quantum imaging [19–21], microscopy and astronomy [22,23], and sensor networks [24,25], as well as the detection of inhomogeneous forces, vector fields, and gradients [26–28]. All these tasks go beyond single-parameter estimation. Only a clear identification of relevant quantum resources can lead to a quantum advancement of these technologies [29–39].

In this Letter, we present the precision limits for multiparameter quantum metrology in multimode interferometers (see Fig. 1), unveiling the nontrivial interplay of mode and particle entanglement. The precision limits are given in matrix form, as bounds for the covariance matrix for the estimators of multiple parameters. As in the single-parameter case, the shot-noise limit is found by maximizing

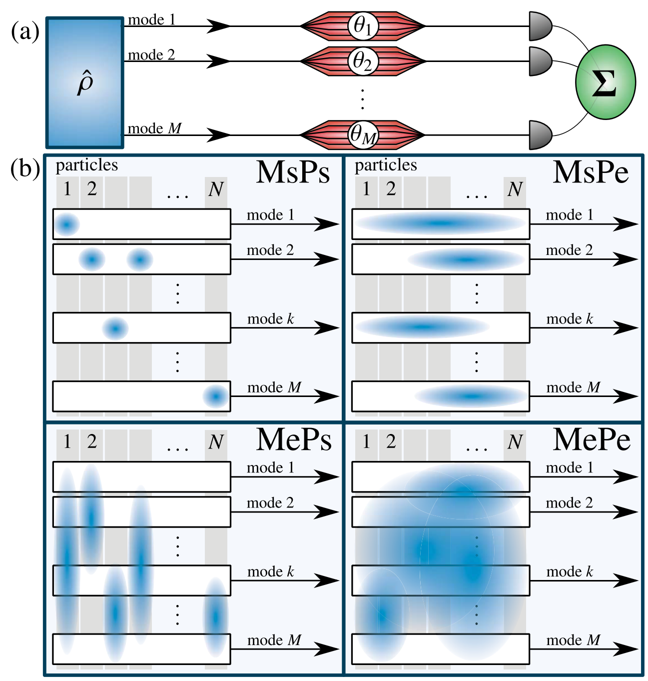

FIG. 1. General scheme for multiparameter quantum metrology with commuting generators of phase shifts. (a) The probe state $\hat{\rho}$ of N particles is distributed among M modes. In each mode $k = 1, \ldots, M$, a parameter θ_k is encoded as a relative phase shift between sublevels. The sensitivity is quantified by the covariance matrix of the estimators $\boldsymbol{\Sigma}$. The probe state $\hat{\rho}$ can be prepared as schematically shown in (b): mode and particle separable (MsPs), mode separable and particle entangled (MsPe), mode entangled and particle separable (MePs), and mode and particle entangled (MePe). The gray bars represent the particle partition of the quantum state, the white bars the mode partition. Mode entanglement is illustrated by vertical blue delocalized distributions, particle entanglement by horizontal delocalization.

the multiparameter sensitivity over all particle-separable states. While particle-separable strategies that use mode entanglement [MePs in Fig. 1(b)] can overcome the sensitivity achievable by states that are particle separable and mode separable (MsPs), mode entanglement is not necessary to overcome the multiparameter shot-noise limit. The highest sensitivity achievable by mode-separable states is obtained in the presence of particle entanglement (MsPe). Finally, the multiparameter Heisenberg limit, defined as the sensitivity bound optimized over all quantum states, can only be reached if both particle entanglement and mode entanglement (MePe) are present. We identify the respective states that saturate the discussed bounds.

Multimode interferometers for multiphase estimation.—In the interferometer scheme of Fig. 1(a), each parameter θ_k is imprinted in one of the M separate modes via the unitary evolution $\hat{U}(\boldsymbol{\theta}) = \exp(-i\sum_{k=1}^{M}\hat{H}_k\theta_k) = \exp(-i\hat{\mathbf{H}}\cdot\boldsymbol{\theta})$. Here, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$ and $\hat{\mathbf{H}} = (\hat{H}_1, \ldots, \hat{H}_M)$ are the vectors of unknown phases and local Hamiltonians, respectively. The initial probe state $\hat{\rho}$ evolves into $\hat{\rho}(\boldsymbol{\theta}) = \hat{U}(\boldsymbol{\theta})\hat{\rho}\hat{U}^{\dagger}(\boldsymbol{\theta})$ and it is finally detected. We indicate with $\mathbf{x} = (x_1, \ldots, x_\mu)$ a sequence of μ independent measurement results that occurs with probability $p(\mathbf{x}|\boldsymbol{\theta}) = \prod_{s=1}^{\mu} p(x_s|\boldsymbol{\theta})$. The sensitivity of the multiparameter estimation is determined by the $M \times M$ covariance matrix $\boldsymbol{\Sigma}$ with elements $\boldsymbol{\Sigma}_{kl} = \mathrm{Cov}(\theta_{\mathrm{est},k}, \theta_{\mathrm{est},l})$, where $\theta_{\mathrm{est},k}(\mathbf{x})$ is a locally unbiased estimator for θ_k, with $\langle\theta_{\mathrm{est},k}\rangle = \theta_k$ and $d\langle\theta_{\mathrm{est},k}\rangle/\theta_l = \delta_{kl}$ [1]. Any linear combination of the M parameters $\mathbf{n}\cdot\boldsymbol{\theta} = \sum_{k=1}^{M} n_k\theta_k$ is estimated with variance $\Delta^2(\sum_{k=1}^{M} n_k\theta_{\mathrm{est},k}) = \sum_{kl=1}^{M} n_k n_l \mathrm{Cov}(\theta_{\mathrm{est},k}, \theta_{\mathrm{est},l}) = \mathbf{n}^T\boldsymbol{\Sigma}\mathbf{n}$. The matrix $\boldsymbol{\Sigma}$ fulfills the chain of inequalities

$$\boldsymbol{\Sigma} \geq \mathbf{F}^{-1}/\mu \geq \mathbf{F}_Q^{-1}/\mu, \tag{1}$$

that identify the Cramér-Rao (CRB) and quantum Cramér-Rao (QCRB) bounds [1], respectively, meaning that $\mathbf{n}^T\boldsymbol{\Sigma}\mathbf{n} \geq \mathbf{n}^T\mathbf{F}^{-1}\mathbf{n}/\mu \geq \mathbf{n}^T\mathbf{F}_Q^{-1}\mathbf{n}/\mu$ for arbitrary $\mathbf{n}$. Here $\mathbf{F}^{-1}$ is the inverse of the classical Fisher matrix with elements $(\mathbf{F})_{kl} = \sum_x p(x|\boldsymbol{\theta})[(\partial/\partial\theta_k)\log p(x|\boldsymbol{\theta})][(\partial/\partial\theta_l)\log p(x|\boldsymbol{\theta})]$, and $(\mathbf{F}_Q[\hat{\rho}])_{kl} = \mathrm{Tr}[\hat{\rho}\hat{L}_k\hat{L}_l]$, where $d\hat{\rho}/d\theta_k = (\hat{L}_k\hat{\rho} + \hat{\rho}\hat{L}_k)/2$, are the elements of the quantum Fisher matrix [1,2]. $\mathbf{F}$ and $\mathbf{F}_Q$ are positive semidefinite matrices and the chain of inequalities (1) is defined only if $\mathbf{F}$ and $\mathbf{F}_Q$ are invertible. Since in the multimode setting considered here all local Hamiltonians $\hat{H}_k$ commute with each other, the bound $\mathbf{F} = \mathbf{F}_Q$ can always be saturated by an optimally chosen set of local projectors in each mode [40,41], for instance, by the projectors onto the eigenstates of $\hat{L}_k$ [42].

We consider probe states of N particles and collective local operators $\hat{H}_k = \sum_{i=1}^{N}\hat{h}_k^{(i)}$, where $\hat{h}_k^{(i)}$ is a local Hamiltonian for the ith particle in the kth mode. The $\hat{h}_k^{(i)}$ have the same spectrum λ_{kj} with eigenvectors $|\lambda_{kj}^{(i)}\rangle$ for all i, where j labels the eigenvalues. For simplicity, we limit the discussion in the main text to the case of two sublevels per mode ($j = \pm$) with $\lambda_{k\pm} = \pm\frac{1}{2}$. A detailed demonstration of all bounds reported below as well as a direct generalization to multilevel systems is given in the Supplemental Material [43].

Sensitivity bounds for particle-separable states.—Here we derive the sensitivity bound for particle-separable states $\hat{\rho}_{\text{p-sep}} = \sum_\gamma p_\gamma \hat{\rho}_\gamma^{(1)} \otimes \cdots \otimes \hat{\rho}_\gamma^{(N)}$, where p_γ is a probability distribution and the $\hat{\rho}_\gamma^{(i)}$ are arbitrary single-particle density matrices of the ith particle. The quantum Fisher matrix of any particle-separable probe state is bounded by

$$\mathbf{F}_Q[\hat{\rho}_{\text{p-sep}}, \hat{\mathbf{H}}] \leq 4\sum_{i=1}^{N}\boldsymbol{\Gamma}[\hat{\rho}^{(i)}, \hat{\mathbf{H}}^{(i)}],$$

where $\boldsymbol{\Gamma}[\hat{\rho}^{(i)}, \hat{\mathbf{H}}^{(i)}]$ is the covariance matrix of the reduced density matrix $\hat{\rho}^{(i)} = \sum_\gamma p_\gamma \hat{\rho}_\gamma^{(i)}$ of particle i with elements $(\boldsymbol{\Gamma}[\hat{\rho}^{(i)}, \hat{\mathbf{H}}^{(i)}])_{kl} = \langle\hat{h}_k^{(i)}\hat{h}_l^{(i)}\rangle_{\hat{\rho}^{(i)}} - \langle\hat{h}_k^{(i)}\rangle_{\hat{\rho}^{(i)}}\langle\hat{h}_l^{(i)}\rangle_{\hat{\rho}^{(i)}}$ and $\hat{\mathbf{H}}^{(i)} = (\hat{h}_1^{(i)}, \ldots, \hat{h}_M^{(i)})$. To find the multiparameter shot noise (SN), we maximize $\mathbf{F}_Q[\hat{\rho}_{\text{p-sep}}, \hat{\mathbf{H}}]$ over all $\hat{\rho}_{\text{p-sep}}$ with given average particle numbers $\langle\hat{N}_k\rangle$ and $\sum_{k=1}^{M}\langle\hat{N}_k\rangle = N$. We obtain

$$\mathbf{F}_{\mathrm{SN}} \equiv \max_{\hat{\rho}_{\text{p-sep}}}\mathbf{F}_Q[\hat{\rho}_{\text{p-sep}}, \hat{\mathbf{H}}] = \begin{pmatrix} \langle\hat{N}_1\rangle & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \langle\hat{N}_M\rangle \end{pmatrix}. \tag{2}$$

The convexity of the quantum Fisher matrix ensures that the bound (2) is achieved by a product of pure single-particle states $|\Psi^{(1)}\rangle \otimes \cdots \otimes |\Psi^{(N)}\rangle$. Optimal states must have the property $\langle\hat{h}_k^{(i)}\rangle_{|\Psi^{(i)}\rangle} = 0$ for all k and i, due to $\lambda_{k+} + \lambda_{k-} = 0$, which leads to the diagonal form of $\mathbf{F}_{\mathrm{SN}}$. If all $\langle\hat{N}_k\rangle > 0$, $\mathbf{F}_{\mathrm{SN}}$ is invertible and, according to Eq. (1), defines the multiparameter shot-noise limit $\boldsymbol{\Sigma} \geq \boldsymbol{\Sigma}_{\mathrm{SN}}/\mu \equiv \mathbf{F}_{\mathrm{SN}}^{-1}/\mu = \mathrm{diag}(1/\langle\hat{N}_1\rangle, 1/\langle\hat{N}_2\rangle, \ldots, 1/\langle\hat{N}_M\rangle)/\mu$, i.e., the smallest covariance matrix $\boldsymbol{\Sigma}$ for particle-separable probe states. In particular, we recover the shot-noise $(\Delta\theta_{\mathrm{est}})^2 = 1/\mu N$ [6,7] in the case of a single parameter ($M = 1$). The shot-noise rank $0 \leq r_{\mathrm{SN}} \leq M$, defined as the number of positive eigenvalues of the matrix $\mathbf{F}_Q[\hat{\rho}, \hat{\mathbf{H}}] - \mathbf{F}_{\mathrm{SN}}$, provides the number of linearly independent combinations of the M parameters that can be estimated with sub-shot-noise sensitivity. A rank $r_{\mathrm{SN}} > 0$ can only be achieved by particle-entangled states.

Let us now gain a better understanding of the role of mode entanglement in determining the sensitivity of particle-separable states. Considering a pure particle-product state formally corresponds to sending the N

particles one-by-one (without any classical correlations) through the M-mode interferometer. Each of the particles can be localized in a single mode [MsPs strategy depicted in Fig. 1(b)], or delocalized over several modes (mode entanglement, MePs). We find

$$\mathbf{F}_{\mathrm{MsPs}} \leq \mathbf{F}_Q[|\Psi^{(1)}\rangle \otimes \cdots \otimes |\Psi^{(N)}\rangle, \hat{\mathbf{H}}] \leq \mathbf{F}_{\mathrm{MePs}}. \quad (3)$$

Here $\mathbf{F}_{\mathrm{MePs}}$ is the quantum Fisher matrix obtained by delocalizing each of the particles over all modes according to the weights $p_k^{(i)} = \langle \hat{N}_k \rangle / N$, where $p_k^{(i)} = |\langle \Psi^{(i)} | \lambda_{k+}^{(i)} \rangle|^2 + |\langle \Psi^{(i)} | \lambda_{k-}^{(i)} \rangle|^2$ is the probability to find particle i in mode k. Moreover, $\mathbf{F}_{\mathrm{MsPs}}$ in Eq. (3) is the quantum Fisher matrix obtained from fully localized single-particle states, i.e., $p_k^{(i)} = \delta_{kk_i}$ such that $\sum_{i=1}^N \delta_{kk_i} = \langle \hat{N}_k \rangle$, which is only defined for integer $\langle \hat{N}_k \rangle$. In the inequalities (3) we vary only the distribution of particles among modes, while considering an arbitrary, fixed state preparation within the modes. The result (3) states that, for pure particle-product states, mode entanglement generally leads to a higher sensitivity than strategies based on mode separability.

Both inequalities in (3) become equalities for states with the property $\langle \hat{h}_k^{(i)} \rangle_{|\Psi^{(i)}\rangle} = 0$ for all k and i and, in this case, no advantage due to mode entanglement can be achieved. Optimal states that reach the sensitivity limit (2) are prepared in a balanced superposition of largest and smallest eigenstate within the modes, which ensures that $\langle \hat{h}_k^{(i)} \rangle_{|\Psi^{(i)}\rangle} = 0$. Hence, if $\langle \hat{N}_k \rangle$ is integer, we obtain the same sensitivity for the optimal MePs states [48]

$$|\Psi_{\mathrm{MePs}}\rangle = \bigotimes_{i=1}^{N} \sum_{k=1}^{M} \sqrt{\frac{\langle \hat{N}_k \rangle}{2N}} (|\lambda_{k+}^{(i)}\rangle + |\lambda_{k-}^{(i)}\rangle),$$

where each particle is delocalized over all modes, and optimal MsPs states

$$|\Psi_{\mathrm{MsPs}}\rangle = \bigotimes_{i=1}^{N} \frac{|\lambda_{k_i+}^{(i)}\rangle + |\lambda_{k_i-}^{(i)}\rangle}{\sqrt{2}},$$

where each particle is localized on a single mode k_i such that $\sum_{i=1}^N \delta_{kk_i} = \langle \hat{N}_k \rangle$.

Sensitivity bounds for mode-separable states.—Let us now determine the upper sensitivity limits for general mode-separable states $\hat{\rho}_{\text{m-sep}} = \sum_\gamma p_\gamma \hat{\rho}_{1,\gamma} \otimes \cdots \otimes \hat{\rho}_{M,\gamma}$, where $\hat{\rho}_{k,\gamma}$ is an arbitrary density matrix of mode k. The state-dependent bound

$$\mathbf{F}_Q[\hat{\rho}_{\text{m-sep}}, \hat{\mathbf{H}}] \leq 4\mathbf{\Gamma}[\hat{\rho}_1 \otimes \cdots \otimes \hat{\rho}_M, \hat{\mathbf{H}}] \quad (4)$$

holds, where $\mathbf{\Gamma}[\hat{\rho}_1 \otimes \cdots \otimes \hat{\rho}_M, \hat{\mathbf{H}}] = \mathrm{diag}((\Delta \hat{H}_1)^2_{\hat{\rho}_1}, \ldots, (\Delta \hat{H}_M)^2_{\hat{\rho}_M})$ is the covariance matrix of the product state of reduced density matrices $\hat{\rho}_k = \sum_\gamma p_\gamma \hat{\rho}_{\gamma,k}$ for the different modes k [46]. A maximization of the quantum Fisher matrix over all mode-separable (MS) states with fixed $\langle \hat{N}_k^2 \rangle$ yields

$$\mathbf{F}_{\mathrm{MS}} \equiv \max_{\hat{\rho}_{\text{m-sep}}} \mathbf{F}_Q[\hat{\rho}_{\text{m-sep}}, \hat{\mathbf{H}}] = \begin{pmatrix} \langle \hat{N}_1^2 \rangle & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \langle \hat{N}_M^2 \rangle \end{pmatrix}. \quad (5)$$

This sensitivity limit is thus determined by the fluctuations of the number of particles in all modes. It should be noticed that $\mathbf{F}_{\mathrm{MS}} \geq \mathbf{F}_{\mathrm{SN}}$, since $\langle \hat{N}_k^2 \rangle \geq \langle \hat{N}_k \rangle$. Mode entanglement is therefore not necessary to overcome the multiparameter shot noise.

For a fixed number of particles N_k in each mode, Eq. (5) reduces to $\mathbf{F}_{\mathrm{MS}} = \mathrm{diag}(N_1^2, \ldots, N_M^2)$. The bound is saturated by a product of NOON states,

$$|\Psi_{\mathrm{MsPe}}\rangle = \bigotimes_{k=1}^{M} \frac{|N_k, +\rangle + |N_k, -\rangle}{\sqrt{2}},$$

with full N_k-particle entanglement in each mode k. Here $|N_k, \pm\rangle_k$ describes N_k particles in the state with eigenvalue $\lambda_{k\pm}$. In the single-parameter case ($M = 1$), the notion of entanglement among different parameter-encoding modes does not exist, and strategies with maximal particle entanglement recover the Heisenberg limit, i.e., $(\Delta \theta_{\mathrm{est}})^2 = 1/\mu N^2$, achieved by NOON states [6,7].

Furthermore, for fixed N_k, the stepwise enhancement of sensitivity from the bound $\mathbf{F}_{\mathrm{SN}}$ for particle-separable states to the bound $\mathbf{F}_{\mathrm{MS}}$ involving full particle entanglement can be probed by deriving bounds for quantum states with a maximal number of entangled particles [8] in each mode. Specifically, $\mathbf{P}$-producible states $\hat{\rho}_{\mathbf{P}\text{-prod}}$ are those that contain not more than $1 \leq P_k \leq N_k$ entangled particles in mode k with $\mathbf{P} = \{P_1, \ldots, P_M\}$. We obtain $\mathbf{F}_{\mathrm{MS}}^{\mathbf{P}} \equiv \max_{\hat{\rho}_{\mathbf{P}\text{-prod}}} \mathbf{F}_Q[\hat{\rho}_{\mathbf{P}\text{-prod}}]$ with $\mathbf{F}_{\mathrm{MS}}^{\mathbf{P}} = \mathrm{diag}(s_1 P_1^2 + r_1^2, \ldots, s_M P_M^2 + r_M^2)$, where $s_k = \lfloor N_k / P_k \rfloor$ and $r_k = N_k - s_k P_k$. These bounds are saturated by products of s_k NOON states of P_k particles and a single NOON state of r_k particles in each mode. In general, we obtain the hierarchy

$$\mathbf{F}_{\mathrm{MS}} \geq \mathbf{F}_{\mathrm{MS}}^{\mathbf{P}} \geq \mathbf{F}_{\mathrm{MS}}^{\mathbf{P}'} \geq \mathbf{F}_{\mathrm{SN}}, \quad (6)$$

if $P_k \geq P'_k$ for all $k = 1, \ldots, M$. We recover $\mathbf{F}_{\mathrm{SN}}$ for $\mathbf{P} = \{1, \ldots, 1\}$, i.e., in the complete absence of particle entanglement and $\mathbf{F}_{\mathrm{MS}}$ for $\mathbf{P} = \{N_1, \ldots, N_M\}$, i.e., maximal particle entanglement in each mode.

The multiparameter Heisenberg limit.—In the following, we identify an ultimate, saturable lower bound on $\mathbf{n}^T \mathbf{\Sigma} \mathbf{n}$ for arbitrary $\mathbf{n}$, minimized over all quantum states. We first derive a weak form of the multiparameter CRB and QCRB,

$$\mathbf{n}^T\mathbf{\Sigma}\mathbf{n} \geq \frac{1}{\mu\mathbf{n}^T\mathbf{F}\mathbf{n}} \geq \frac{1}{\mu\mathbf{n}^T\mathbf{F}_Q\mathbf{n}}, \quad (7)$$

respectively, where we chose the normalization $|\mathbf{n}|^2 = 1$. The inequalities (7) can be derived without assuming the existence of the inverse of $\mathbf{F}$ and $\mathbf{F}_Q$ [43]. While Eq. (1) is a matrix inequality and provides bounds for all possible $\mathbf{n}^T\mathbf{\Sigma}\mathbf{n} = \Delta^2(\sum_k n_k\theta_{\mathrm{est},k})$ at once, Eq. (7) expresses a bound for a single, specific, but arbitrary, linear combination of parameters specified by the vector $\mathbf{n}$ [36,37,39]. Since $\mathbf{n}^T\mathbf{A}^{-1}\mathbf{n} \geq (\mathbf{n}^T\mathbf{A}\mathbf{n})^{-1}$ holds for all $\mathbf{n}$ and all matrices $\mathbf{A}$, whenever $\mathbf{A}^{-1}$ exists, the chain of inequalities (7) is weaker than (1). This also means that saturation of the weak bound (7) implies saturation of (1) whenever it exists.

The state-dependent bound $\mathbf{F}_Q[\hat{\rho}, \hat{\mathbf{H}}] \leq 4\mathbf{\Gamma}[\hat{\rho}, \hat{\mathbf{H}}]$ holds for arbitrary quantum states $\hat{\rho}$, where $\mathbf{\Gamma}[\hat{\rho}, \hat{\mathbf{H}}]$ is the full covariance matrix. Furthermore, an achievable upper limit on the covariances is given as $\mathbf{n}^T\mathbf{\Gamma}[\hat{\rho}, \hat{\mathbf{H}}]\mathbf{n} \leq \mathbf{n}^T\mathbf{\Gamma}^{\mathbf{n}}[\hat{\rho}, \hat{\mathbf{H}}]\mathbf{n}$ for arbitrary $\mathbf{n}$, where $\mathbf{\Gamma}^{\mathbf{n}}[\hat{\rho}, \hat{\mathbf{H}}] = \mathbf{v}^{\mathbf{n}}_{\hat{\rho}}\mathbf{v}^{\mathbf{n}T}_{\hat{\rho}}$, and $\mathbf{v}^{\mathbf{n}}_{\hat{\rho}}$ is a vector with elements $\epsilon_k(\Delta\hat{H}_k)_{\hat{\rho}}$, for $k=1,\ldots,M$ and $\epsilon_k = \mathrm{sgn}(n_k)$. Maximizing over all quantum states with fixed $\langle\hat{N}_k^2\rangle$ yields $\mathbf{n}^T\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}}\mathbf{n} \equiv \max_{\hat{\rho}}\mathbf{n}^T\mathbf{F}_Q[\hat{\rho}, \hat{\mathbf{H}}]\mathbf{n}$ with

$$\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}} = \begin{pmatrix} \langle\hat{N}_1^2\rangle & \cdots & \epsilon_1\epsilon_M\sqrt{\langle\hat{N}_1^2\rangle\langle\hat{N}_M^2\rangle} \\ \vdots & \ddots & \vdots \\ \epsilon_1\epsilon_M\sqrt{\langle\hat{N}_1^2\rangle\langle\hat{N}_M^2\rangle} & \cdots & \langle\hat{N}_M^2\rangle \end{pmatrix}, \quad (8)$$

where subscript HL represents the Heisenberg limit. Notice that Eq. (8) can be written as $\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}} = \mathbf{v}^{\mathbf{n}}\mathbf{v}^{\mathbf{n}T}$, where $\mathbf{v}^{\mathbf{n}} = (\epsilon_1\sqrt{\langle\hat{N}_1^2\rangle}, \ldots, \epsilon_M\sqrt{\langle\hat{N}_M^2\rangle})$. $\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}}$ is a singular rank-one matrix that cannot be inverted: this implies that the multiparameter Cramér-Rao bound (1) is not defined, while its weaker form (7) is.

The multiparameter Heisenberg limit is defined on the basis of Eqs. (7) and (8) as $\mathbf{n}^T\mathbf{\Sigma}\mathbf{n} \geq \mathbf{n}^T\mathbf{\Sigma}^{\mathbf{n}}_{\mathrm{HL}}\mathbf{n} \equiv (\mu\mathbf{n}^T\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}}\mathbf{n})^{-1}$, and is saturated by the states

$$|\Psi^{\mathbf{n}}_{\mathrm{MePe}}\rangle = \frac{1}{\sqrt{2}}(|N_1, \epsilon_1\rangle \otimes |N_2, \epsilon_2\rangle \otimes \ldots \otimes |N_M, \epsilon_M\rangle + |N_1, -\epsilon_1\rangle \otimes |N_2, -\epsilon_2\rangle \otimes \ldots \otimes |N_M, -\epsilon_M\rangle), \quad (9)$$

for arbitrary $\mathbf{n}$. Both the states (9) and the matrix (8) depend on the sign of the components of $\mathbf{n}$. The states (9) contain entanglement among all modes and among all of the N_k particles in each mode. In the single-mode case ($M = 1$) this reduces to the standard NOON state and we again recover the Heisenberg limit $(\Delta\theta_{\mathrm{est}})^2 = 1/\mu N^2$.

Sensitivity bounds for separability among specific modes.—To probe the transition from complete mode separability to full M-mode entanglement, we derive bounds for quantum states that contain entanglement only between specific subsets of the M modes. States that are mode separable in the partition $\Lambda = \mathcal{A}_1|\ldots|\mathcal{A}_L$, where the $\mathcal{A}_m$ describe groups of modes, can be written as $\hat{\rho}_{\Lambda\text{-sep}} = \sum_\gamma p_\gamma\hat{\rho}_{\gamma,\mathcal{A}_1} \otimes \ldots \otimes \hat{\rho}_{\gamma,\mathcal{A}_L}$, with density matrices $\hat{\rho}_{\gamma,\mathcal{A}_m}$ on $\mathcal{A}_m$. Following [47,49], we obtain the state-dependent upper bound

$$\mathbf{F}_Q[\hat{\rho}_{\Lambda\text{-sep}}, \hat{\mathbf{H}}] \leq 4\mathbf{\Gamma}[\hat{\rho}_{\mathcal{A}_1} \otimes \cdots \otimes \hat{\rho}_{\mathcal{A}_L}, \hat{\mathbf{H}}],$$

where $\hat{\rho}_{\mathcal{A}_m} = \sum_\gamma p_\gamma\hat{\rho}_{\gamma,\mathcal{A}_m}$ is the reduced density matrix for $\mathcal{A}_m$. This matrix is obtained from the full covariance matrix $\mathbf{\Gamma}[\hat{\rho}_{\Lambda\text{-sep}}, \hat{\mathbf{H}}]$ by removing all off-diagonal elements that describe correlations between the $\mathcal{A}_m$, while retaining the correlations within each of the $\mathcal{A}_m$.

By combining the methods used for the derivation of Eqs. (4) and (8), the sensitivity limits $\mathbf{F}^{\mathbf{n}}_{\Lambda}$ for the states $\hat{\rho}_{\Lambda\text{-sep}}$ can be obtained. The result is obtained from $\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}}$ by setting to zero the off-diagonal elements that describe mode correlations across different groups $\mathcal{A}_m$. These matrices interpolate between the sensitivity limits of fully M-mode entangled states $\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}}$ and fully mode-separable states $\mathbf{F}_{\mathrm{MS}}$. This is expressed by the hierarchy

$$\mathbf{n}^T\mathbf{F}^{\mathbf{n}}_{\mathrm{HL}}\mathbf{n} \geq \mathbf{n}^T\mathbf{F}^{\mathbf{n}}_{\Lambda_A}\mathbf{n} \geq \mathbf{n}^T\mathbf{F}^{\mathbf{n}}_{\Lambda_B}\mathbf{n} \geq \mathbf{n}^T\mathbf{F}_{\mathrm{MS}}\mathbf{n}, \quad (10)$$

which holds for all $\mathbf{n}$ and any pair of partitions Λ_A, Λ_B, such that the subsets in Λ_A can be obtained by joining subsets of Λ_B. The sensitivity $\mathbf{F}^{\mathbf{n}}_{\Lambda}$ can be reached by mode products of states of the form (9) for each of the $\mathcal{A}_m$. For a fixed number of particles, the lowest (fully mode separable) bound in (10) constitutes the largest bound in the hierarchy (6) as a function of the number of entangled particles.

Enhancement of sensitivity by multimode and multiparticle entanglement.—The role of mode entanglement for quantum multiparameter estimation has been studied intensively over recent years [29–39]. No general consensus on the possible advantage of mode entanglement has been reached. Many studies have focused their analysis on the sum $\sum_{k=1}^M(\Delta\theta_{\mathrm{est},k})^2$ of single-parameter sensitivities or the weighted sum $\sum_{k=1}^M w_k^2(\Delta\theta_{\mathrm{est},k})^2$ with $w_k \geq 0$. Both these figures of merit ignore possible correlations between the parameters and lead to the result that mode correlations can only have a detrimental influence on the sensitivity. This can be seen by taking the trace on the QCRB (1), $\sum_{k=1}^M(\Delta\theta_{\mathrm{est},k})^2 \geq \sum_{k=1}^M(\mathbf{F}_Q^{-1})_{kk}$, which is always larger or equal to the sum of single-parameter sensitivities $\sum_{k=1}^M(\mathbf{F}_Q)_{kk}^{-1}$ (see, e.g., [45]). Mode entanglement establishes correlations that can lead to an enhancement of phase sensitivity only when considering a figure of merit that includes the covariances among the parameters. This possibility is fully accounted for when studying bounds for $\boldsymbol{\Sigma}$ in full matrix form, as done in this Letter.

The figure of merit $\mathbf{n}^T\boldsymbol{\Sigma}\mathbf{n} = \sum_{kl=1}^{M} n_k n_l \mathrm{Cov}(\theta_{\mathrm{est},k}, \theta_{\mathrm{est},l})$ may include covariances between the parameters, in addition to the weighted sum of single-parameter variances. Let us illustrate the quantum gain due to multimode and multiparticle entanglement in (7) using the example of an equally weighted linear combination of parameters, $|n_k| = 1/\sqrt{M}$ with arbitrary signs, and an equal and integer number of $N_k = \bar{N} = N/M$ particles in each mode. We determine the maximal sensitivity $S^{\max}_{M_e,P_e} = \max_{\hat{\rho}_{M_e,P_e}} \mathbf{n}^T \mathbf{F}_Q[\hat{\rho}_{M_e,P_e}, \hat{\mathbf{H}}]\mathbf{n}$ for quantum states $\hat{\rho}_{M_e,P_e}$ with up to $P_e \leq N/M$ entangled particles in each mode and up to $M_e \leq M$ entangled modes. Notice that $P_e = 1$ does not necessarily imply full particle separability since it only demands that there is no entanglement among the particles that enter the same mode. If additionally $M_e = 1$, we have a fully mode- and particle-separable state with shot-noise sensitivity $S^{\max}_{1,1} = N$. The gain factor $G_{M_e,P_e} = S^{\max}_{M_e,P_e}/S^{\max}_{1,1} = (sP_e^2 + r^2)(uM_e^2 + v^2)/(NM)$ expresses the largest achievable quantum enhancement over the shot-noise limit, where $s = \lfloor \bar{N}/P_e \rfloor$, $r = \bar{N} - sP_e$, $u = \lfloor M/M_e \rfloor$, and $v = M - uM_e$. Special cases of interest are given by

$$G_{1,1} = 1, \qquad G_{1,\bar{N}} = \bar{N},$$
$$G_{M,1} = M, \qquad G_{M,\bar{N}} = \bar{N}M.$$

We observe that local particle entanglement in each mode can achieve an enhancement of up to $\bar{N}$ (corresponding to the number of entangled particles per mode), while mode entanglement can increase the sensitivity by a factor of M (corresponding to the number of entangled modes). By combining both, we can achieve a gain factor up to $\bar{N}M$.

Finally, we remark that our results can be extended to provide bounds on more general figures of merit $\mathrm{Tr}\{\mathbf{W}\boldsymbol{\Sigma}\}$, where $\mathbf{W} \geq 0$ is an arbitrary weight matrix. The sensitivity bounds and optimal states are obtained by performing a mode transformation that diagonalizes the matrix $\mathbf{W}$ [43].

Conclusions.—We identified sensitivity bounds and optimal states for the simultaneous estimation of multiple parameters in multimode interferometers and characterized the interplay between mode and particle entanglement. Our bounds are given in terms of the full Fisher matrix and are valid for any linear combination of estimators taking into account correlations between parameters. In particular, this led to the identification of the multiparameter shot-noise limit in matrix form—corresponding to the maximum sensitivity achievable by particle-separable states—and the Heisenberg limit—corresponding to the maximum sensitivity achievable for any probe state. Particle entanglement is thus necessary to overcome the multiparameter shot-noise limit with a fixed number of probe particles. When correlations between the parameters are present, the multiparameter sensitivity further grows with the number of entangled modes. This reveals the possibility to achieve a collective quantum enhancement for the estimation of multiple parameters beyond an optimized point-by-point estimation of individual parameters.

Our results build the foundation for the development of genuine quantum technological strategies in applications that rely on the precise acquisition of an ensemble of parameters, such as sensing of spatially distributed fields and imaging techniques. Experimental realizations are possible with existing technology in a wide range of atomic and photonic systems that provide coherent access to multiple modes (see, e.g., [5,20,25,26,33,49]).

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