

Dynamics of Finite-Temperature Conformal Field Theories from Operator Product Expansion Inversion Formulas

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We apply the operator product expansion inversion formula to thermal two-point functions of bosonic and fermionic conformal field theories in general odd dimensions. This allows us to analyze in detail the operator spectrum of these theories. We find that nontrivial thermal conformal field theories arise when the thermal mass satisfies an algebraic transcendental equation that ensures the absence of an infinite set of operators from the spectrum. The solutions of these gap equations for general odd dimensions are in general complex numbers and follow a particular pattern. We argue that this pattern unveils the large- N vacuum structure of the corresponding theories at zero temperature.

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Introduction.—The description of critical systems in nontrivial backgrounds requires data not present in the plane geometry. Perhaps the simplest example is that of conformal field theories (CFTs) on $S^1_\beta \times \mathbb{R}^{d-1}$, with β the radius of the circle, that describe finite-size or finite-temperature critical systems. In such a case, the two-point function of a scalar operator $\phi(x)$ will in principle depend on the one-point functions of all operators that appear in its operator product expansion (OPE) with itself, since the latter can be nonzero. In particular, for an operator $\mathcal{O}(x)$ with dimension $\Delta_{\mathcal{O}}$ we schematically have $\langle \mathcal{O}(x) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} \propto b_{\mathcal{O}}/\beta^{\Delta_{\mathcal{O}}}$, where $b_{\mathcal{O}}$ is a dimensionless parameter.

In $d = 2$ the plane is conformally related to the cylinder and, although one-point functions of conformal primaries vanish on the latter, there exist operators such as the energy-momentum tensor which transform anomalously under a conformal map. This fixes their one-point functions on the cylinder, and therefore the CFT data on \mathbb{R}^2 determine the finite-size or finite-temperature corrections to correlation functions on $S^1_\beta \times \mathbb{R}$ [1,2].

For $d > 2$ there is no conformal transformation between \mathbb{R}^d and $S^1_\beta \times \mathbb{R}^{d-1}$, and generically one needs to find other ways to determine the additional data $b_{\mathcal{O}}$. A first step in this direction was described in Ref. [3], where the leading anisotropic finite-size corrections to the two-point function of scalars in \mathbb{R}^d were connected to the ratio of the thermal free-energy density of the system and the normalization C_T

of the energy-momentum tensor two-point function. An extension of these ideas to the nontrivial 3D $O(N)$ vector model was performed in Refs. [4,5], where the relevance of the planar OPE to the description of the finite-size or finite-temperature CFTs was demonstrated. In the latter works the crucial point was that parameters such as $b_{\mathcal{O}}$ were independently determined by the gap equation of the vector model. In particular, once the bosonic thermal mass was determined, all one-point functions could be evaluated and hence the full finite-temperature two-point function could be reconstructed.

In more recent developments, the improved understanding of CFTs on \mathbb{R}^d using numerical and analytic bootstrap methods (see Ref. [6] for a recent review) calls for an extension of these advances to finite-size or finite-temperature critical systems. In this context an interesting work has recently appeared [7], whose main result is a Lorentzian inversion formula for the thermal two-point function of a scalar $\phi(x)$ with dimension Δ_ϕ . Using the OPE one can show that the Euclidean position-space [8] thermal two-point function takes the generic form

$$\langle \phi(x)\phi(0) \rangle_\beta \equiv g(r, \cos \theta) = \sum_{\mathcal{O}_s} a_{\mathcal{O}_s} \left(\frac{r}{\beta} \right)^{\Delta_{\mathcal{O}_s}} \frac{C_s^\nu(\cos \theta)}{r^{2\Delta_\phi}}, \quad (1)$$

where $x^\mu = (\tau, \mathbf{x})$ are coordinates on $S^1_\beta \times \mathbb{R}^{d-1}$ with period $\tau \sim \tau + \beta$, $r = |\mathbf{x}|$, and $\theta \in [0, \pi]$ is a polar angle when \mathbb{R}^{d-1} is written in spherical coordinates. $C_s^\nu(\cos \theta)$ are Gegenbauer polynomials with $\nu = d/2 - 1$. The sum in Eq. (1) runs over all operators \mathcal{O}_s in the OPE $\phi \times \phi$ with spin s and dimension $\Delta_{\mathcal{O}_s}$. The coefficients $a_{\mathcal{O}_s}$ are given by (following the conventions of Ref. [7])

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$$a_{\mathcal{O}_s} = \frac{s!}{2^s(\nu)_s} \frac{g_{\phi\phi\mathcal{O}_s} b_{\mathcal{O}_s}}{C_{\mathcal{O}_s}}, \quad (2)$$

with $C_{\mathcal{O}_s}$ and $g_{\phi\phi\mathcal{O}_s}$ the corresponding two- and three-point function coefficients, and $(a)_n$ the Pochhammer symbol. The unit operator 1 is the unique operator with dimension zero, and here

$$a_1 = \frac{2^{2\Delta_\phi-d}\Gamma(\Delta_\phi)}{\pi^{d/2}\Gamma(\frac{d}{2}-\Delta_\phi)}, \quad (3)$$

so that the momentum-space two-point function is unit normalized.

Complexifying Δ , one defines the spectral function $a(\Delta, s)$ via

$$g(r, \cos\theta) = \sum_s \oint_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\Delta}{2\pi i} a(\Delta, s) \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\phi-\Delta}}, \quad (4)$$

whose poles at $\Delta = \Delta_{\mathcal{O}_s}$ with residues $-a_{\mathcal{O}_s}$ yield the physical spectrum. Assuming that the physical poles lie on the right of the imaginary axis, one can close the contour clockwise for $r < 1$ (we set $\beta = 1$ from now on) if $a(\Delta, s)$ does not grow exponentially at infinity. One can then use the orthogonality of Gegenbauer polynomials (see, e.g., Ref. [9], Sec. 7.313) to project the right-hand side of Eq. (4) on a spin- s state and then integrate with a suitable power in the region of convergence $r \in [0, 1]$ to obtain $a(\Delta, s)$ as

$$a(\Delta, s) = \frac{1}{N_{s,\nu}} \int_0^1 \frac{dr}{r^{\Delta-2\Delta_\phi+1}} \times \int_{-1}^1 dx (1-x^2)^{\nu-1/2} C_s^\nu(x) g(r, x), \quad (5)$$

where

$$N_{s,\nu} = \frac{2^{1-2\nu} \pi \Gamma(s+2\nu)}{(s+\nu)\Gamma(s+1)\Gamma^2(\nu)}. \quad (6)$$

This is termed the Euclidean inversion formula in Ref. [7].

Writing $x = \cos\theta = (w+1/w)/2$ with $w = e^{i\theta}$ one can transform Eq. (5) into a contour integral over the unit circle in the complex- w plane. To exploit the analytic structure of the two-point function $g(r, \cos\theta)$, one would like to allow w to explore the full complex plane. This can be done by a suitable complexification of the Euclidean variables r, θ , defining $z = rw$ and $\bar{z} = r/w$, which are now independent real variables. As a function of w , $g(r, w)$ is assumed to have the cuts $(-\infty, -1/r)$, $(-r, 0)$, $(0, r)$, and $(1/r, \infty)$, and to grow not faster than w^{s_0} ($1/w^{s_0}$) for large (small) w for some constant s_0 . Moreover, one needs to use the analytic extension of the Gegenbauer polynomials to the whole complex plane as

$$C_s^\nu(w) = \frac{\Gamma(s+2\nu)}{\Gamma(\nu)\Gamma(s+\nu+1)} (F_s(1/w)e^{i\nu\pi} + F_s(w)e^{-i\nu\pi}), \quad (7)$$

where

$$F_s(w) = w^{s+2\nu} {}_2F_1(s+2\nu, \nu; s+\nu+1; w^2). \quad (8)$$

Then, the integral giving $a(\Delta, s)$ will receive contributions from the discontinuities across the cuts of $g(r, w)$ as well as from the arcs at infinity. The final result is

$$a(\Delta, s) = a_{\text{disc}}(\Delta, s) + \theta(s_0 - s) a_{\text{arcs}}(\Delta, s), \quad (9)$$

where

$$a_{\text{disc}}(\Delta, s) = K_s \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^\infty \frac{dz}{z} \left[(z\bar{z})^{\Delta_\phi - (\Delta/2) - \nu} \times (z - \bar{z})^{2\nu} F_s\left(\sqrt{\frac{\bar{z}}{z}}\right) \text{disc}[g(z, \bar{z})] \right], \quad (10)$$

with

$$K_s = (1 + (-1)^s) \frac{\Gamma(s+1)\Gamma(\nu)}{4\pi\Gamma(s+\nu)}. \quad (11)$$

The discontinuity relevant for the evaluation of Eq. (10) is the one across the cut $(1/r, \infty)$, since all others are related to it.

Gap equations from the inversion formula.—The OPE inversion formulas are powerful tools when they are applied to already known correlation functions. In this context, one needs an ansatz for the thermal two-point function before applying OPE inversion. For bosons, one of the simplest choices is to consider the momentum-space two-point function

$$G^{(d)}(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m_{\text{th}}^2}, \quad (12)$$

where $\omega_n = 2\pi n$, $n = 0, \pm 1, \pm 2, \dots$, are the bosonic Matsubara frequencies along the finite direction. Clearly, Eq. (12) is motivated by known work on thermal field theory which shows that fields develop generically a thermal mass m_{th} at finite temperature. From our point of view we are asking whether the simple ansatz Eq. (12) can define a thermal CFT. We make no reference to a Lagrangian, although it is known that Eq. (12) can be obtained, e.g., in the large- N limit of the $O(N)$ model.

In arbitrary d , Eq. (12) can be Fourier transformed to

$$G^{(d)}(\tau, \mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \sum_{n=-\infty}^{\infty} \left(\frac{m_{\text{th}}}{|X_n|} \right)^{d/2-1} K_{d/2-1}(m_{\text{th}}|X_n|), \quad (13)$$

$$X_n = (\tau - n, \mathbf{x}),$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind. Defining $z = \tau + i|\mathbf{x}|$, we have $|X_n| = \sqrt{(n-z)(n-\bar{z})}$. From now on we focus on odd $d = 2k + 1$, $k = 1, 2, \dots$, and in that case we may write [9], Sec. 8.468, [10]

$$G^{(2k+1)}(\tau, \mathbf{x}) = \frac{1}{2^{k+1}\pi^k} \sum_{n=-\infty}^{\infty} \frac{m_{\text{th}}^{k-1}}{|X_n|^k} e^{-m_{\text{th}}|X_n|} \times \sum_{p=0}^{k-1} \frac{L_{k,p}}{(m_{\text{th}}|X_n|)^p}, \quad (14)$$

with

$$L_{k,p} = \frac{(k-1+p)!}{2^p p!(k-1-p)!}. \quad (15)$$

These coefficients also appear in the Bessel polynomials [11]

$$y_n(x) = \sum_{p=0}^n L_{n+1,p} x^p = \sqrt{\frac{2}{\pi x}} e^{1/x} K_{n+1/2}(1/x). \quad (16)$$

The relevant discontinuity $\text{disc}(G^{(d)})$ now follows simply from understanding the discontinuity of the function

$$f^{(k)}(x) = \frac{a^{k-1}}{(\sqrt{x})^k} e^{-a\sqrt{x}} \sum_{p=0}^{k-1} \frac{L_{k,p}}{(a\sqrt{x})^p} \quad (17)$$

across the cut due to the square-root branch point at $x = 0$. Assuming that the cut goes from $x = 0$ to $x = \infty$, it can be verified that

$$\text{disc}(f^{(k)}(x)) = \frac{2}{x^{k-1}} \left(\frac{1}{\sqrt{-x}} U_k(x) \cos(a\sqrt{-x}) + V_k(x) \sin(a\sqrt{-x}) \right), \quad (18)$$

where

$$U_k(x) = \frac{1}{2} [\theta_{k-1}(\sqrt{x}) + \theta_{k-1}(-\sqrt{x})],$$

$$V_k(x) = \frac{1}{2\sqrt{x}} [\theta_{k-1}(\sqrt{x}) - \theta_{k-1}(-\sqrt{x})], \quad (19)$$

with $\theta_n(x) = x^n y_n(1/x)$ the so-called reverse Bessel polynomials [12].

Using the results Eqs. (18) and (19), we can now calculate Eq. (9). For the discontinuity part we find

$$a_{\text{disc},0}^{(k)}(\Delta, s) = [1 + (-1)^s] \frac{1}{2^{2s+k} s! \Gamma(k+s-\frac{1}{2})} \times \sum_{n=0}^{k-1+s} \frac{2^{n+1}}{n!} \frac{[2(k-1+s)-n]!}{(k-1+s-n)!} \times m_{\text{th}}^n Li_{2k-1+s-n}(e^{-m_{\text{th}}}), \quad (20)$$

in the conventions of Ref. [7], where $Li_\alpha(z) = \sum_{n=1}^{\infty} z^n/n^\alpha$ is the polylogarithm. The result Eq. (20) only pertains to the leading term in a \bar{z} expansion of the quantity under the integral in Eq. (9) [13], reproducing contributions of operators with $\Delta = d - 2 + s$. These are higher-spin conserved currents saturating the unitarity bound. Subleading terms in the \bar{z} expansion can also be considered and would lead to expressions that could be denoted by $a_{\text{disc},1}^{(k)}, a_{\text{disc},2}^{(k)}, \dots$, corresponding to higher-twist operators.

The arc part $a_{\text{arcs}}^{(d)}(\Delta, s)$ is nonzero only for $s = 0$, and in that case it needs to be taken into account carefully. We find

$$a_{\text{arcs}}^{(d)}(\Delta, 0) = \frac{1}{2^{\Delta-(d-5)/2} \sqrt{\pi}} m_{\text{th}}^\Delta \Gamma\left(-\frac{\Delta}{2}\right) \Gamma\left(-\frac{\Delta-d+2}{2}\right). \quad (21)$$

Notice that for $m_{\text{th}} = 0$, only the $\Delta = 0$ term survives, giving the contribution of the identity operator. This, along with the corresponding $m_{\text{th}} = 0$ contributions from $a_{\text{disc}}^{(k)}(\Delta, s)$, yields the spectrum of generalized free CFTs. When $m_{\text{th}} \neq 0$ and for $\Delta > 0$, Eq. (21) yields contributions of an infinite tower of scalar operators with $\Delta = 2m$, $m = 1, 2, \dots$, as well as contributions with $\Delta = d - 2 + 2l$, $l = 0, 1, 2, \dots$. The former correspond to operators of the form σ^m , $m = 1, 2, \dots$, where σ is the shadow of ϕ^2 .

For the latter operators we will first focus on the $l = 0$ case, corresponding to the ϕ^2 operator, which appears both from Eqs. (21) and (20). If we demand the absence of this operator from the spectrum, as required by the fact that it is substituted by the σ operator, then the residue of the $\Delta = d - 2$ arc contribution should cancel the $s = 0$ contribution in Eq. (20). This turns out to give rise to a condition that determines m_{th} , namely,

$$\sum_{n=0}^{k-1} \frac{2^{n+1}}{n!} \frac{(2(k-1)-n)!}{(k-1-n)!} m_{\text{th}}^n Li_{2k-1-n}(e^{-m_{\text{th}}}) = -\frac{1}{2\sqrt{\pi}} m_{\text{th}}^{2k-1} \Gamma\left(-k + \frac{1}{2}\right). \quad (22)$$

This is called the gap equation and it is here presented for any $d = 2k + 1$, $k = 1, 2, \dots$.

Higher poles in Eq. (21) at $\Delta = d - 2 + 2l$, $l = 1, 2, \dots$, correspond to scalar operators of the form $\phi \partial^{2l} \phi$. Such operators also arise from subleading terms in the \bar{z} expansion of the quantity under the integral in Eq. (9), from expressions we previously referred to as $a_{\text{disc},1}^{(k)}, a_{\text{disc},2}^{(k)}, \dots$. These operators should also disappear from the spectrum when the gap equation (22) is satisfied. Although we have verified this in a couple of cases, we do not have a general proof for it.

The arc contribution of the identity operator provides a quick consistency check of our computations. Since the

identity operator has $\Delta = 0$, we see that the pole associated with it appears due to $\Gamma[-(\Delta/2)]$ in Eq. (21). For the residue of that pole, we find

$$\text{Res}_{\Delta=0}^{(d)}[a_{\text{arcs}}(\Delta, 0)] = -\frac{2^{(d-3)/2}}{\sqrt{\pi}} \Gamma\left(\frac{d}{2} - 1\right), \quad (23)$$

exactly as required to reproduce the correct normalization of the identity operator in our conventions—for this we need to take into account a_1 from Eq. (3) and recall that we are working in conventions where the $1/2^{k+1}\pi^k$ in Eq. (14) has been rescaled away.

It is also possible to study finite-temperature fermionic two-point functions using the inversion formula. The simplest case to consider is the singlet projection of the two-point functions of Dirac fermions $\psi_i(x)$, $\bar{\psi}_i(x)$ in odd dimensions,

$$\langle \psi_i(x) \bar{\psi}_i(0) \rangle_\beta \equiv \tilde{g}(r, \cos \theta) = \sum_{\tilde{\mathcal{O}}_s \neq 1} \tilde{a}_{\tilde{\mathcal{O}}_s} \left(\frac{r}{\beta}\right)^{\Delta_{\tilde{\mathcal{O}}_s}} \frac{C_s^{\nu}(\cos \theta)}{r^{2\Delta_\psi}}, \quad (24)$$

with $\Delta_\psi = \Delta_\phi + 1/2$. We denote by $i, j = 1, 2, \dots, 2^{(d-1)/2}$ the spinor indices. Notice that Eq. (24) vanishes at zero temperature, which means that the unit operator is absent in the finite-temperature OPE. The corresponding unit-normalized momentum-space two-point function is

$$\tilde{G}^{(d)}(\omega_n, \mathbf{p}) = \frac{\tilde{m}_{\text{th}}}{\omega_n^2 + \mathbf{p}^2 + \tilde{m}_{\text{th}}^2}, \quad (25)$$

where the fermionic Matsubara frequencies are $\omega_n = 2\pi(n + 1/2)$, $n = 0, \pm 1, \pm 2, \dots$. The propagator Eq. (25) vanishes for $\tilde{m}_{\text{th}} = 0$, so we will only consider $\tilde{m}_{\text{th}} \neq 0$ in the fermionic case from now on. The calculations follow closely the bosonic case—e.g., it is known that fermionic Matsubara sums reduce to a linear combination of bosonic ones. We then notice that by virtue of the relationship $\Delta_\psi = \Delta_\phi + 1/2$, the fermionic formulas can all be obtained from the bosonic ones by the simple shift $\Delta \rightarrow \Delta - 1$. The arc contributions in the fermionic case are thus given by

$$\tilde{a}_{\text{arcs}}^{(d)}(\Delta, 0) = -\frac{1}{2^{\Delta-(d-3)/2}\sqrt{\pi}} \tilde{m}_{\text{th}}^{\Delta-1} \Gamma\left(-\frac{\Delta-1}{2}\right) \times \Gamma\left(-\frac{\Delta-d+1}{2}\right), \quad (26)$$

relevant for operators of dimension $\Delta = 2m + 1$ and $\Delta = d - 1 + 2m$, $m = 0, 1, 2, \dots$. The former are contributions that do not arise from the discontinuity part, having the form $\tilde{\sigma}^m$ with $\tilde{\sigma}$ the shadow field of $\bar{\psi}\psi$. Note that, as expected, there is no contribution from the unit operator.

The latter provide contributions from operators of the form $\bar{\psi}\partial^{2m}\psi$ that coincide with those coming from the discontinuity. The fermionic gap equation is the condition for the cancellation of the latter operators from the spectrum, and it reads

$$\sum_{n=0}^{k-1} \frac{2^{n+1}}{n!} \frac{[2(k-1)-n]!}{(k-1-n)!} \tilde{m}_{\text{th}}^{n+1} \text{Li}_{2k-1-n}(-e^{-\tilde{m}_{\text{th}}}) = -\frac{1}{2\sqrt{\pi}} \tilde{m}_{\text{th}}^{2k} \Gamma\left(-k + \frac{1}{2}\right). \quad (27)$$

Discussion.—One of the messages of this work is that OPE inversion formulas can reveal the nontrivial dynamics of finite-temperature CFTs. In the simple examples we have studied, the dynamics effect a rearrangement in the operator spectrum which is ensured by the gap equations (22) and (27). An analysis of the gap equations shows that their solutions follow a pattern which, as we will argue below, is intimately related to the vacuum structure of scalar and fermionic theories near *even* dimensions.

In the bosonic case the gap equation (22) in $d = 3$ reads

$$-m_{\text{th}} = 2 \log(1 - e^{-m_{\text{th}}}), \quad (28)$$

with the well-known solution

$$m_{\text{th}}^{(d=3)} = 2 \log\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.96242. \quad (29)$$

In $d = 5$ the gap equation becomes [14]

$$-\frac{1}{6} m_{\text{th}}^3 = \text{Li}_3(e^{-m_{\text{th}}}) + m_{\text{th}} \text{Li}_2(e^{-m_{\text{th}}}). \quad (30)$$

This has a complex conjugate pair of solutions given numerically by

$$m_{\text{th}}^{(d=5)} \approx 1.17431 \pm 1.19808i. \quad (31)$$

In fact, we find that for $d = 3, 7, 11, \dots$, the bosonic gap equation (22) has a unique real solution for m_{th} and complex solutions that come in conjugate pairs, except in the case $d = 3$, where there are no complex solutions. To give another example, in $d = 7$ we find a real and a pair of complex conjugate solutions. For $d = 5, 9, 13, \dots$, we do not find any real solutions, and the gap equation only has pairs of complex conjugate solutions. In $d = 5$ we only find the solutions Eq. (31), while in $d = 9$ we find four complex conjugate pairs of solutions. Notice also that $m_{\text{th}} = 0$ is never a solution of the bosonic gap equations.

The fermionic gap equations in $d = 3, 5$ are given, respectively, by [14]

$$-\tilde{m}_{\text{th}}^2 = 2\tilde{m}_{\text{th}} \log(1 + e^{-\tilde{m}_{\text{th}}}), \quad (32)$$

$$-\frac{1}{6}\tilde{m}_{\text{th}}^4 = \tilde{m}_{\text{th}}\text{Li}_3(-e^{-\tilde{m}_{\text{th}}}) + \tilde{m}_{\text{th}}^2\text{Li}_2(-e^{-\tilde{m}_{\text{th}}}). \quad (33)$$

For $d = 3$ and $\tilde{m}_{\text{th}} \neq 0$, Eq. (32) has only a pair of complex conjugate imaginary solutions $\tilde{m}_{\text{th}}^{(d=3)} = \pm 2\pi i/3$. For $d = 5$, Eq. (33) has a pair of opposite real solutions, as well as a pair of complex conjugate imaginary ones which can be found numerically. This pattern continues to higher dimensions; namely, for $d = 7, 11, 15, \dots$ there is no real solution to the corresponding fermionic gap equation, while for $d = 9, 13, 17, \dots$ there is always a pair of opposite real solutions and an increasing number of complex conjugate ones.

The above pattern for the solutions of bosonic and fermionic gap equations for all *odd* d fits nicely with a renormalization-group understanding of universality classes of scalars and fermions in general dimensions. In the bosonic case the standard lore is that the large- N universality class for scalars in $d = 2k + 1$, $k = 1, 2, \dots$, is accessible via the ϵ expansion starting from $d = 2k + 2$. Using the general- d large- N results of Refs. [15–22], this has been verified in specific cases in Refs. [23–25]. The key ingredient in such studies is the Hubbard-Stratonovich transformation, which introduces a field σ via the classically marginal interaction $\sigma\phi^2$. This way σ has dimension $\Delta_\sigma = 2$ in all d , and the scalars ϕ can be integrated out resulting in an effective potential for σ of the general form

$$V_{\text{eff}}(\sigma) \sim \text{Tr}_d \log(-\partial^2 + \sigma) + g_*\sigma^{d/2} + \dots, \quad (34)$$

where g_* is some critical dimensionless coupling. For general d the effective potential can also receive contributions from terms involving derivatives of σ , but the term $\sigma^{d/2}$ is universal. Performing the $\text{Tr}_d \log$ calculation in $d - \epsilon$ one finds that for $d = 4, 8, 12, \dots$ there is a resulting contribution of the form $\sigma^{d/2} \log \sigma^2$, which is positive and dominates for large σ . Thus, besides possible local minima, the effective potential has a global minimum. On the other hand, for $d = 6, 10, 14, \dots$ the term $\sigma^{d/2}$ leads to an unbounded potential, and hence to the absence of a global minimum, regardless of the sign of the $\text{Tr}_d \log$ contribution. This matches exactly the pattern we see for m_{th} . A real m_{th} implies a global minimum, while a complex m_{th} signals unstable local extrema with nonzero decay width.

In the fermionic case our results are consistent with the understanding that the corresponding large- N universality classes in $d = 2k + 1$, $k = 1, 2, \dots$, are also accessible via the ϵ expansion starting from a generalization of the Gross-Neveu-Yukawa model to $d = 2k + 2$ [26]. The corresponding Hubbard-Stratonovich transformation introduces the field $\tilde{\sigma}$ via the classically marginal interaction $\tilde{\sigma}\bar{\psi}\psi$. Here, $\tilde{\sigma}$ has dimension $\Delta_{\tilde{\sigma}} = 1$ in all d , and integrating out the fermions leads to an effective potential of the form

$$V_{\text{eff}}(\tilde{\sigma}) \sim -\text{Tr}_d \log(\not{\partial} + \tilde{\sigma}) + \tilde{g}_*\tilde{\sigma}^d + \dots \quad (35)$$

Notice that the $\text{Tr}_d \log$ term enters with the opposite sign compared to the bosonic case. In this case the universal term $\tilde{\sigma}^d$ gives always a bounded from below contribution (recall d is even). However, the $\text{Tr}_d \log$ term alters the form of the effective potential in $d - \epsilon$. More specifically, for $d = 4, 8, 12, \dots$, this term gives a negative contribution that dominates at infinity leading to an unstable vacuum structure, while for $d = 6, 10, 14, \dots$, it gives a positive contribution that guarantees the presence of a global minimum. In either case there can be a number of unstable extrema. This matches exactly the pattern for the \tilde{m}_{th} solutions to the fermionic gap equations.

To summarize, OPE inversion formulas applied to CFTs in nontrivial geometries reveal crucial dynamical properties of critical systems at the level of the operator spectrum. The consistency of the lift to the nontrivial geometry requires that CFTs develop thermal masses that solve a gap equation. Remarkably, these thermal masses also encode information about the vacuum structure of CFTs even at zero temperature.

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- [1] J. L. Cardy, *Nucl. Phys.* **B270**, 186 (1986).
 - [2] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.* **56**, 742 (1986).
 - [3] J. L. Cardy, *Nucl. Phys.* **B290**, 355 (1987).
 - [4] A. C. Petkou and N. D. Vlachos, *Phys. Lett. B* **446**, 306 (1999).
 - [5] A. C. Petkou and N. D. Vlachos, [arXiv:hep-th/9809096](https://arxiv.org/abs/hep-th/9809096).
 - [6] D. Poland, S. Rychkov, and A. Vichi, [arXiv:1805.04405](https://arxiv.org/abs/1805.04405).
 - [7] L. Iliesiu, M. Koloğlu, R. Mahajan, E. Perlmutter, and D. Simmons-Duffin, [arXiv:1802.10266](https://arxiv.org/abs/1802.10266).
 - [8] For the corresponding momentum-space expression, see Refs. [4,5].
 - [9] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 8th ed. (Academic Press, New York, 2015).
 - [10] In the conventions of Ref. [7] the normalization $1/2^{k+1}\pi^k$ in Eq. (14) is rescaled away.
 - [11] H. L. Krall and O. Frink, *Trans. Am. Math. Soc.* **65**, 100 (1949).
 - [12] E. Grosswald, *Bessel Polynomials* (Springer-Verlag, Berlin, 1978).
 - [13] This is denoted by the subscript “0.”
 - [14] E. G. Filothodoros, A. C. Petkou, and N. D. Vlachos, [arXiv:1803.05950](https://arxiv.org/abs/1803.05950).
 - [15] A. N. Vasil’ev, Yu. M. Pis’mak, and Yu. R. Khonkonen, *Theor. Math. Phys.* **46**, 104 (1981).
 - [16] A. N. Vasil’ev, Yu. M. Pis’mak, and Yu. R. Khonkonen, *Theor. Math. Phys.* **47**, 465 (1981).
 - [17] A. N. Vasil’ev, Yu. M. Pis’mak, and Yu. R. Khonkonen, *Theor. Math. Phys.* **50**, 127 (1982).

- [18] K. Lang and W. Rühl, *Z. Phys. C* **50**, 285 (1991).
- [19] K. Lang and W. Rühl, *Nucl. Phys.* **B402**, 573 (1993).
- [20] K. Lang and W. Rühl, *Nucl. Phys.* **B400**, 597 (1993).
- [21] A. Petkou, *Ann. Phys. (N.Y.)* **249**, 180 (1996).
- [22] A. C. Petkou, *Phys. Lett. B* **359**, 101 (1995).
- [23] L. Fei, S. Giombi, and I. R. Klebanov, *Phys. Rev. D* **90**, 025018 (2014).
- [24] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, *Phys. Rev. D* **91**, 045011 (2015).
- [25] J. A. Gracey, *Phys. Rev. D* **93**, 025025 (2016).
- [26] J. Zinn-Justin, *Nucl. Phys.* **B367**, 105 (1991).