Monogamy of Particle Statistics in Tripartite Systems Simulating Bosons and Fermions

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In the quantum world, correlations can take the form of entanglement which is known to be monogamous. In this Letter we show that another type of correlation, indistinguishability, is also restricted by some form of monogamy. Namely, if particles A and B simulate bosons, then A and C cannot perfectly imitate fermions. Our main result consists in demonstrating to what extent it is possible.

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Introduction.—If a particle in box A is indistinguishable from a particle in box B, then all the parameters describing their states, apart from their positions, are the same. Such particles are strongly correlated because the state of one of them automatically determines the state of the other. In quantum theory, the above system is described by quantum correlations—i.e., by an entangled state which is either symmetric (bosons) or antisymmetric (fermions).

Quantum entanglement cannot be efficiently explained within any reasonable classical theory [1]. Even within the quantum theory, the description of entanglement is not simple if it occurs between more than two particles. First, in these cases entanglement can take many inequivalent forms [1–3]. For instance, already for three particles there is more than one way to define a maximally entangled state. In addition, quantum correlations are restricted by the so-called monogamy bounds [4–6]. Specifically, if particles A and B are maximally entangled, then no other particle C can be entangled with either A or B.

In multipartite systems, indistinguishability can also take different forms. For example, three particles can be prepared in a tripartite symmetric state (three bosons), an antisymmetric state (three fermions), or some general state allowing different pairs of particles to obey different statistics. It is intuitively clear that while particles in boxes A and B behave like bosons, it is not possible for particles in boxes B and C to behave like fermions. This could be interpreted as a manifestation of some form of a monogamy.

However, in order to truly speak of a monogamy between different types of statistics, one needs to consider a more detailed problem: given that particles in boxes A and B behave like bosons with probability p, what is the probability that particles in B and C behave like fermions? Here we answer the above question. More precisely, we derive trade-off bounds on the simulability of bosons and fermions in tripartite systems and represent them in a simple graphical way.

In this Letter, we say that particles in a given state simulate some type of particle statistics, rather than that they are particles of a certain type (bosons, fermions, etc.). This is because in our considerations we assume that the same system can be prepared in states corresponding to different statistics. In general, this assumption requires the application of asymmetric operations, which is only possible if the corresponding particles are distinguishable. Therefore, we prefer to say that here we discuss the simulation of indistinguishability with entanglement.

To justify our approach, let us observe that although in most cases one deals with fundamentally indistinguishable bosons or fermions, there are many situations in which particles can be effectively distinguished. This is because in reality each particle has more than one degree of freedom (d.o.f.) (e.g., position and spin or polarization), and the symmetry or antisymmetry requirement applies to the whole state of the system. If one observes only a single d.o.f. and ignores (or is unaware of) the remaining ones, the observed state does not need to obey the fundamental symmetry of the investigated particles.

In fact, various effects can occur when auxiliary d.o.f. act as effective particle labels. For instance, two polarization-entangled photons can be distinguished by different momenta—say a photon moving to the right and one moving to the left. In this case, the polarization-entangled state does not need to be symmetric. It can even be antisymmetric, provided that the same is true for the spatial state of the two photons, as the total state must be symmetric. Two photons in such a state would antibunch on a beam splitter [7]. An analogous example can be provided for fermions. Although quarks were expected to be fermions, they were also predicted to occupy the same state, which would seemingly violate the exclusion principle. This led to the discovery of a new *hidden* property, color, which for three quarks making a baryon is always in the antisymmetric state [8].

Finally, let us briefly discuss our work in the context of other related research. First, the formulation of a resource theory concerning indistinguishability (and symmetry or asymmetry) is an active research topic (see e.g., Ref. [9]). Our Letter contributes to these investigations by showing that in multipartite systems, symmetry and antisymmetry is affected by monogamy-like relations. In addition, we link indistinguishability with another well-established resource—entanglement.

Second, our work has implications on foundations of quantum theory and quantum information theory. We show that imperfect bosonic or fermionic behavior does not need to be absolute. It is possible that in some cases particle A is an imperfect boson when considered together with particle B and an imperfect fermion when considered with particle C. This can have potential use in quantum information tasks based on indistinguishability, like boson sampling (see also Ref. [10]).

Third, our work could find applications in quantum optics, cold atom physics, solid state physics, or even in high-energy physics—in simple words, in any field dealing with multipartite states of elementary particles. Due to the arguments provided above, the detection of states studied in our work—i.e., the states that depart from perfect symmetry or antisymmetry—can be considered as a signature of a hidden structure of a fundamentally symmetric or anti-symmetric system. On the other hand, the capability of manipulating with one d.o.f. of a composite globally symmetric or antisymmetric system can be used as a remote control of some other d.o.f.

Preliminaries.—Let us consider a set of three modes A, B, and C to which we refer as "boxes." Each box is occupied by a single particle, so that the general pure state of the system is of the form

$$|\Psi\rangle = \alpha_1|A,B,C\rangle + \alpha_2|B,A,C\rangle + \alpha_3|C,A,B\rangle + \alpha_4|C,B,A\rangle + \alpha_5|A,C,B\rangle + \alpha_6|B,C,A\rangle. \quad (1)$$

Here we assume that the three particles are distinguishable; therefore the notation $|A,B,C\rangle$ means that the first particle is in box A, the second in B, and the third in C. Moreover, for simplicity we do not consider mixed states of the system. Nonetheless, our reasoning applies to them as well (see the Supplemental Material [11]).

Next, we introduce permutation operators Π_{XY} swapping the labels of boxes X and Y, for instance

$$\Pi_{AB}|A,B,C\rangle = |B,A,C\rangle. \quad (2)$$

The average value of these operators, denoted as

$$v_{XY} = \langle\Psi|\Pi_{XY}|\Psi\rangle, \quad (3)$$

has a clear operational meaning. If $v_{XY} = \pm 1$, then the Hong-Ou-Mandel experiment [12] conducted on particles from boxes X and Y would result in perfect bunching (or antibunching). Since bunching (antibunching) is a typical bosonic (fermionic) property that can be used as an indicator of particle statistics, in this Letter we say that if $v_{XY} = \pm 1$, then the particles in boxes X and Y simulate bosons (fermions). In general, for $-1 < v_{XY} < 1$, the bunching (antibunching) is not perfect, and the particles in boxes X and Y simulate bosons (fermions) with probability $(1 \pm v_{XY})/2$.

Let us also observe that a cyclic permutation operator S, whose matrix representation is given by

$$\begin{array}{ccc|cccccc} & & & A & B & C & C & A & B \\ & & & B & A & A & B & C & C \\ & & & C & C & B & A & B & A \\ \hline A & B & C & 0 & 0 & 1 & 0 & 0 & 0 \\ B & A & C & 0 & 0 & 0 & 0 & 1 & 0 \\ C & A & B & 0 & 0 & 0 & 0 & 0 & 1 \\ C & B & A & 0 & 1 & 0 & 0 & 0 & 0 \\ A & C & B & 0 & 0 & 0 & 1 & 0 & 0 \\ B & C & A & 1 & 0 & 0 & 0 & 0 & 0 \end{array}, \quad (4)$$

can be expressed in terms of permutation operators Π_{AB}, Π_{BC}, and Π_{AC}:

$$S = \Pi_{AB}\Pi_{BC} = \Pi_{AC}\Pi_{AB} = \Pi_{BC}\Pi_{AC}. \quad (5)$$

Since $S^3 = \mathbb{1}$, its eigenvalues are 1 and $e^{\pm i(2\pi/3)} = -\frac{1}{2} \pm i(\sqrt{3}/2)$.

Monogamy between simulation of bosons and fermions.—Let us consider the relations between the permutation properties of different subsystems. We start with a simple example of $v_{AB} = 1$, which means that the particles occupying boxes A and B simulate bosons. Then $\langle S\rangle = \langle\Psi|\Pi_{AB}\Pi_{BC}|\Psi\rangle = \langle\Psi|\Pi_{BC}|\Psi\rangle = v_{BC}$. Since the spectrum of S is 1 and $-\frac{1}{2} \pm i(\sqrt{3}/2)$, the smallest real value attainable by $\langle S\rangle$ is $-\frac{1}{2}$. Therefore, the maximal possible fermionic behavior in this case is given by $v_{BC} = -\frac{1}{2}$. As a result, $\langle S\rangle = -\frac{1}{2}$ and, because of Eq. (5), $v_{AC} = -\frac{1}{2}$. An example of a state leading to the above values is

$$\frac{1}{2}(|A,B,C\rangle + |B,A,C\rangle - |A,C,B\rangle - |B,C,A\rangle). \quad (6)$$

Similarly, one can ask about the maximal bosonic behavior of B and C, provided that A and B simulate fermions. In this case, $v_{AB} = -1$ and $\langle S\rangle = -v_{BC} = -v_{AC}$. In order to maximize v_{BC} we need to minimize $\langle S\rangle$, which is again $-\frac{1}{2}$. Therefore, the maximal bosonic behavior is $v_{BC} = v_{AC} = \frac{1}{2}$. The possible corresponding state is of the form

$$\frac{1}{2}(|A,B,C\rangle - |B,A,C\rangle + |A,C,B\rangle - |B,C,A\rangle). \quad (7)$$

Note that if particles in A and B simulate bosons and so do particles in B and C, then particles in A and C simulate bosons too. This is because in this case $\langle S\rangle = v_{AB} = v_{BC} = v_{AC}$. Therefore, the simulability of bosons is transitive. This is also true for the simulability of fermions, in which case $-\langle S\rangle = v_{AB} = v_{BC} = v_{AC}$.

General case.—The situation gets more complicated when neither pair of particles is perfectly bosonic or fermionic; i.e., v_{AB}, v_{BC}, and v_{AC} are between -1 and 1. To analyze this case, we introduce the following three operators:

$$W_1 = \frac{1}{3}(\Pi_{AB} + \Pi_{BC} + \Pi_{AC}), \quad (8)$$

$$W_2 = \frac{1}{3}(2\Pi_{AB} - \Pi_{BC} - \Pi_{AC}), \quad (9)$$

$$W_3 = \frac{1}{\sqrt{3}}(\Pi_{BC} - \Pi_{AC}). \quad (10)$$

These operators have eigenvalues ± 1 and 0. For W_1, the eigenvalue 0 is four times degenerate and the eigenvalue $+1$ corresponds to the symmetric state of the three particles (bosonic state), whereas -1 corresponds to the antisymmetric one (fermionic state). Interestingly, W_1 commutes with both W_2 and W_3: $[W_1, W_2] = [W_1, W_3] = 0$, and in addition $W_1 W_2 = W_1 W_3 = 0$.

There are two interesting properties of W_2 and W_3. First, the two operators anticommute: $\{W_2, W_3\} = W_2 W_3 + W_3 W_2 = 0$. Second, $W_2^2 = W_3^2$. Because of these two properties, the operator

$$W_\theta = W_2 \cos\theta + W_3 \sin\theta, \quad (11)$$

where $\theta \in [0, 2\pi)$, obeys

$$W_\theta^2 = W_2^2 \cos^2\theta + W_3^2 \sin^2\theta + \{W_2, W_3\}\cos\theta \sin\theta = W_2^2 = W_3^2. \quad (12)$$

Note that the above resembles the properties of Pauli spin-1/2 operators. The anticommutation of Pauli operators lies at the heart of the Bloch vector representation of spin-1/2 states [13], and we will see in a moment that the properties of W_2 and W_3 also allow one to propose a simple graphical representation of constraints on v_{AB}, v_{BC}, and v_{AC}.

The operator W_1 is supported on the subspace that is orthogonal to the one on which W_2 and W_3 are supported. Therefore,

$$|\langle W_1\rangle| + |\langle W_\theta\rangle| \le 1, \quad (13)$$

or explicitly,

$$|v_{AB} + v_{BC} + v_{AC}| + |(2v_{AB} - v_{BC} - v_{AC})\cos\theta + \sqrt{3}(v_{BC} - v_{AC})\sin\theta| \le 3. \quad (14)$$

The above constitutes a family of monogamy relations for v_{AB}, v_{BC}, and v_{AC} parametrized by θ. Any quantum state of the form (1) satisfies these relations for all θ. We stress that Eq. (14) has a clear operational meaning stemming from the direct relation of the average values v_{XY} with the probabilities of bunching on a beam splitter.

The graphical representation of the relations (13) and (14) defines a region in a three-dimensional space. With each state of form (1) (or a mixture of such states), one can associate the vector $\mathbf{v} = (v_{AB}, v_{BC}, v_{AC})$ which needs to lie inside this region. In order to find the shape, let us first give a new representation of the operators W_1, W_2, and W_3. We define a vector of operators

$$\Pi = (\Pi_{AB}, \Pi_{BC}, \Pi_{AC}) \quad (15)$$

and real vectors $\mathbf{w}_i$ $(i = 1, 2, 3, \theta)$ such that

$$W_i = \mathbf{w}_i \cdot \mathbf{\Pi}. \quad (16)$$

The average value of W_i can then be represented as

$$\langle W_i\rangle = \mathbf{w}_i \cdot \mathbf{v}. \quad (17)$$

Therefore, the relations (13) take the form

$$|\mathbf{w}_1 \cdot \mathbf{v}| + |\mathbf{w}_\theta \cdot \mathbf{v}| \le 1. \quad (18)$$

The above formula defines two conical regions whose circular bases are connected (see Fig. 1). The bases lie in the plane spanned by $\mathbf{w}_2$ and $\mathbf{w}_3$, and the whole region has a rotational symmetry with respect to the $\mathbf{w}_1$ axis.

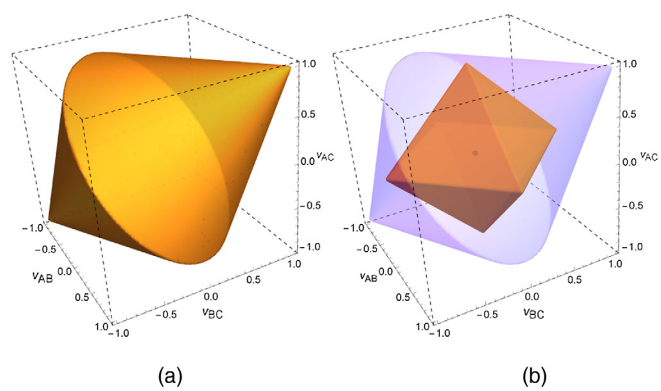

FIG. 1. Graphical representation of the tradeoff relation (18). (a) The vector $\mathbf{v} = (v_{AB}, v_{BC}, v_{AC})$ lies inside the conical region. (b) The octahedron inscribed into the cones corresponds to vectors $\mathbf{v}$ obtainable with only bipartite entanglement. The dot at $\mathbf{v} = (0, 0, 0)$ is the only possibility resulting from fully separable states. The inverse of this statement, however, is not true—there are entangled states for which all the values v_{XY} vanish.

Because of circular symmetry, one can get rid of the parameter θ and write a single monogamy relation

$$|\mathbf{w}_1 \cdot \mathbf{v}| + \sqrt{(\mathbf{w}_2 \cdot \mathbf{v})^2 + (\mathbf{w}_3 \cdot \mathbf{v})^2} \leq 1. \tag{19}$$

However, the parameter θ allows one to show that the above monogamy relation is tight; i.e., one can find a quantum state corresponding to each point on the surface of the region denoted by Eq. (18).

Consider the state

$$|\chi_{\theta,\varphi}^{(\pm,\pm)}\rangle = \cos\varphi|\pm\rangle + \sin\varphi|\Psi_\theta^\pm\rangle, \tag{20}$$

such that

$$\langle\pm|W_1|\pm\rangle = \pm 1, \tag{21}$$

$$\langle\Psi_\theta^\pm|W_\theta|\Psi_\theta^\pm\rangle = \pm 1. \tag{22}$$

In the above equations, $|\pm\rangle$ correspond to the symmetric (bosonic) and antisymmetric (fermionic) states of the three particles, and $|\Psi_\theta^\pm\rangle$ are the ± 1 eigenstates of the operator W_θ. Because W_1 and W_θ span orthogonal subspaces, we get $\langle\pm|\Psi_\theta^\pm\rangle = 0$, and therefore

$$|\langle W_1\rangle| + |\langle W_\theta\rangle| = |\cos\varphi|^2 + |\sin\varphi|^2 = 1. \tag{23}$$

The states in Eq. (20) cover the whole surface for $\theta \in [0, \pi)$ and $\varphi \in [0, \pi/2]$ (see the Supplemental Material [11] for more details).

Simulability and entanglement.—Having described the allowed triples $\{v_{AB}, v_{BC}, v_{AC}\}$, we now focus on their dependence on the type of entanglement in the tripartite system described by the pure state $|\Psi\rangle$ [see Eq. (1)]. If this state is not entangled, the three reduced single-particle density matrices need to be pure. Since they all have diagonal forms given by

$$\rho_1 = \mathrm{diag}(|\alpha_1|^2 + |\alpha_5|^2, |\alpha_2|^2 + |\alpha_6|^2, |\alpha_3|^2 + |\alpha_4|^2),$$
$$\rho_2 = \mathrm{diag}(|\alpha_2|^2 + |\alpha_3|^2, |\alpha_1|^2 + |\alpha_4|^2, |\alpha_5|^2 + |\alpha_6|^2),$$
$$\rho_3 = \mathrm{diag}(|\alpha_4|^2 + |\alpha_6|^2, |\alpha_3|^2 + |\alpha_5|^2, |\alpha_1|^2 + |\alpha_2|^2), \tag{24}$$

this means that only a single α_i is nonzero. But this implies that all the average permutation values v_{XY} vanish, since

$$v_{AB} = 2Re(\alpha_1^*\alpha_2 + \alpha_3^*\alpha_4 + \alpha_5^*\alpha_6),$$
$$v_{AC} = 2Re(\alpha_1^*\alpha_4 + \alpha_2^*\alpha_6 + \alpha_3^*\alpha_5),$$
$$v_{BC} = 2Re(\alpha_1^*\alpha_5 + \alpha_2^*\alpha_3 + \alpha_4^*\alpha_6). \tag{25}$$

If the state $|\Psi\rangle$ is bipartite entangled, it can be written as $|\psi\rangle_1 \otimes |\phi\rangle_{23}$, $|\psi\rangle_2 \otimes |\phi\rangle_{13}$, or $|\psi\rangle_3 \otimes |\phi\rangle_{12}$. In any of these cases, only one of the single-particle density matrices needs to be pure. It can be easily checked that in such a situation exactly one parameter v_{XY} can be nonzero, and that its maximum value can attain ± 1. Thus, the bipartite entangled states and their mixtures span the polyhedron in the space $\{v_{AB}, v_{AC}, v_{BC}\}$ defined by the vertices $\{\pm 1, 0, 0\}$, $\{0, \pm 1, 0\}$, and $\{0, 0, \pm 1\}$.

Finally, the presence of genuine tripartite entanglement corresponds to the situation in which all single-partite density matrices are mixed. This is the only situation in which more than one parameter v_{XY} can be nonzero. However, the inverse of this statement is not true, as showcased by the state $(1/\sqrt{3})(|ABC\rangle + |CAB\rangle + |BCA\rangle)$, for which all the average values v_{XY} vanish.

Simulability of imperfect bosons and fermions is not transitive.—Now we discuss properties of a particular state. Let us consider a state corresponding to the vector $\mathbf{v} = (x, x, -x)$. We are looking for the maximal x for which $\mathbf{v}$ satisfies the monogamy relation. The inequality (19) implies that the maximal value is $x = \frac{3}{5}$. The corresponding state is $\sqrt{\frac{4}{5}}|\phi\rangle + (1/\sqrt{5})|+\rangle$, which is a superposition of the symmetric (bosonic) state and

$$|\phi\rangle = \frac{1}{2}(|A,B,C\rangle + |B,A,C\rangle - |C,B,A\rangle - |B,C,A\rangle). \tag{26}$$

This example shows substantial departure from the transitivity of simulability of imperfect bosons and fermions. Although a pair of particles in boxes A and B (or B and C) simulates bosons with the probability $(1 + v_{AB})/2 = \frac{4}{5}$, the pair in boxes A and C simulates fermions with the probability $(1 - v_{AC})/2 = \frac{4}{5}$. Note that the probability of bunching greater than $\frac{3}{4}$ is a sufficient condition to violate the noncontextuality-like inequality [14–16]. The violation of this inequality implies that bunching properties of particles cannot be explained by a classical hidden variable model for which it is predetermined whether the particle goes through or reflects from a beam splitter.

Because the monogamy relation (18) is symmetric under the transformation $\mathbf{v} \to -\mathbf{v}$, a state for which $\mathbf{v} = (-\frac{3}{5}, -\frac{3}{5}, \frac{3}{5})$ exists as well.

Glance at the four-partite scenario.—We now turn our attention to the case of four particles in four boxes, denoted by letters from A to D (see the Supplemental Material [11] for more details). By analogy with the three-box scenario, we consider six v_{XY} parameters corresponding to average values of permutations Π_{XY}. In particular, we ask how the minimal v_{XY} changes when some pairs are set to simulate bosons.

Figure 2 presents two scenarios with significantly different properties. In case (a), v_{AB} and v_{CD} are set to 1 and we ask what is the minimal value of

$$-x = v_{AC} = v_{AD} = v_{BC} = v_{BD}, \qquad 0 < x < 1. \tag{27}$$

The tripartite monogamy relations stemming from triangles ABC and ABD imply that $-x \geq -\frac{1}{2}$. One can show that this bound is attained by the eigenstate corresponding to the largest eigenvalue of the operator

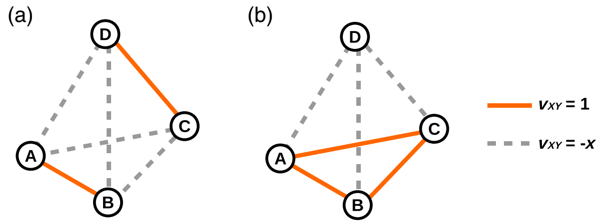

FIG. 2. Two examples of four-partite scenarios. The solid edges indicate the simulation of bosons ($v_{XY} = 1$), and the dashed edges denote the simulation of imperfect fermions ($v_{XY} = -x$, $0 < x < 1$). The tripartite trade-off relations (18) can be applied to such systems by considering subsets of all particles. In case (a), these relations are enough to derive the maximal value of x, but in case (b), they need to be supplemented with additional four-partite relations.

$$\Pi_{AB} + \Pi_{CD} - \Pi_{AC} - \Pi_{AD} - \Pi_{BC} - \Pi_{BD}. \quad (28)$$

Note that the same minimal value of x would be true if we demanded that only v_{AB} be set to 1.

On the other hand, it is not possible to fully explain scenario (b) with only tripartite relations. Once again, they imply that $-x \geq -\frac{1}{2}$. This time, however, $-x$ cannot be smaller than $-\frac{1}{3}$, as

$$\langle \Pi_{AB} + \Pi_{AC} + \Pi_{BC} - \Pi_{AD} - \Pi_{BD} - \Pi_{CD} \rangle = 3 + 3x, \quad (29)$$

and the largest eigenvalue of the above operator is 4.

Summary and outlook.—Entangled particles can simulate indistinguishable particles, provided they are prepared in a proper state. If the state is symmetric, the particles behave like bosons, and if it is antisymmetric, they behave like fermions. In general, the particles can simulate various combinations of imperfect bosonic and fermionic properties. Here we show that these combinations are restricted by fundamental bounds. These bounds take the form of monogamy relations, a tripartite version of which was derived in this Letter. Our relations are tight and can be represented using a simple three-dimensional visualization. One of the conclusions that stems from the relations is that simulability of imperfect bosons or fermions need not be objective. Instead, it can be relative; i.e., the particle in box A can simulate an imperfect boson when considered together with the particle in box B and, at the same time, an imperfect fermion when considered together with the particle in box C. Finally, the tripartite relations can only partially explain the behavior of four-partite systems, which implies that indistinguishability, similarly to entanglement, depends on the number of particles.

We considered permutation properties of all three particle pairs originating from a tripartite system. These properties can be related to the probabilities of bunching and antibunching in the Hong-Ou-Mandel experiment. However, following Menssen *et al.* [17], it would be interesting to consider genuine tripartite relations of the states studied in this Letter. For example, one can study the counting statistics for the states in Eq. (1) on a symmetric three-port (tritter).

Another interesting extension of this Letter could stem from relaxing the superselection rule which states that each box is occupied by a single particle. In general, one could study the constraints on closeness to the symmetric and antisymmetric subspaces of pairs (or larger numbers) of subsystems for arbitrary n-qudit states. This line of research might have implications for quantum marginal problems.

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