Topology of One-Dimensional Quantum Systems Out of Equilibrium

Max McGinley and Nigel R. Cooper
T.C.M. Group, Cavendish Laboratory, University of Cambridge, JJ Thomson Avenue, Cambridge, CB3 0HE, United Kingdom

(Received 16 April 2018; published 29 August 2018)

We study the topological properties of one-dimensional systems undergoing unitary time evolution. We show that symmetries possessed both by the initial wave function and by the Hamiltonian at all times may not be present in the time-dependent wave function—a phenomenon which we dub "dynamically induced symmetry breaking." This leads to the possibility of a time-varying bulk index after quenching within noninteracting gapped topological phases. The consequences are observable experimentally through particle transport measurements. With reference to the entanglement spectrum, we explain how the topology of the wave function can change out of equilibrium, both for noninteracting fermions and for symmetry-protected topological phases protected by antiunitary symmetries.

DOI: 10.1103/PhysRevLett.121.090401

In the past few decades, numerous examples of gapped quantum many-particle systems with topologically nontrivial ground states have been discovered [1,2]. Despite the lack of a local order parameter, these states cannot be smoothly connected to their topologically trivial counterparts without closing the bulk energy gap and removing their characteristic gapless edge modes.

Central to the modern understanding of these phases is the importance of symmetry constraints on the Hamiltonian, through which a rich "periodic table" of noninteracting fermionic topological phases emerges [3–6]. Such systems can be characterized by bulk indices which capture global features of the Bloch bands, generalizing the Chern number for two-dimensional systems [7]. These indices are topological invariants: they are unchanged under symmetry-respecting deformations of the Hamiltonian, provided the gap does not close. More general symmetry-protected topological (SPT) phases are also known to exist beyond free fermions [8].

More recently, the topological properties of quantum states far from equilibrium have been examined [9–21], motivated by possibilities to study coherent dynamics in cold atom experiments [22–24]. The Chern number after a quantum quench has been shown to be constant in time [15–18], a result that has often been assumed to be a universal feature of all bulk invariants in noninteracting fermionic systems [19,21]. However, existing studies leave open the role of symmetry in the postquench state.

In this Letter we address the effects of symmetries on the topology of one-dimensional (1D) quantum systems that are out of equilibrium. We show that the bulk index of the time-evolved wave function *can* vary in time. Surprisingly, this can occur even when the Hamiltonian retains the required symmetries at all times and remains within the same phase. This behavior stems from a phenomenon that we call dynamically induced symmetry breaking: after a quantum quench, the symmetries of the time-dependent state do not necessarily match those of the governing Hamiltonian. We determine the dynamical behavior of the bulk index in all symmetry classes for noninteracting fermions in one dimension, and show that the predicted dynamics of the bulk index can be directly measured in experiment. We also describe how the bulk index relates to the topology of the wave function out of equilibrium, using the entanglement spectrum [25].

We conclude by explaining the relevance of dynamically induced symmetry breaking to interacting SPT phases, and numerically demonstrate the consequences for the entanglement spectrum of time-reversal protected Haldane phases. Our work highlights the difference between static and dynamic protection of topological phases in general: while the topological properties of a ground state may be robust against time-independent symmetry-respecting perturbations, the same is not necessarily true of time-dependent symmetry-respecting perturbations.

Symmetry under dynamics.— At equilibrium, noninteracting fermionic topological insulators are classified into ten symmetry classes according to the presence of the "generic" symmetries of time-reversal (TRS), particle-hole (PHS), and chiral (or sublattice) symmetry [4,26]. Note that in superconducting systems, PHS is not a physical symmetry, but represents a redundancy in the Bogoliubov–de Gennes equations [27]. Each of these symmetries imposes a constraint on the matrix H_{ij} that defines the Hamiltonian $\hat{\mathcal{H}}$ via $\hat{\mathcal{H}} = \hat{\psi}_i^\dagger H_{ij} \hat{\psi}_j$, where $\hat{\psi}_j^\dagger$ creates a fermion in a state j. These are [28]

$$\mathsf{T}\mathsf{H}^*\mathsf{T}^\dagger = \mathsf{H} \qquad \text{TRS}, \tag{1a}$$

$$\mathsf{C}\mathsf{H}^*\mathsf{C}^\dagger = -\mathsf{H} \qquad \text{PHS}, \tag{1b}$$

$$\mathsf{S}\mathsf{H}\mathsf{S}^\dagger = -\mathsf{H} \qquad \text{Chiral}, \tag{1c}$$

where T, C, and S are unitary matrices that satisfy $T^*T = \pm 1$, $C^*C = \pm 1$, and $S^*S = 1$.

For systems with a unique ground state, the symmetries (1) of the Hamiltonian are inherited by the ground state wave function $|\Psi\rangle$, and therefore by the single-particle density matrix $\rho_{ij} = \langle\Psi|\hat{\psi}_i^\dagger \hat{\psi}_j|\Psi\rangle$, which itself fully characterizes the state of noninteracting fermions. One finds [6]

$$T\rho^* T^\dagger = \rho \qquad \text{TRS}, \tag{2a}$$

$$C\rho^* C^\dagger = 1 - \rho \qquad \text{PHS}, \tag{2b}$$

$$S\rho S^\dagger = 1 - \rho \qquad \text{Chiral}. \tag{2c}$$

This characterization of the symmetry properties of the state (2) admits a natural generalization out of equilibrium. We consider nonequilibrium states arising from a very general quench protocol: the system is prepared in the ground state of an initial Hamiltonian H^i at time $t = 0$ and then evolves under some other Hamiltonian $H^f(t)$, which may itself vary in time in an arbitrary manner. The single particle density matrix evolves as $\rho(t) = U(t)\rho(0)U(t)^\dagger$ under the time evolution matrix $U(t) = \mathcal{T}\exp[-i\int_0^t dt' H^f(t')]$ ($\mathcal{T}$ denotes time-ordering). By replacing ρ with $\rho(t)$ in Eq. (2), we can determine the symmetries of the state at time t.

We find two general mechanisms by which the symmetries of the initial state can be broken for $t > 0$.

Explicit symmetry breaking.—If a symmetry of the Hamiltonian changes between H^i and $H^f(t)$, this symmetry will not appear in the state at $t > 0$ [29]. This applies in simple situations where a generic symmetry of the Hamiltonian is lost, e.g., if H^i has chiral (sublattice) symmetry but $H^f(t)$ does not. However, it also applies in situations where a generic symmetry is preserved, but the matrix (T, C, or S) that realizes the symmetry changes. For example, even if chiral (sublattice) symmetry is preserved, the sets of sites that constitute the two sublattices could differ between H^i and $H^f(t)$. (We provide other examples in the Supplemental Material [30].)

Dynamically induced symmetry breaking.—Even if there is no change in symmetry of the Hamiltonian—i.e., initial and final Hamiltonians have the same symmetries, realized by the same unitary matrices—we find that there can be a change in the symmetry of the state purely due to unitary dynamics. In this case the density matrix satisfies

$$T\rho(t)^* T^\dagger = \rho(-t) \qquad \text{TRS}, \tag{3a}$$

$$C\rho(t)^* C^\dagger = 1 - \rho(t) \qquad \text{PHS}, \tag{3b}$$

$$S\rho(t) S^\dagger = 1 - \rho(-t) \qquad \text{Chiral}, \tag{3c}$$

where we have used $\rho(-t)$ to denote a fictitious system time evolved by a time $+t$ under the Hamiltonian $-H^f(t)$. Because in general $\rho(-t) \neq \rho(t)$, we infer that, surprisingly, TRS and chiral symmetries of the state are *not* preserved under dynamics, because Eqs. (3a), (3c) are not equivalent to the symmetry conditions Eqs. (2a), (2c). On the other hand, the time-dependent PHS condition Eq. (3b) *is* equivalent to the equilibrium case Eq. (2b), so PHS is the one generic symmetry that is retained at all times.

In the following we will focus on quantum quenches without explicit symmetry breaking.

Dynamics of the bulk index.—At equilibrium, the bulk index that characterizes topology in one dimension is the Chern-Simons (CS) invariant [6], or, equivalently, the Zak phase α_Z [36]

$$\mathrm{CS}_1 \equiv \frac{\alpha_Z}{2\pi} \coloneqq \frac{i}{2\pi}\int_{\mathrm{BZ}} dk \langle u_k^\alpha | \partial_k u_k^\alpha \rangle, \tag{4}$$

expressed in terms of the ground state Bloch functions $|u_k^\alpha\rangle$ for occupied bands α (a sum over α for all occupied bands is to be understood). The functions are assumed to vary smoothly with wave vector k and are chosen to be periodic in the Brillouin zone (BZ).

The CS invariant is only defined modulo 1, since gauge transformations of the occupied Bloch states can change CS_1 by an integer. However, in the presence of TRS and/or chiral symmetry, the integer part can be given physical meaning through the use of certain symmetry-related gauge choices [6]. Under such gauges, all equilibrium topological invariants in one dimension can be deduced from quantized (integer or half-integer) values of the CS invariant. These quantized values, and hence the topological classification, arise only when particular symmetry combinations are imposed. The five nontrivial classes are listed in Table I with their topological classifications under $\mathrm{CS}_1(t = 0)$.

TABLE I. Topological characterizations of 1D insulators in and out of equilibrium. The five nontrivial classes in one dimension are defined by the presence of TRS, PHS, and chiral symmetries (T, C, S) according to Eq. (1), and their topologically distinct values of CS_1 in equilibrium are given. Asterisks denote cases for which CS_1 must be evaluated in a gauge specified by the TRS or chiral symmetries. After time evolving under a Hamiltonian in the same symmetry class, the fractional part of $\mathrm{CS}_1(t)$ either varies in time, or stays fixed to its initial value. The possible values of $\mathrm{CS}_1(t)$ mod 1 are given, which determine the topological classification (Class.) out of equilibrium. Nontrivial wave functions within this classification will also have degenerate entanglement spectra (Ent.).

Class	T	C	S	$\mathrm{CS}_1(t=0)$	$\mathrm{CS}_1(t)$ mod 1	Class./Ent.
AIII	0	0	1	$\mathbb{Z}/2^*$	Varies [0, 1)	0
BDI	+	+	1	$\mathbb{Z}/2^*$	Const. $\{0, 1/2\}$	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}/2$ mod 1	Const. $\{0, 1/2\}$	$\mathbb{Z}_2$
DIII	−	+	1	$\mathbb{Z}$mod 2^*	Const. 0	0
CII	−	−	1	$\mathbb{Z}^*$	Const. 0	0

We consider the effects of dynamically induced symmetry breaking on CS_1 in these five classes.

All states which possess PHS (classes BDI, D, DIII, and CII) must have a CS invariant quantized to 0 or 1/2 up to the addition of an integer [37]. As we have shown, PHS is preserved under time evolution, and so the time-dependent $\mathrm{CS}_1(t)$ must also be quantized for $t > 0$. Moreover, assuming that all Hamiltonians are smooth in k space, one can define a continuous PHS-preserving interpolation between the initial and final states parametrized by the time t, under which the fractional part of $\mathrm{CS}_1(t)$ cannot change. The fractional part of $\mathrm{CS}_1(t)$ is therefore constant when PHS is present.

States which do not possess PHS can have a CS invariant quantized to half-integer values if there is a chiral symmetry (class AIII). We have argued above that chiral symmetry will in general undergo dynamically induced symmetry breaking. Thus, for $t > 0$ the CS invariant need no longer be quantized, and one expects $\mathrm{CS}_1(t)$ to vary in time. This leads to the surprising finding that even when the initial and final Hamiltonians satisfy the same (chiral) symmetry at all times the bulk index becomes time dependent.

Relation to physical observables.—Remarkably, the dynamics of the bulk index has directly observable consequences even far from equilibrium. (This contrasts with the Chern index for which the relationship with the Hall conductance does not hold out of equilibrium [38,39].) Specifically, the identification of CS_1 with the bulk polarization of the system (i.e. the centers of Wannier states) [40] still holds beyond the adiabatic limit. To show this, we calculate the mean current

$$\begin{aligned}\langle j(t)\rangle &= \frac{1}{2\pi}\int_{\mathrm{BZ}} dk \langle u_k^\alpha(t)|\partial_k \hat{H}_k^f(t)|u_k^\alpha(t)\rangle \\ &= \frac{1}{2\pi}\int_{\mathrm{BZ}} dk \{\partial_k[\langle u_k^\alpha(t)|\hat{H}_k^f(t)|u_k^\alpha(t)\rangle] \\ &\quad - \langle u_k^\alpha(t)|\hat{H}_k^f(t)|\partial_k u_k^\alpha(t)\rangle - \langle \partial_k u_k^\alpha(t)|\hat{H}_k^f(t)|u_k^\alpha(t)\rangle\} \\ &= \frac{i}{2\pi}\int_{\mathrm{BZ}} dk [\langle \partial_t u_k^\alpha(t)|\partial_k u_k^\alpha(t)\rangle + \langle u_k^\alpha(t)|\partial_t \partial_k u_k^\alpha(t)\rangle] \\ &= \frac{d}{dt}\mathrm{CS}_1. \end{aligned} \tag{5}$$

We have integrated by parts, and used the periodicity of $|u_k^\alpha\rangle$ in the BZ. Thus, the time variation of $\mathrm{CS}_1(t)$ is reflected in the postquench current and bulk polarization, which can be measured in experiment. Note that no assumption of any form of adiabaticity is required.

We have numerically verified that this relationship between the CS invariant and local current holds, even within the bulk of a finite system. We consider spinless fermions, represented by operators $\hat{\psi}_j^{(\dagger)}$ acting on the sites labeled by j, with a hopping Hamiltonian $\hat{\mathcal{H}} = -\sum_j (J_1 \hat{\psi}_{2j+1}^\dagger \hat{\psi}_{2j} + J_2 \hat{\psi}_{2j+2}^\dagger \hat{\psi}_{2j+1} + B_1 \hat{\psi}_{2j+3}^\dagger \hat{\psi}_{2j} + B_2 \hat{\psi}_{2j+4}^\dagger \hat{\psi}_{2j+1} + \mathrm{H.c.})$. In general, the model possesses only a chiral sublattice symmetry (class AIII), but if all hopping amplitudes are real, TRS and PHS are also present (class BDI). Figure 1(a) shows the time variation of the CS invariants for AIII and BDI systems, calculated as bulk integrals. This is compared to the bulk polarization $Q_B(t)$ in a finite system with the same hopping amplitudes, calculated as the particle number in the right subsystem B (see inset). The gauge-invariant $Q_B(t)$ equals $\mathrm{CS}_1(t)$ up to an integer, until correlations span the whole system. Thus in one dimension, the change in the CS invariant is directly measurable as particle accumulation.

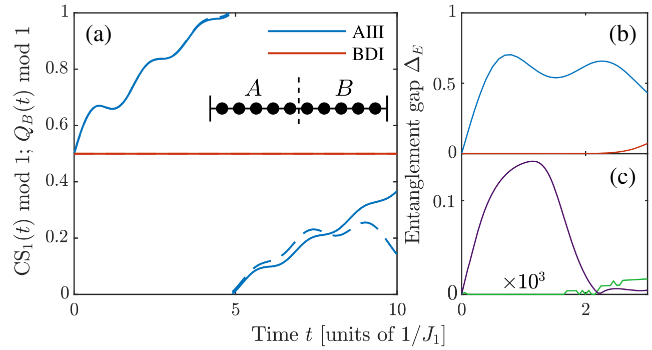

FIG. 1. Panel (a): Time-dependent CS invariant of a hopping model of spinless fermions, calculated as a bulk integral in k space (solid lines), compared with the polarization $Q_B(t)$ of a 24-site open boundary system with the same parameters (dashed lines). $Q_B(t)$ is calculated as the expected particle number within the right-hand half, subsystem B of the inset. The red lines are for a BDI system and the blue lines are for an AIII system. The parameters for the quenches are $(J_1, J_2) = (0.3, e^{i\alpha}) \to (0.8e^{i\alpha}, 1)$ with $B_{1,2} = 0.05$ throughout; $\alpha = 0$ for class BDI and $\alpha = 0.4$ for class AIII. The observables in the finite sample match the dynamics of the bulk invariants even out of equilibrium, until correlations traverse the whole system at which point the discrete nature of k space invalidates Eq. (5). Panel (b): dynamics of the entanglement gap Δ_E for the same systems as above with the entanglement cut between A and B [inset of (a)]. In the BDI case, the entanglement gap remains close to zero until correlations span the system size, whereas the AIII system immediately becomes gapped. Panel (c): dynamics of the entanglement gap for a spin-1 chain initialized in a Haldane phase, possessing TRS only (purple line), and both TRS and dihedral symmetry (green line, scaled by 10^3). When the Haldane phase is supported by TRS only, the entanglement spectrum becomes gapped for $t > 0$.

Note that the time variation of CS_1 can be seen in even simpler models such as the SSH model [41] with complex hopping amplitudes, i.e., our model with $B_1 = B_2 = 0$ (which is in class BDI). If the phases of either $J_{1,2}$ change across the quench, then TRS and PHS undergo explicit symmetry breaking, and one finds the same behavior as expected for an AIII quench: the dynamically induced breaking of chiral symmetry allows $\mathrm{CS}_1(t)$ to vary (see the Supplemental Material [30] for details).

Topological characterizations out of equilibrium.—We have determined general features of the dynamics of the bulk index. To what extent does this bulk index encode *topological* features of the time-evolving state? One may naïvely expect that the topology of the state is preserved as long as $\mathrm{CS}_1(t)$ does not vary in time (as occurs for all nontrivial classes other than AIII). However, this approach overlooks the gauge dependence of $\mathrm{CS}_1(t)$. An individual measurement of $\mathrm{CS}_1(t)$ at some time t is still only defined modulo 1. Unlike in equilibrium, this ambiguity cannot be resolved by a symmetry-related gauge choice, since TRS and chiral symmetries are broken by the dynamics. Therefore, wave functions with the same CS_1 modulo 1 cannot be distinguished by the bulk index and are thus topologically equivalent.

Once we restrict ourselves to consider only $\mathrm{CS}_1(t)$ mod 1, we can determine a new classification of states which can be topologically distinguished out of equilibrium; this is given in the last column of Table I. Note that systems in classes DIII and CII must be initialized with CS_1 mod $1 = 0$, and hence all such systems are topologically trivial for $t > 0$. A striking consequence of this is that two initial equilibrium states with different topology can time evolve into the same wave function, even though CS_1 mod 1 does not exhibit any time dependence; see the Supplemental Material [30] for an example in class DIII.

One of the clearest signatures of topological nontriviality in equilibrium is the presence of gapless edge excitations [42], connected to the nontrivial bulk index through the bulk-boundary correspondence. These edge modes also manifest themselves within the ground state entanglement spectrum [25], which mimics any physical edge modes that would be present at a boundary [43,44]. In the present nonequilibrium setting, the many-body wave function $|\Psi(t)\rangle$ can be thought of as the ground state of some fictitious Hamiltonian $\hat{\mathcal{H}}^{\mathrm{fic}}(t)$ which possesses the same symmetries as the state. For concreteness we can choose (in a second-quantized language) [21,45]

$$\hat{\mathcal{H}}^{\mathrm{fic}}(t) = \hat{\mathcal{U}}(t)\hat{\mathcal{H}}^{\mathrm{i}}\hat{\mathcal{U}}(t)^{\dagger}, \tag{6}$$

where $\hat{\mathcal{U}}(t)$ is the many-body time evolution operator. The equilibrium entanglement spectrum is a property of the ground state only; therefore the entanglement spectrum of $|\Psi(t)\rangle$ encodes the equilibrium topology of $\hat{\mathcal{H}}^{\mathrm{fic}}(t)$, which is independent of our specific choice (6). If $\hat{\mathcal{H}}^{\mathrm{fic}}(t)$ cannot be deformed to some trivial Hamiltonian without breaking the enforced symmetries, then it must possess gapless boundary modes [42], which themselves will show up in the entanglement spectrum of $|\Psi(t)\rangle$—this allows us to probe the bulk-boundary correspondence out of equilibrium.

We now apply the equilibrium classification to $\hat{\mathcal{H}}^{\mathrm{fic}}(t)$, which, due to dynamically induced symmetry breaking, will at most possess PHS only. When PHS is enforced, $\hat{\mathcal{H}}^{\mathrm{fic}}(t)$ will be topological if and only if the CS invariant of its ground state $|\Psi(t)\rangle$ is a half-odd integer. We conclude that in one dimension a vanishing entanglement gap Δ_E may only be supported for $t > 0$ in PHS systems which are initialized with a noninteger CS invariant. This is exactly the condition for topological nontriviality that we deduced purely from $\mathrm{CS}_1(t)$, summarized in the last column of Table I. Thus we expect the bulk-boundary correspondence to hold out of equilibrium, once $\mathrm{CS}_1(t)$ is interpreted modulo 1.

We have verified these predictions by numerical calculations of the time evolution of the entanglement spectrum for all symmetry classes in one dimension. Results for the contrasting cases of classes AIII and BDI are shown in Fig. 1(b). While our arguments have focused on translationally invariant noninteracting systems, our results on the entanglement spectrum and topological classification should be robust against symmetry-preserving disorder, as well as weak interactions.

In passing, we note that the quench protocol we have used throughout includes Floquet systems as a subset. Indeed, in that context PHS is found to play a different role to TRS and chiral symmetries [46,47]. Our results show that the connection between bulk indices and particle transport, which appears in Floquet systems as adiabatic pumping [46], holds much more generally, not requiring periodicity or adiabaticity. However, our topological characterization of the instantaneous wave function is distinct from the recently classified Floquet SPT orders [48–50], which refer to micromotion over a whole period, and cannot be inferred from, e.g., the entanglement spectrum at some fixed time [50]. The preservation of entanglement degeneracies in class D Floquet systems (where no dynamically induced symmetry breaking occurs) has also previously been observed numerically [51].

Interacting SPT phases.—Our consideration of noninteracting fermionic phases reveals the existence of a nonequilibrium topological classification which differs from equilibrium. One expects a similar nonequilibrium classification also for interacting systems, e.g., SPT phases of bosons protected by more general symmetries. Indeed dynamically induced symmetry breaking, which is a crucial ingredient, can occur in any system: we show in the Supplemental Material [30] that if a symmetry of the Hamiltonians is realized by an *antiunitary* second-quantized operator $\hat{\mathcal{O}}$ [52], then $|\Psi(t)\rangle$ will generically not respect that symmetry. Of the three symmetries considered in the main text, only PHS is unitary [53], and so the results agree.

Unlike for free fermions, a universal bulk index analogous to Eq. (4) does not exist for all 1D SPT phases. Nevertheless, one can still derive a nonequilibrium classification of SPT phases (using, e.g., projective symmetry representations [54–56]) which will be reflected in the dynamics of the entanglement spectrum. For example, in the case where only one symmetry is present, it is clear that topology is lost (preserved) if the symmetry is antiunitary (unitary).

We demonstrate this behavior for the spin-1 Haldane phase, which can be protected by a unitary dihedral symmetry, or by antiunitary TRS [57]. We have numerically investigated the fate of entanglement degeneracies after a quench that does not explicitly break symmetry. We plot the results in Fig. 1(c). (Details are given in the Supplemental Material [30].) In the TRS-protected case, the entanglement degeneracy is lifted for $t > 0$, indicating the expected breakdown of the Haldane phase due to dynamically induced symmetry breaking.

In summary, we have studied the role played by symmetries in the topological classification of 1D systems that are out of equilibrium, and identified the important phenomenon of dynamically-induced symmetry breaking. It will be of interest to extend these studies to other dimensions and to classify all SPT phases out of equilibrium in future work.

We gratefully acknowledge helpful discussions with Jan Carl Budich and Claudio Chamon. This work was supported by an EPSRC studentship, by EPSRC Grants No. EP/P009565/1 and No. EP/K030094/1, and by the Simons Foundation. Statement of compliance with EPSRC policy framework on research data: All data are directly available within the publication.

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[4] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
[5] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).
[6] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New J. Phys. **12**, 065010 (2010).
[7] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[8] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[9] A. A. Patel, S. Sharma, and A. Dutta, Eur. Phys. J. B **86**, 367 (2013).
[10] G. Kells, D. Sen, J. K. Slingerland, and S. Vishveshwara, Phys. Rev. B **89**, 235130 (2014).
[11] P. Hauke, M. Lewenstein, and A. Eckardt, Phys. Rev. Lett. **113**, 045303 (2014).
[12] L. Mazza, D. Rossini, M. Endres, and R. Fazio, Phys. Rev. B **90**, 020301 (2014).
[13] M. C. Strinati, L. Mazza, M. Endres, D. Rossini, and R. Fazio, Phys. Rev. B **94**, 024302 (2016).
[14] C. Wang, P. Zhang, X. Chen, J. Yu, and H. Zhai, Phys. Rev. Lett. **118**, 185701 (2017).
[15] M. S. Foster, M. Dzero, V. Gurarie, and E. A. Yuzbashyan, Phys. Rev. B **88**, 104511 (2013).
[16] P. D. Sacramento, Phys. Rev. E **90**, 032138 (2014).
[17] L. D'Alessio and M. Rigol, Nat. Commun. **6**, 8336 (2015).
[18] M. D. Caio, N. R. Cooper, and M. J. Bhaseen, Phys. Rev. Lett. **115**, 236403 (2015).
[19] C. Yang, L. Li, and S. Chen, Phys. Rev. B **97**, 060304 (2018).
[20] Z. Gong and M. Ueda, arXiv:1710.05289.
[21] Z.-G. Liu, L. Xiong, B. Huang, G.-C. Guo, and M. Gong, arXiv:1802.02931.
[22] N. Fläschner, D. Vogel, M. Tarnowski, B. S. Rem, D. S. Lühmann, M. Heyl, J. C. Budich, L. Mathey, K. Sengstock, and C. Weitenberg, Nat. Phys. **14**, 265 (2018).
[23] M. Tarnowski, F. N. Ünal, N. Fläschner, B. S. Rem, A. Eckardt, K. Sengstock, and C. Weitenberg, arXiv: 1709.01046.
[24] N. R. Cooper, J. Dalibard, and I. B. Spielman, arXiv:1803.00249.
[25] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[26] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[27] B. A. Bernevig, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).
[28] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).
[29] There do exist fine-tuned scenarios where symmetry is preserved even when $H^f(t)$ breaks the symmetry [12], but we consider only general cases here.
[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.090401 for a more complete discussion on the gauge dependence of $\mathrm{CS}_1(t)$, examples of explicit symmetry breaking, a treatment of symmetry properties for interacting systems, and details of the numerics used to calculate the data in Fig. 1(c). Contains Refs. [31–35].
[31] J. C. Budich and M. Heyl, Phys. Rev. B **93**, 085416 (2016).
[32] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
[33] C. G. Velasco and B. Paredes, Phys. Rev. Lett. **119**, 115301 (2017).
[34] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[35] Calculations performed using the ITensor C++ library (version 1.2) http://itensor.org/.
[36] J. Zak, Phys. Rev. Lett. **62**, 2747 (1989).
[37] J. C. Budich and E. Ardonne, Phys. Rev. B **88**, 075419 (2013).
[38] M. D. Caio, N. R. Cooper, and M. J. Bhaseen, Phys. Rev. B **94**, 155104 (2016).
[39] F. N. Ünal, E. J. Mueller, and M. O. Oktel, Phys. Rev. A **94**, 053604 (2016).
[40] R. D. King-Smith and D. Vanderbilt, Phys. Rev. B **47**, 1651 (1993).
[41] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).
[42] A. M. Essin and V. Gurarie, Phys. Rev. B **84**, 125132 (2011).
[43] L. Fidkowski, Phys. Rev. Lett. **104**, 130502 (2010).
[44] X.-L. Qi, H. Katsura, and A. W. W. Ludwig, Phys. Rev. Lett. **108**, 196402 (2012).
[45] L. Vidmar, D. Iyer, and M. Rigol, Phys. Rev. X **7**, 021012 (2017).

[46] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).

[47] R. Roy and F. Harper, Phys. Rev. B **96**, 155118 (2017).

[48] D. V. Else and C. Nayak, Phys. Rev. B **93**, 201103 (2016).

[49] C. W. von Keyserlingk and S. L. Sondhi, Phys. Rev. B **93**, 245145 (2016).

[50] A. C. Potter, T. Morimoto, and A. Vishwanath, Phys. Rev. X **6**, 041001 (2016).

[51] D. J. Yates and A. Mitra, Phys. Rev. B **96**, 115108 (2017).

[52] While the first-quantized PHS and chiral symmetries in Eq. (1) anticommute with the matrix H, they can be represented as many-body operators which commute with the operator $\hat{\mathcal{H}} = \hat{\psi}_i^\dagger H_{ij} \hat{\psi}_j$, and are unitary and antiunitary on the many-body Hilbert space, respectively. See section II of Ref. [28] for a more complete discussion.

[53] A. W. W. Ludwig, Phys. Scr. **2016**, 014001 (2016).

[54] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011).

[55] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).

[56] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **84**, 235128 (2011).

[57] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).