

Violation of the Harris-Barghathi-Vojta Criterion

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In 1974, Harris proposed his celebrated criterion: Continuous phase transitions in d -dimensional systems are stable against quenched spatial randomness whenever $d\nu > 2$, where ν is the clean critical exponent of the correlation length. Forty years later, motivated by violations of the Harris criterion for certain lattices such as Voronoi-Delaunay triangulations of random point clouds, Barghathi and Vojta put forth a modified criterion for topologically disordered systems: $a\nu > 1$, where a is the disorder decay exponent, which measures how fast coordination number fluctuations decay with increasing length scale. Here we present a topologically disordered lattice with coordination number fluctuations that decay as slowly as those of conventional uncorrelated randomness, but for which the clean universal behavior is preserved, hence violating even the modified criterion.

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Spatial disorder arising from impurities, defects, and other inhomogeneities is intrinsic to most real systems. Moreover, disorder, in the form of random lattices, provides a useful theoretical tool for the discretization of nontrivial spaces in fields ranging from classical equilibrium statistical systems to quantum gravity [1], and references therein]. It is therefore important from both theoretical and experimental standpoints to understand the influence of quenched randomness. In particular, with respect to the critical behavior of physical systems, the fundamental question is whether disorder affects the nature of phase transitions, and a number of arguments and criteria have been offered.

The original relevance criterion, proposed by Harris [2,3], states that a second-order transition in a d -dimensional system, with clean correlation length exponent ν [4], is stable against quenched spatial disorder if $d\nu > 2$. A relevant example is the 2D contact process (CP), for which that inequality is violated [5], and references therein]. When placed on a regular lattice with random dilutions, the CP phase transition does not show clean universal behavior, but is instead controlled by an infinite-randomness critical point, where the usual algebraic scaling relations are replaced by a logarithmic time evolution [5,6]. However, a different behavior emerges when the CP is placed on another kind of spatially disordered lattice, namely a Voronoi-Delaunay (VD) construction, where any two of the (randomly distributed) sites are connected whenever their Voronoi (or proximity) cells share a common edge. Here, clean universal properties have been verified [7], in clear contradiction with Harris' criterion. In order to address these puzzling results, Barghathi and Vojta [8] refined the original Harris criterion in terms of spatial anticorrelations in the coordination number fluctuations, offering an explanation for the preserved universality of VD lattices. This modified condition, which we call Harris-Barghathi-Vojta (HBV) criterion, posits that

quenched topological disorder is irrelevant with respect to the stability of the phase transition in systems that satisfy

$$a\nu > 1, \quad (1)$$

where a is the dimension-dependent *disorder decay exponent*, defined by the relation

$$\sigma_Q \sim L_b^{-a}. \quad (2)$$

In Eq. (2), the lattice under consideration is subdivided into spatial blocks of length L_b and σ_Q is the standard deviation of the block-averaged coordination numbers from the asymptotic mean value. Note that for uncorrelated disorder $a = d/2$ and Eq. (1) corresponds to the original Harris criterion. In particular, in Ref. [8] it is shown that in 2D, $a = 1$ for uncorrelated disorder (e.g., in regular lattices with random dilutions), whereas $a = 3/2$ for VD lattices. Hence, the modified criterion Eq. (1) successfully explains earlier results of the Ising model and the contact process on those geometries (see Table I for a compilation of predictions and observations in 2D).

TABLE I. Predictions of the HBV criterion for the 2D Ising and directed percolation (DP) universality classes as well as observations from numerical studies with respect to whether the clean universal behavior is present (\checkmark) or not (\times) for the respective model.

d	Class	Lattice	$a\nu$	Prediction	Observation
2	Ising	VD	3/2	\checkmark	\checkmark [9–11]
2	Ising	Diluted	1	\dots	marginal [12–27]
2	DP	VD	1.100	\checkmark	\checkmark [7,28]
2	DP	Diluted	0.733	\times	\times [5,6,29]
2	DP	VD ⁺	0.733	\times	\checkmark [this work]

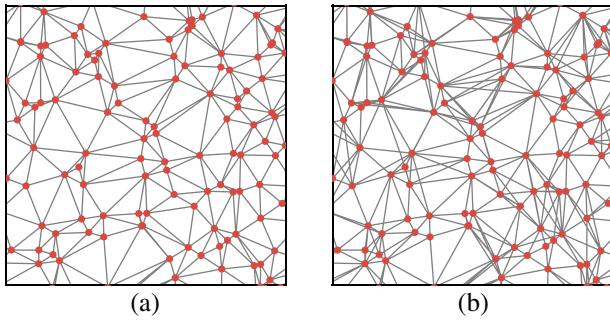


FIG. 1. (a) Sample of a usual VD triangulation of a Poissonian point cloud. (b) The lattice we introduce in this Letter, VD^+ , constructed from (a) by adding random local bonds.

In this Letter, we define the VD^+ lattice: a Voronoi-Delaunay triangulation of a Poissonian point cloud, furnished with additional local bonds, see Fig. 1. This lattice is constructed from a VD triangulation of N sites, to which kN bonds between next-nearest neighbors are randomly added (we select $k = 1$), resulting in a lattice with a total coordination number of exactly $(6 + 2k)N$. This latter constraint is relevant for the applicability of the HBV criterion [8,30].

We find the disorder decay exponent for the VD^+ to be $a = 1$ (see Fig. 2). This follows from the additional bonds being a source of uncorrelated disorder, which decays more slowly than the coordination number fluctuations of the original VD lattice. This interpretation is corroborated by the measurement of $\sigma_Q(L_b)$ for the additional bonds alone: as can be seen in Fig. 2, both curves are almost identical, demonstrating the dominance of the additional bonds. The HBV criterion therefore predicts that our construction should not display clean universal behavior for any universality class with $\nu < 1$, such as directed percolation

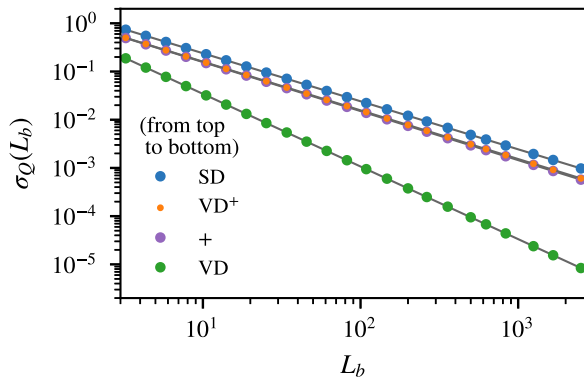


FIG. 2. Coordination number fluctuations on different length scales for VD^+ , ordinary VD, the set of additional bonds that distinguishes both (+), and a site-diluted square lattice (SD). The curves are found to decay as $\sigma_Q \sim L_b^{-3/2}$ for the Voronoi-Delaunay (VD) triangulation, and as $\sigma_Q \sim L_b^{-1}$ for the other lattices. For the analysis, 100 independent realizations of size $L = 5000$ have been used for each lattice. Note that the data points for VD^+ and (+) almost coincide.

(DP). In order to verify this prediction, we perform extensive numerical simulations of the contact process on the 2D VD^+ lattice and find strong indication of clean universal behavior, contradicting the HBV prediction.

The contact process is a simple nonequilibrium lattice model which exhibits a continuous phase transition belonging to the directed percolation universality class. Each site is in either of two states, active (infected) or inactive (healthy). During the time evolution of the system, active sites can infect random neighbors with probability p or recover (i.e., become inactive) with complementary probability $1 - p$. After each time step (infection attempt or self-recovery) the simulation time is incremented by $\Delta t = 1/N_a$, where N_a is the number of active particles in the system prior to the update. As soon as the system enters the so-called absorbing state where the whole lattice is empty, the dynamic terminates.

In order to determine the critical point p_c that separates the absorbing (subcritical) and active (supercritical) phase, we conduct simulations starting from a single active seed [31,32] on an otherwise empty VD^+ lattice of size $L = 6000$, allowing for times up to $T = 10^5$. Unwanted finite size effects are avoided by ensuring that cluster diameters remain smaller than L . We use 2000 independent disorder realizations with 2000 runs each. For $p_c = 0.589775(3)$ we find that the average cluster size follows a power-law behavior, $\langle N_a(t) \rangle \sim t^\theta$, as expected in the case of clean universality. The uncertainty of the critical point is estimated from simulations at probabilities $p = p_c \pm \delta p$, $\delta p \ll 1$, such that the corresponding time evolution barely, but noticeably, bends away from a straight line in a double-logarithmic plot. A linear fit in the region $10^3 < t < 10^5$ yields an exponent $\theta = 0.230(9)$, where the error stems from the uncertainty of the critical point, a result compatible with the reference value $\theta = 0.2293(4)$ of the clean DP class [33].

Once the critical point is known, we can obtain the quasistationary (QS) density ρ , as well as the average lifetime of the quasistationary state, τ , directly at p_c . For details on the QS method, we refer the reader to Refs. [34,35]. As emphasized in Ref. [36], very long

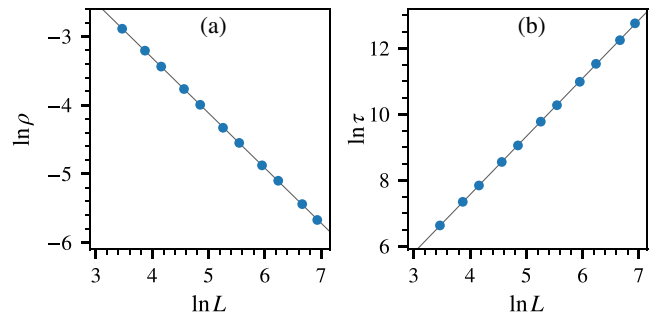


FIG. 3. Quasistationary density (a) and lifetime (b) as a function of the linear lattice size L , ranging from 32 to 1024. The slope of $\ln \rho$ yields β/ν , whereas $\ln \tau$ yields z . The dots represent averages over 320 independent disorder realizations. The error bars are smaller than the symbol sizes.

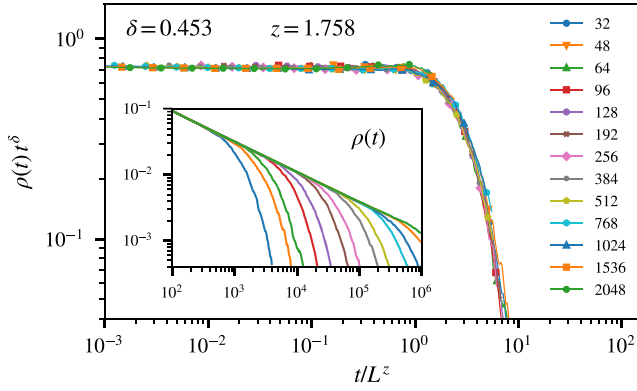


FIG. 4. Finite size data collapse of simulations starting from a fully occupied VD^+ lattice at the critical point $p_c = 0.589775$, using the critical exponent estimates stated in the figure. All curves are averages over 500 disorder realizations with 5 runs per realization. L denotes the linear system size. The inset shows the nonrescaled density as a function of time.

simulation times are needed to reliably detect a deviation from the universal behavior. We therefore use 320 independent realizations of the VD^+ lattice, ranging from $L = 32$ to 1024, and simulate the contact process for a time of 10^8 . The measurements are taken after a generously sized relaxation period of 7×10^7 . The resulting data, shown in Fig. 3, reveal that $\ln \rho$ and $\ln \tau$ follow straight lines, and linear fits yield the exponents

$$\beta/\nu = 0.800(5), \quad (3)$$

$$z = 1.758(14), \quad (4)$$

where the errors reflect both the uncertainty of the critical point and the fluctuations of the individual data points, with the latter accounting for roughly one third of the total uncertainties. Both exponents are compatible with reference values, $\beta/\nu = 0.797(3)$ and $z = 1.7674(6)$ [33]. In fact, the power law dependence displayed by τ is in itself an indication of clean universal behavior, since an exponential scaling, $\ln \tau \sim L^\psi$, is expected for uncorrelated randomness [36].

In another set of simulations we measured the density of particles $\rho(t)$, starting from a fully occupied lattice. For finite-size systems at the critical point it is expected to follow

$$\rho(L, t) = t^{-\delta} \tilde{\rho}(t/L^z), \quad (5)$$

where $\tilde{\rho}$ is a universal scaling function. Performing simulations on lattices from $L = 32$ to 2048, running for times up to 10^6 and using 500 disorder realizations with 5 runs on each realization, we obtain the data presented in Fig. 4, which show the finite-size data collapse, as well as the original measurement of $\rho(t)$ in the inset. The data collapse is performed by, using the estimate [Eq. (4)] for z ,

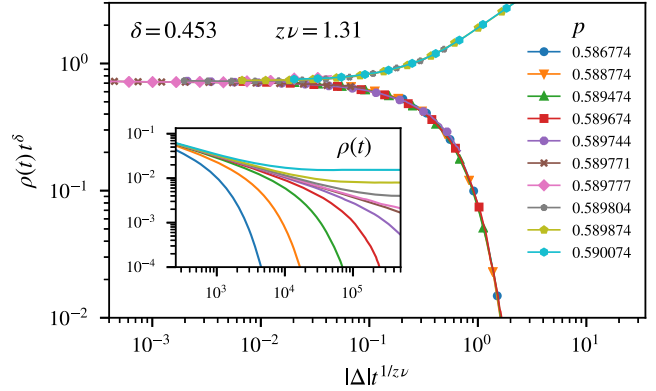


FIG. 5. Data collapse in the off-critical region, starting from a fully occupied VD^+ lattice of size $L = 2048$ using the critical exponent estimates given in the figure. All curves are averages over 250 disorder realizations, with 5 runs per realization. The symbol Δ denotes the distance from the critical point $p_c = 0.589775$. The inset shows the nonrescaled density as a function of time.

determining the value of δ for which the curves in Fig. 4 superpose each other. This gives us the result

$$\delta = 0.453(5). \quad (6)$$

This estimate is again compatible with the clean reference value $\delta = 0.4523(10)$ from Ref. [33].

We can obtain a further exponent, $\nu_{\parallel} = z\nu$, from off-critical simulations, using the relation

$$\rho(\Delta, t) = t^{-\delta} \hat{\rho}(\Delta t^{1/\nu_{\parallel}}), \quad (7)$$

where $\Delta = p - p_c$ denotes the distance from criticality and $\hat{\rho}$ is a scaling function. Using the estimate for δ obtained above, we produce the curve collapse of Fig. 5 when we set

$$\nu_{\parallel} = 1.31(2), \quad (8)$$

in accordance with the reference value, $\nu_{\parallel} = 1.292(4)$ [33].

In summary, all the exponents we obtain for the two-dimensional VD^+ lattice in numerical simulations of the contact process, turn out to be fully compatible with the ones for the clean universality class. In particular, the data collapse plots are flawless within numerical precision. This provides strong evidence that the phase transition on the VD^+ is in fact controlled by the clean universal behavior of the DP class, hence indicating that the VD^+ violates the HBV criterion. Clearly, regardless of our extensive numerical effort, including large lattice sizes and long simulation times, the possibility of a crossover away from the clean universal behavior for extremely long times can not be ruled out, though very large crossover times are unlikely for VD , as reasoned in Ref. [8]. Moreover, it could be argued that the ‘‘canonical’’ constraint (i.e., fixed number of additional bonds) imposed in the VD^+ construction leads to spurious results. In particular, Ref. [37] shows that although

the canonical and its “grand-canonical” counterpart (i.e., without global constraints) belong to the same universality class, the approach to the asymptotic behavior can be much slower in the former. We therefore also implemented a grand-canonical construction of the VD^+ lattice, VD_{GC}^+ , where the additional bonds are now drawn independently with a certain probability for each node, resulting in a total coordination number which fluctuates around $(6 + 2)N$. When we repeat all simulations described above on the VD_{GC}^+ , we again obtain results fully compatible with the clean universal behavior of the DP class [38], revealing that the constraint has no influence on our conclusions.

Furthermore our findings strengthen an assumption two of us made recently [39]: that fluctuations in the coordination number do not exclusively determine the stability of the phase transition against quenched disorder. A further study on this subject, which considers the contact process on the topologically disordered lattice with constant coordination number (CC) also introduced in Ref. [39], is currently in preparation. In particular, since the VD^+ lattice is clearly not planar (triangulations are maximal sets of nonintersecting edges [40], thus added bonds necessarily cross existing ones), our results answer a question raised in Ref. [39], as planarity is shown *not* to be a necessary condition for stability of the phase transition against quenched spacial disorder.

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[1] W. Janke and R. Villanova, *Phys. Rev. B* **66**, 134208 (2002).
 [2] A. B. Harris, *J. Phys. C* **7**, 1671 (1974).
 [3] A. B. Harris, *J. Phys. Condens. Matter* **28**, 421006 (2016).
 [4] Note that in the context of nonequilibrium systems the spatial correlation length exponent is usually denoted by ν_{\perp} .
 [5] T. Vojta, A. Farquhar, and J. Mast, *Phys. Rev. E* **79**, 011111 (2009).
 [6] T. Vojta and M. Dickison, *Phys. Rev. E* **72**, 036126 (2005).
 [7] M. M. de Oliveira, S. G. Alves, S. C. Ferreira, and R. Dickman, *Phys. Rev. E* **78**, 031133 (2008).
 [8] H. Barghathi and T. Vojta, *Phys. Rev. Lett.* **113**, 120602 (2014).
 [9] W. Janke, M. Katoot, and R. Villanova, *Phys. Lett. B* **315**, 412 (1993).

[10] W. Janke, M. Katoot, and R. Villanova, *Phys. Rev. B* **49**, 9644 (1994).
 [11] F. Lima, J. Moreira, J. Andrade, and U. Costa, *Physica (Amsterdam)* **283A**, 100 (2000).
 [12] V. S. Dotsenko and V. Dotsenko, *ZhETF* **33**, 40 (1981) [*V. S. Dotsenko and V. Dotsenko, JETP Lett.* **33**, 37 (1981)].
 [13] G. Jug, *Phys. Rev. B* **27**, 4518 (1983).
 [14] R. Shankar, *Phys. Rev. Lett.* **58**, 2466 (1987).
 [15] B. Shalaev, *Phys. Rep.* **237**, 129 (1994).
 [16] A. W. Ludwig, *Nucl. Phys.* **B285**, 97 (1987).
 [17] J.-K. Kim and A. Patrascioiu, *Phys. Rev. B* **49**, 15764 (1994).
 [18] J.-K. Kim and A. Patrascioiu, *Phys. Rev. Lett.* **72**, 2785 (1994).
 [19] W. Selke, *Phys. Rev. Lett.* **73**, 3487 (1994).
 [20] K. Ziegler, *Phys. Rev. Lett.* **73**, 3488 (1994).
 [21] R. Kühn, *Phys. Rev. Lett.* **73**, 2268 (1994).
 [22] J.-K. Kim, *Phys. Rev. B* **61**, 1246 (2000).
 [23] A. Gordillo-Guerrero, R. Kenna, and J. Ruiz Lorenzo, *AIP Conf. Proc.* **1198**, 42 (2009).
 [24] P. H. L. Martins and J. A. Plascak, *Phys. Rev. E* **76**, 012102 (2007).
 [25] N. G. Fytas and A. Malakis, *Phys. Rev. E* **81**, 041109 (2010).
 [26] N. G. Fytas and P. E. Theodorakis, *Eur. Phys. J. A* **86**, 30 (2013).
 [27] Q. Zhu, X. Wan, R. Narayanan, J. A. Hoyos, and T. Vojta, *Phys. Rev. B* **91**, 224201 (2015).
 [28] M. M. de Oliveira, S. G. Alves, and S. C. Ferreira, *Phys. Rev. E* **93**, 012110 (2016).
 [29] T. Vojta and M. Y. Lee, *Phys. Rev. Lett.* **96**, 035701 (2006).
 [30] Note that, from geometrical constraints, any triangular lattice with N vertices on a torus has exactly $6N$ bonds [8].
 [31] H. Hinrichsen, *Adv. Phys.* **49**, 815 (2000).
 [32] M. Henkel, H. Hinrichsen, and S. Lübeck, *Non-Equilibrium Phase Transitions—Volume 1: Absorbing Phase Transitions* (Springer, New York, 2009).
 [33] R. Dickman, *Phys. Rev. E* **60**, R2441 (1999).
 [34] R. Dickman and M. M. de Oliveira, *Physica (Amsterdam)* **357A**, 134 (2005).
 [35] M. M. de Oliveira and R. Dickman, *Phys. Rev. E* **71**, 016129 (2005).
 [36] M. M. de Oliveira and S. C. Ferreira, *J. Stat. Mech.* (2008) P11001.
 [37] A. Aharony, A. B. Harris, and S. Wiseman, *Phys. Rev. Lett.* **81**, 252 (1998).
 [38] The location of the critical point of the VD_{GC}^+ is slightly lower (by about 0.2%) compared to the ordinary (canonical) VD^+ .
 [39] M. Schrauth, J. A. J. Richter, and J. S. E. Portela, *Phys. Rev. E* **97**, 022144 (2018).
 [40] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd ed. (Wiley, Chichester, 2000).