Scattering Approach to Anderson Localization

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We develop a novel approach to the Anderson localization problem in a d-dimensional disordered sample of dimension $L \times M^{d-1}$. Attaching a perfect lead with the cross section M^{d-1} to one side of the sample, we derive evolution equations for the scattering matrix and the Wigner-Smith time delay matrix as a function of L. Using them one obtains the Fokker-Planck equation for the distribution of the proper delay times and the evolution equation for their density at weak disorder. The latter can be mapped onto a nonlinear partial differential equation of the Burgers type, for which a complete analytical solution for arbitrary L is constructed. Analyzing the solution for a cubic sample with $M = L$ in the limit $L \to \infty$, we find that for $d < 2$ the solution tends to the localized fixed point, while for $d > 2$ to the metallic fixed point, and provide explicit results for the density of the delay times in these two limits.

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Introduction.—Sixty years ago Anderson discovered that the classical diffusion in a random potential can be totally suppressed by quantum interference effects [1]. Since that time the problem of Anderson localization has remained in the focus of very active research and recently it has received a lot of attention in the context of topological insulators and many-body localization [2].

Apart from the strictly one-dimensional case, the most developed nonperturbative theory of Anderson localization is available for disordered wires. The only important parameters of such a system are the length L, the mean free path l, the number of the propagating modes N at the Fermi energy E, and the localization length $\xi = Nl$. The disorder is usually assumed to be weak, so that $L, l \gg \lambda_F$, where λ_F is the Fermi wavelength. There are two powerful analytical approaches which can solve the problem of Anderson localization in a wire for an arbitrary ratio L/ξ: the Dorokhov, Mello, Pereyra, and Kumar (DMPK) equation [3,4] and the supersymmetric nonlinear σ-model [5,6]. Both solutions are restricted to the quasi-one-dimensional geometry of a wire, for which the transverse dimension M is much smaller than L. Despite a lot of efforts, a similar theory for higher dimensional systems has not been developed so far and it is the purpose of this Letter to take the first step towards this long-standing goal.

We consider a d-dimensional weakly disordered sample of the length L in the x direction and the width M in all other transverse directions. A perfect lead is attached to one side of the sample along the x direction, which has the same cross section M^{d-1} as the sample. The scattering setup allows one to introduce the scattering matrix S and the Wigner-Smith time-delay matrix $Q = -i\hbar S^{-1/2}(\partial S/\partial E)S^{-1/2}$, whose eigenvalues $\tilde{\tau}_i$ are referred to as the proper delay times (see Refs. [7,8] for reviews). Generalizing the approach developed for the one-dimensional systems [9,10] we derive the Fokker-Planck equation for the evolution of the distribution function $P(\{\tilde{\tau}_i\}, r)$ in fictitious time $r \propto L/l$, provided that $L, M, l \gg \lambda_F$. Then we focus on the time-dependent equation for the density $\rho(\tilde{\tau}, r)$ of the delay times, which contains important information about localization in the corresponding closed system. Mapping this equation onto a nonlinear partial differential equation of the Burgers type, we construct its complete analytical solution for arbitrary L, M, and l.

Our general solution, which is valid for any dimensionality d, allows us in particular to consider a d-dimensional cubic sample with $M = L$. Analyzing such a system in the limit $L/\lambda_F \to \infty$, we find that for $d < 2$ the solution tends to the localized fixed point, while for $d > 2$ to the metallic (diffusive) fixed point and derive explicit analytical results for the density of the delay times in these two limits. Thus our approach provides a solid nonperturbative foundation for the arguments of the scaling theory of Anderson localization [11].

As the derivation of our results involves a lot of technical steps, in this Letter we only outline its main points and leave the technical details for a more specialized publication [12].

Model.—We consider the Hamiltonian for a particle moving in the d-dimensional δ-correlated disordered potential:

$$H = -\sum_{i=0}^{d-1} \frac{\partial^2}{\partial x_i^2} + V(\boldsymbol{r}), \qquad \boldsymbol{r} = (x, \boldsymbol{\rho}),$$

$$\langle V(\boldsymbol{r})V(\boldsymbol{r}')\rangle = \sigma\delta(\boldsymbol{r} - \boldsymbol{r}'), \qquad \sigma = \frac{1}{2\pi\nu\tau_s}, \tag{1}$$

where $x \equiv x_0, \boldsymbol{\rho} \equiv (x_1, \ldots x_{d-1})$, ν is the density of states, τ_s is the scattering mean free time and we set $\hbar = 2m = 1$.

A sample is assumed to be finite with $-L \le x \le 0$ and $0 \le x_i \le M$ for $i = 1, \ldots, d-1$, and the Dirichlet boundary condition is imposed in all directions.

By attaching a perfect lead to one side of the sample at $x = 0$, we obtain a scattering system characterized by the $N \times N$ S matrix, which is unitary $S^\dagger = S^{-1}$ and symmetric $S^T = S$ due to the time reversal symmetry. The eigenfunctions in the transverse directions $u_{\boldsymbol{n}}(\boldsymbol{\rho}) = (2/M)^{(d-1/2)} \prod_{i=1}^{d-1} \sin(\pi n_i x_i/M)$, $n_i \in \mathbb{N}$, correspond to the eigenenergies $E_{\boldsymbol{n}} = (\pi \boldsymbol{n}/M)^2$. The number of open channels at the energy E is equal to $N = \gamma_{d-1}(M\sqrt{E}/\pi)^{d-1}$, where $\gamma_d = \{[\pi^{(d/2)}]/[2^d\Gamma(d/2+1)]\}$.

Recursion relations for S and Q matrices.—In order to derive an equation for the evolution of S by increasing L to $L + \delta L$, we first consider the scattering matrix of a thin slice of the length $\delta L \ll \lambda_F$. Using the Lippmann-Schwinger equation, one can show that the reflection and the transmission matrices from the left and from the right coincide, respectively, $r' = r$, $t' = t$, and to the leading order in $\delta L/\lambda_F$ are given by

$$r = -B(I+B)^{-1}, \quad t = I + r, \quad B \equiv \frac{i}{2}\hat{q}^{-1/2}\bar{V}(0)\hat{q}^{-1/2}, \tag{2}$$

where $\hat{q}$ is the diagonal matrix, whose elements are the quantized longitudinal momenta $q_{\boldsymbol{n}} = \sqrt{E - E_{\boldsymbol{n}}}$ and $\bar{V}_{\boldsymbol{nm}}(x) \equiv \int_x^{x+\delta L} dx' \int d\boldsymbol{\rho} V(x', \boldsymbol{\rho}) u_{\boldsymbol{n}}(\boldsymbol{\rho}) u_{\boldsymbol{m}}(\boldsymbol{\rho})$.

Applying the standard formula for the composition of the scattering matrices and using the fact that $r = t - I$ we derive the relation between $S_{n+1} \equiv S(L + \delta L)$ and $S_n \equiv S(L)$, the scattering matrices corresponding to the system of the length $L = n\delta L$ and $L + \delta L = (n+1)\delta L$:

$$f(S_{n+1}) = f(e^{i\hat{q}\delta L} S_n e^{i\hat{q}\delta L}) + A_{n+1}, \tag{3}$$

where $A_{n+1} \equiv \hat{q}^{-1/2}\bar{V}(L)\hat{q}^{-1/2}$, and $f(S) \equiv i[(S-I)/(S+I)]$. The above equation is a direct generalization of the one-dimensional relation [9]. Differentiating it with respect to E one obtains the recursion relation for Q:

$$W_n Q_{n+1} W_n^T = C_n (J_n Q_n J_n^T + K_n) C_n + H_n. \tag{4}$$

All the matrices involved in this equation can be expressed through S_n, A_{n+1}, and $\hat{q}$ and their definitions are given in the Supplemental Material [13]. Both relations preserve the symmetries of the scattering and Wigner-Smith matrices, respectively: $S^\dagger = S^{-1}$, $S^T = S$, $Q^\dagger = Q$, $Q^T = Q$. They hold for any strength of disorder σ and are very convenient for numerical simulations, as they deal with the matrices corresponding to $d-1$ rather than d-dimensional systems.

Now we assume that disorder is weak, i.e., $l \gg \lambda_F$. Then an analysis of the relations (3) and (4) suggests that the change of S and the eigenvectors of Q at each step of the recursion is governed by the parameter $\delta L/\lambda_F$, while the change of the eigenvalues of Q by the parameter $\delta L/l$ [13]. As $\delta L/\lambda_F \gg \delta L/l$, this implies that S and the matrix of the eigenvectors of Q, O, represent fast variables, while $\tilde{\tau}_i$ are slow variables. Therefore, in the following we assume that for $L \gg \lambda_F$, S and O are statistically independent random matrices and the first two moments of the distribution of their matrix elements satisfy the following conditions

$$\langle S_{ij}\rangle = 0, \qquad \langle O_{ij}\rangle = 0, \qquad \langle S_{ij}S_{kl}\rangle = 0,$$
$$\langle S_{ij}S_{kl}^*\rangle = \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{N+1}, \qquad \langle O_{ij}O_{kl}\rangle = \frac{\delta_{ik}\delta_{jl}}{N}. \tag{5}$$

These relations can be justified by two observations: (i) the phases of the S-matrix elements are fast oscillating even in the absence of disorder, (ii) the momentum of a reflected particle is completely randomized for weak disorder. In Supplemental Material we explain why these conditions are strongly motivated by the recursion relations and check their validity by numerical simulations.

Fokker-Planck equation and the evolution equation for the density.—The recursion relation (4) can be transformed into the Fokker-Planck equation for the joint probability distribution function $P(\{\tilde{\tau}_i\})$ in the continuum limit $\delta L \to 0$. To this end, we first use the general relation between $P(\{\tilde{\tau}_i\})$ calculated at two consequent steps:

$$P_{n+1}(\{\tilde{\tau}_i\}) = P_n(\{\tilde{\tau}_i\}) + \left(-\sum_i \frac{\partial}{\partial\tilde{\tau}_i}\langle\delta\tilde{\tau}_i\rangle + \frac{1}{2}\sum_{ik}\frac{\partial^2}{\partial\tilde{\tau}_i\partial\tilde{\tau}_k}\langle\delta\tilde{\tau}_i\delta\tilde{\tau}_k\rangle\right) \times P_n(\{\tilde{\tau}_i\}), \tag{6}$$

where $\langle\ldots\rangle$ stands for the averaging over S, O, and $V(\boldsymbol{r})$ and only the terms up to the first order in δL must be retained on the right-hand side (r.h.s.). The averages $\langle\delta\tilde{\tau}_i\rangle$ and $\langle\delta\tilde{\tau}_i\delta\tilde{\tau}_k\rangle$ can be computed with the help of the perturbation theory:

$$\delta\tilde{\tau}_i = \langle i|O^T\delta Q O|i\rangle + \sum_{k\neq i}\frac{|\langle k|O^T\delta Q O|i\rangle|^2}{\tilde{\tau}_i - \tilde{\tau}_k}, \tag{7}$$

where $\{|i\rangle\}$ is the standard basis in $\mathbb{R}^N$ and we omit the index n for all variables to lighten the notation. The matrix $\delta Q \equiv Q_{n+1} - Q_n$ can be found from Eq. (4).

Introducing the scaled variables $\tau = \tilde{\tau}/\tau_s$ and $r = A_d(L/l)$, with $A_d \equiv \{[\sqrt{\pi}\Gamma(d+1/2)]/[\Gamma(d/2)]\}$, and taking the limit $\delta L \to 0$, we derive (see the Supplemental Material [13] for details) the Fokker-Planck equation for the distribution function $P(\{\tau_i\}, r)$:

$$\frac{\partial P}{\partial r} = \frac{1}{N}\sum_i \frac{\partial}{\partial\tau_i}\left[(N-1)\tau_i - 2N - \sum_{k\neq i}\frac{\tau_i^2}{\tau_i - \tau_k} + \frac{\partial}{\partial\tau_i}\tau_i^2\right]P. \tag{8}$$

The distribution function $P(\{\tau_i\}, r)$ contains the full information about the delay times; however in order to distinguish between the localized and delocalized phases of the closed system, it is sufficient to study a simpler quantity—the density of the delay times $\rho(\tau, r) = (1/N)\sum_i \langle \delta(\tau - \tau_i) \rangle$, which can be obtained from $P(\{\tau_i\}, r)$ by integrating out all but one variables τ_i.

The evolution equation for $\rho(\tau, r)$, which can be derived from Eq. (8) in the standard way [14], reads

$$\frac{\partial \rho}{\partial r} = \frac{\partial}{\partial \tau}\left[\rho\left(\tau - 2 - \tau^2 \int d\tau' \frac{\rho(\tau', t)}{\tau - \tau'}\right) + \frac{\partial}{\partial \tau}\frac{\tau^2 \rho}{2N}\right]. \quad (9)$$

Burgers equation and the stationary solution.—The integrodifferential equation for the density can be mapped onto a nonlinear partial differential equation employing the method used in Ref. [15]. We introduce the Stieltjes transform of $\rho(\tau, r)$ defined as

$$F(z, r) = \int_0^\infty d\tau' \frac{\rho(\tau', r)}{z - \tau'}. \quad (10)$$

The function $F(z, r)$ is analytic in the complex plane for all z except the positive real axis, where it is discontinuous:

$$F_\pm \equiv \lim_{\epsilon \to 0^+} F(\tau \pm i\epsilon) = \pm \frac{\pi}{i}\rho(\tau, r) + \int_0^\infty d\tau' \frac{\rho(\tau', r)}{\tau - \tau'}. \quad (11)$$

Using this formula, the analyticity of F and Eq. (9) one can show that F satisfies the nonlinear differential equation of the Burgers type

$$\frac{\partial F}{\partial r} = \frac{1}{2N}\frac{\partial}{\partial z}\left(N[2(z-2)F - z^2 F^2] + \frac{\partial}{\partial z} z^2 F\right), \quad (12)$$

whose solution allows us to find ρ through the relation $\rho(\tau, r) = (i/2\pi)(F_+ - F_-)$.

Hopf-Cole transformation and the nonstationary solution.—In order to find a solution of Eq. (12) we employ a variant of the Hopf-Cole transformation:

$$F(z, r) = \frac{z-2}{z^2} - \frac{4}{z^2}\frac{u'_s(s, r)}{u(s, r)}, \qquad s = -\frac{4N}{z} \quad (13)$$

which maps the equation for F onto the generalized diffusion equation:

$$8N \frac{\partial u}{\partial r} = 4s^2 u''_{ss} - s(s + 4N)u. \quad (14)$$

One can look for the general solution of this equation as a linear combination of the eigenfunctions $e^{-(\lambda/2)r} u_\lambda(s)$. It turns out that the spectrum is continuous for $\lambda = (4\mu^2 + 1)/4N$, and the corresponding orthogonal eigenfunctions are given by the Whittaker functions $W_{-N, i\mu}(s)$ with $\mu > 0$ [16]. Additionally to this set of the eigenstates there is another eigenfunction $W_{-N,1/2}(s)$ for $\lambda = 0$ corresponding to the stationary state [17]. Thus the solution of Eq. (14) can be written as

$$u(s, r) = c_0 W_{-N, \frac{1}{2}}(s) + \int_0^\infty d\mu c(\mu) e^{-[(4\mu^2+1)r/8N]} W_{-N, i\mu}(s), \quad (15)$$

where the coefficients $c_0 = \Gamma(N+1)$ and $c(\mu) = \{[8\mu \sinh(\pi\mu)\Gamma(N + \frac{1}{2} + i\mu)\Gamma(N + \frac{1}{2} - i\mu)]/[\pi(1 + 4\mu^2) \times \Gamma(N)]\}$ are determined from the initial condition $u(s, 0) = e^{-s/2}$. This formula along with Eq. (13) and the relation $\rho(\tau, r) = (i/2\pi)(F_+ - F_-)$ provides the general solution for $\rho(\tau, r)$, which is valid for any $L/\lambda_F \gg 1$, $N \propto (M/\lambda_F)^{d-1} \gg 1$ and $l/\lambda_F \gg 1$.

The density of delay times for a cubic sample in the thermodynamic limit.—For a cubic sample $M = L$ and it follows from Eq. (15) that the r dependence of the solution is governed by the parameter $r/N \propto (\lambda_F/l)(L/\lambda_F)^{2-d}$, which has a meaning of the inverse dimensionless conductance g^{-1}. One can see that in the thermodynamic limit $(L/\lambda_F \to \infty)$, $r/N \to \infty$ for $d < 2$ and $r/N \to 0$ for $d > 2$. In the former case, the solution tends to its localized fix point given by $W_{-N,1/2}(s)$, whereas in the latter case it tends to the metallic (diffusive) fixed point, where the contribution from all $W_{-N,i\mu}(s)$ is important. The $d = 2$ case is a marginal one and requires more careful treatment [12].

Localized regime.—In the localized regime, where the solution is determined by the stationary state, the density can be found from the asymptotics of $W_{-N,1/2}(s)$ at $N \to \infty$. As $s \propto N/\tau$, such asymptotics depend generally on the value of τ. It turns out that one needs to consider separately two different regimes: $\tau \sim N^0$ and $\tau \sim N^2$, for which the asymptotics of $W_{-N,1/2}(s)$ and hence the expressions for the density are different:

$$\rho_{st}(\tau) = \begin{cases} \frac{2}{\pi}\frac{\sqrt{\tau-1}}{\tau^2}, & \tau \sim N^0, \quad \tau \geq 1 \\ \frac{4N}{\tau^2}, & \tau \gtrsim N^2. \end{cases} \quad (16)$$

A long τ^{-2} tail in the distribution of the delay times in the localized regime was previously found analytically for 1D and quasi-one-dimensional systems [7,18]. In the numerical simulations for the 2D Anderson model both power laws $\tau^{-3/2}$ and τ^{-2}, which follow from our result, were identified [19].

The localization length can be estimated as $\xi \propto v_F \tau_W^{\mathrm{typ}}$, where v_F is the Fermi velocity and τ_W^{typ} is a typical value of the Wigner delay time $\tau_W = \sum_{i=1}^N \tilde{\tau}_i$. According to Eq. (16) a typical value of $\tilde{\tau}$ is of order of τ_s and therefore $\xi \propto N v_F \tau_s = Nl$. This result is in agreement with the quasi-one-dimensional result, where $L \to \infty$ at constant W. For a cubic sample with $d < 2$, $N \propto (L\sqrt{E})^{d-1}$ grows with L, however, $\xi/L \to 0$ in the thermodynamic limit, as expected in the localized regime.

Diffusive and ballistic regimes.—In the metallic regime, where $r/N \ll 1$, a direct analysis of Eq. (15) is complicated, so it is more convenient to derive the limiting solution in a different way. For $r/N \ll 1$ the last term in Eq. (12) is small and hence can be neglected, then introducing the new function $\psi(\xi, r)$, such that $F = (z-2)/z^2 + z^{-1}\psi(\ln z, r)$, one can map Eq. (12) onto the inviscid forced Burgers equation

$$\frac{\partial \psi}{\partial r} + \psi \frac{\partial \psi}{\partial \xi} = 2e^{-\xi} - 4e^{-2\xi}, \tag{17}$$

which can be solved by the method of characteristics:

$$F(z, r) = \frac{z - 2 + 2\sqrt{1 - z + \frac{z^2}{z_0^2}}}{z^2}, \tag{18}$$

where $z_0 = z_0(z, r)$ is determined implicitly by the equation $f(z_0, r) = z$ with $f(x, r) \equiv (x/2)\{x[1 + \cosh(2r/x)] + 2\sinh(2r/x)\}$. This formula gives a solution at an arbitrary value of $r \propto L/l$ in the metallic regime. Now we can analyze it in detail in the ballistic ($L/l \ll 1$) and the diffusive ($L/l \gg 1$) limits.

In the ballistic regime, $r \ll 1$, one can expand $f(x, r)$ in the power series in r/x and find z_0 approximately. The leading order result reads

$$F(z, r) \approx \frac{1}{z - 2r}, \quad \Rightarrow \rho(\tau, r) = \delta(\tau - 2r), \tag{19}$$

which describes a ballistic motion with the Fermi velocity, $L \propto v_F \tilde{\tau}$, as expected.

In the diffusive regime ($r \gg 1$), the solution can be found by scaling $z_0 = yr$, $z = wr^2$, and $F(z, r) = (1/r^2)\tilde{F}(z/r^2, r)$ and keeping only the leading order terms in r. The appearance of such a scaling implies that a typical delay time $\tilde{\tau} \propto L^2/D$ (D is the classical diffusion constant), which is very natural in the diffusive regime. The function $\tilde{F}(w, r)$ is then given by

$$\tilde{F}(w, r) = \frac{1}{w} + \frac{2}{rw^{\frac{3}{2}}}\sqrt{\frac{w}{y^2} - 1}, \tag{20}$$

where $y = y(w)$ satisfies the equation $y \cosh y^{-1} = \sqrt{w}$. This result implies that $\rho(\tau, r) \approx \tilde{\rho}(w)/r^3 \neq 0$ only for $w \in [w_{\min}, w_{\max}]$, where $w_{\min} \approx \pi^2/16r^2$ and $w_{\max} \approx 2.28$. The behavior of $\tilde{\rho}(w)$ can be found analytically at $w \to w_{\min}$, where $\tilde{\rho}(w) \approx 2/\pi w^{3/2}$, and at $w \to w_{\max}$, where $\tilde{\rho}(w) \approx 2\sqrt{(w_{\max} + 1)(w_{\max} - w)}/\pi w_{\max}^2$. For intermediate values of w, $\tilde{\rho}(w)$ can be determined numerically from Eq. (20).

The appearance of the power law $\tau^{-3/2}$ tail in the metallic regime can be related to the classical diffusion [20].

Comparison with the DMPK equation and other approaches.—Since our method works also for a quasi-one-dimensional geometry, it makes sense to compare it with the DMPK equation. In Refs. [21,22] the DMPK equation for the reflection eigenvalues in the presence of absorption was derived. As the proper delay times can be extracted from the reflection eigenvalues in the limit of weak absorption [23], one can obtain the DMPK equation for proper delay times and compare it with our Eq. (8). It turns out that Eq. (8) coincides with the DMPK equation in the quasi-one-dimensional case.

We stress that the scattering isotropy assumption for a thin slice, which is crucial for the derivation of the DMPK equation [14], is not used in our approach, in which the scattering properties of a slice are treated microscopically. This allows us to study the problem in higher dimensions.

In Ref. [24] a similar scattering setup with a single multichannel lead was considered and a relation between the statistics of the partial delay times and certain correlation functions of the nonlinear σ model was derived. In contrast to the present method, such an approach is limited by the available solutions of the σ model: one can either employ a nonperturbative solution for the quasi-one-dimensional geometry or rely on the perturbative expansion in the metallic regime in higher dimensions. These limitations are shared by most of the other known methods, in contrast to our approach.

Another outcome of Ref. [24] is a simple relation between the statistics of the delay times and the local statistics of the wave functions derived for a single-channel lead. It would be of great interest to generalize that relation to a multichannel case; this would allow one to get information about wave functions of a closed sample directly from the results of the present work.

Conclusions.—We have developed a new approach to the d-dimensional Anderson localization problem, which enabled us to obtain in a nonperturbative way the statistics of the delay times in the ballistic, diffusive, and localized regimes at weak disorder. It overcomes the limitations of the existing methods and paves the way for studying analytically Anderson localization in higher dimensional systems.

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